IEEE Control Systems Letters paper presented at
2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Nonlinear Feedback Control Design via NEOC

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Abstract— **Quadratic performance indices associated with linear plants offer simplicity and lead to linear feedback control laws, but they may not adequately capture the complexity and flexibility required to address various practical control problems. One notable example is to improve, by using possibly nonlinear laws, on the trade-off between rise time and overshoot commonly observed in classical regulator problems with linear feedback control laws. To address these issues, non-quadratic terms can be introduced into the performance index, resulting in nonlinear control laws. In this study, we tackle the challenge of solving optimal control problems with non-quadratic performance indices using the closed-loop neighboring extremal optimal control (NEOC) approach and homotopy method. Building upon the foundation of the Linear Quadratic Regulator (LQR) framework, we introduce a parameter associated with the non-quadratic terms in the cost function, which is continuously adjusted from 0 to 1. We propose an iterative algorithm based on a closed-loop NEOC framework to handle each gradual adjustment. Additionally, we discuss and analyze the classical work of Bass and Webber, whose approach involves including additional non-quadratic terms in the performance index to render the resulting Hamilton-Jacobi equation analytically solvable. Our findings are supported by numerical examples.**

I. INTRODUCTION

Linear optimal control offers numerous advantages and serves as a foundational concept in optimal control theory. However, in many scenarios, non-linear controllers are expected to outperform even the best linear controllers. A particularly noteworthy scenario where linear feedback control falls short is the common tradeoff between rise time and overshoot observed in closed-loop system step responses. It is observed that achieving a faster rise time often leads to greater overshoot. Similarly, when a closed-loop system in a non-zero initial state needs to decay to zero without any external input, a similar tradeoff arises: faster decay to zero results in a potentially higher overshoot. Overshoot is one of the key design requirements and poses significant challenges in various applications.

It is widely recognized that non-linear feedback controllers offer a potential improvement in dealing with the tradeoff problem, as demonstrated in [1]–[5]. These controllers exhibit a unique behavior where the control gain increases with larger errors and decreases with smaller errors. Consequently, larger control gains lead to reduced rise/decay times. By also dynamically adjusting the gain based on the magnitude of the error, non-linear controllers have the capability to mitigate overshoot. The introduction of non-quadratic terms in the cost function poses however a significant challenge, as it transforms a Linear Quadratic Regulator (LQR) problem into a general optimal control problem, often involving partial differential equations. The theoretical foundations for solving these problems may rely on constructing a Lyapunov function for the system that is also the steady-state solution of the Hamilton–Jacobi equation [1], [2], [5], an often challenging task.

In seminal works such as [1], [2], researchers demonstrated the analytical solvability of specific non-quadratic control problems. This was accomplished by leveraging the compositional structure of the non-quadratic terms, which consist of finite or infinite series of non-negative definite homogeneous multinomials. Nevertheless, while this theoretical framework is compelling, it mandates the incorporation of even more non-quadratic terms into the performance index simply to secure an analytic solution thereby rendering it somewhat artificial. [2].

In this paper, we provide a different approach to address the optimal control problem with a non-quadratic performance index. Instead of offering analytical solutions, we propose a numerical algorithm to directly tackle the resulting non-linear control challenge. Recent studies by Rai et al. [6], [7] introduced a method to handle neighboring extremal optimal control (NEOC) in cases where the original control law is a closed-loop feedback. This method revolves around adapting the optimal control law to parameter changes within the system dynamics or cost function without necessitating the re-solution of the optimal control problem. Initially, we formulate the problem as an LQR, using only the quadratic term in the cost function to establish a baseline. Subsequently, we introduce a parameter associated with the non-quadratic terms in the cost function. We utilize the NEOC and a homotopy approach to solve the non-quadratic optimal control problem by adjusting the parameter from 0 to 1 in a series of small steps. In contrast to prior methodologies [1], [2], [8], our contribution encompasses two key aspects:

1) We propose a numerical approach capable of

This research is supported in part by the NASA University Leadership Initiative (ULI) project (grant no. 80NSSC20M0161), the NSF project (grant no. 2120430) and a gift funding from Northrop Grumman Corporation.

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handling a wider array of non-quadratic terms in the performance index for a given LQR problem, extending beyond solely non-negative definite homogeneous multinomials.
2) Our approach provides freedom to specify the cost functions arbitrarily and does not necessitate the inclusion of additional terms of unknown consequence in the performance index to obtain the solution.

II. Problem Formulation

We first recall the standard linear result, see e.g. [9]. Consider the linear time-invariant (LTI) system

$$\dot{x} = Ax + Bu \quad x(0) = x_0 \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $u \in \mathbb{R}^m$. We assume that Ω is a compact set and that the system is controllable. Consider also a nominal performance index to be optimized

$$V\left(x_0, u(\cdot)\right) = \lim_{T\to\infty} \int_0^T [\|u(t)\|^2 + x^\top(t)Qx(t)]dt \tag{2}$$

where $Q = Q^\top$ is non-negative definite, and such that $[A, Q^{1/2}]$ is observable. With P the unique positive definite solution of the steady state Riccati equation

$$PA + A^\top P - PBB^\top P + Q = 0, \tag{3}$$

the control law given by $u = -B^\top Px$ is optimal, linear, and provably stabilizing. The optimal performance index is $x^\top(0)Px(0)$. As explained in [9], this result is a particular example of the applicability of a Hamilton-Jacobi equation to solve the optimal control problem. The optimal performance expressed as a function of the (initial) state, call it $\phi(x_0)$, satisfies, with $m(x) = x^\top Qx$, the steady-state Hamilton-Jacobi equation which is

$$[\nabla\phi(x)]^\top Ax - \frac{1}{4}[\nabla\phi(x)]^\top BB^\top\nabla\phi(x) + m(x) = 0 \tag{4}$$

and in this case, $\phi(x) = x^\top Px$. Further, the optimal control law is given by the Hamilton-Jacobi theory as

$$u^* = -\frac{1}{2}B^\top\nabla\phi(x). \tag{5}$$

It is well known that the optimal control law for the nominal performance index (2) can result in significant overshoots in variables of interest. Motivated by [1], we introduce non-quadratic terms into an otherwise quadratic performance index in order to reduce the overshoot. We consider the new performance index

$$V\left(x_0, u(\cdot)\right) = \lim_{T\to\infty} \int_0^T [\|u(t)\|^2 + x^\top(t)Qx(t) + \sum_{\nu=2}^{N} \xi_{2\nu}(x(t))]dt \tag{6}$$

where $\xi_{2\nu}(x)$ is an arbitrary non-negative definite homogenous multinomial of degree 2ν in the entries of x. The purpose of the arbitrary $\xi_{2\nu}(x)$ forms (which are of fourth or higher even degree) is to give extra weighting to certain components of x, at least when they are large.

Remark 1: We restrict the non-quadratic terms to homogeneous multinomials to facilitate a comparative analysis with [1]. For the NEOC approach, we only need to assume that the non-quadratic term is a smooth, Lipschitz continuous on Ω, non-negative definite, radially increasing function that is strictly convex (i.e., has a positive definite Hessian) at the origin. These conditions are used to ensure the existence of a stable and optimal control law and in the development of the NEOC approach [6].

Our objective here is to find the admissible control law[1] that minimizes the performance index given in (6). Note that an admissible control law results in a finite integral in (6) for all $x(0)$.

III. Including nonlinear control terms

In this section, we will review a result due to [1] that shows how the introduction of non-quadratic terms (including but not limited to $\xi_{2\nu}(x)$) into an otherwise quadratic performance index for a linear system can result in an analytically computable feedback control law consisting of a linear part (due to the quadratic-only terms in the performance index) and a nonlinear part associated with the non-quadratic terms.

Reference [1] introduces a modification to the performance index, which becomes for an arbitrary integer $N \geq 2$:

$$V\left(x_0, u(\cdot)\right) = \lim_{T\to\infty} \int_0^T [u^2(t) + x^\top(t)Qx(t) + \sum_{\nu=2}^{N} \xi_{2\nu}(x(t)) + \frac{1}{4}[\sum_{\nu=2}^{N} B^\top\nabla\phi_{2\nu}(x(t))]^2]dt, \tag{7}$$

where $\phi_{2\nu}(x)$ is also a non-negative definite multinomial and homogeneous of degree 2ν in the entries of x, and is defined by

$$[\nabla\phi_{2\nu}(x)]^\top[A - BB^\top P]x = -\xi_{2\nu}(x) \tag{8}$$

or, what is equivalent,

$$\phi_{2\nu}(x) = \int_0^\infty \xi_{2\nu}(y(t))dt \tag{9}$$

where $\dot{y} = (A - BB^\top P)y$ and $y(0) = x$.

The purpose of the fourth summand in the performance index (7) is to adjust the performance index so as to allow a closed-form solution of the steady-state Hamilton-Jacobi equation. Including both the original and the supplementary non-quadratic terms in the performance index introduces nonlinear terms of odd degrees in the optimal control law. The following theorem summarizes the solution to this modified optimal control problem. The claims of the theorem, excluding those referring to bounded-input and bounded-state stability, are developed throughout the paper [1] but are consolidated in a concise manner in our proof.

[1] A control law is admissible if it is continuous on Ω, $u(0, \alpha) = 0$, it stabilizes the system, in the sense that $x(t) \to 0$ as $t \to \infty$, while also ensuring that $x(t, \alpha) \in \Omega\ \forall t$, and it results in a finite integral in (6) for all $x(0, \alpha)$ in Ω.

Theorem 1: Consider the system (1) with $[A, B]$ controllable and associated performance indices (2), where Q is nonnegative definite and such that $[A, Q^{1/2}]$ is observable, and (7), where $\xi_{2\nu}(x)$ is a nonnegative homogeneous multinomial form of degree 2ν. With P the unique positive definite solution of the Riccati equation associated with the linear quadratic problem, the nonnegative homogeneous form of degree 2ν designated by $\phi_{2\nu}(x)$ is defined by (8) and (9), and the optimal performance index $\phi(x_0)$ is given by

$$\phi(x) = x^\top Px + \sum_{\nu=2}^{N} \phi_{2\nu}(x) \tag{10}$$

while the optimal control law is given by

$$u^* = -\frac{1}{2}B^\top \nabla\phi(x) = -B^\top Px - \frac{1}{2}B^\top \sum_{\nu=2}^{N} \nabla\phi_{2\nu}(x).$$

The associated closed-loop system is globally asymptotically stable, and exponentially stable over an arbitrarily large bounded set containing the origin; the associated forced system with external input $v(\cdot)$

$$\dot{x} = Ax - BB^\top Px - \frac{1}{2}BB^\top \sum_{\nu=2}^{N} \nabla\phi_{2\nu}(x) + Bv \tag{11}$$

is bounded-input, bounded-state (BIBS) stable [10], i.e. there exists $\mathcal{KL}$-function β and $\mathcal{K}$-function γ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|v\|). \tag{12}$$

Proof: To prove the theorem, we shall first show that the Hamilton-Jacobi equation (4) is satisfied with $m(x)$ corresponding to the last three summands in the index (7). Then the different stability claims will be addressed.

Observe using the definition of $\phi(x)$ in the theorem statement and the Hamilton-Jacobi equation that

$$\begin{aligned}
&[\nabla\phi(x)]^\top Ax - \frac{1}{4}[\nabla\phi(x)]^\top BB^\top[\nabla\phi(x)] \\
&\quad = x^\top[PA + A^\top P - PBB^\top P]x \\
&\qquad + \sum_{\nu=2}^{N}[\nabla\phi_{2\nu}(x)]^\top[A - BB^\top P]x - \frac{1}{4}[B^\top \sum_{\nu=2}^{N} \nabla\phi_{2\nu}(x)]^2, \\
&\quad = -x^\top Qx - \sum_{\nu=2}^{N} \xi_{2\nu}(x) - \frac{1}{4}[B^\top \sum_{\nu=2}^{N} \nabla\phi_{2\nu}(x)]^2.
\end{aligned}$$

The last equality is obtained from the defining equation (8) and the Riccati equation satisfied by P as given in (3). Since the right-hand side is precisely $-m(x)$, this shows that the Hamilton-Jacobi equation is satisfied.

To establish global asymptotic stability, observe first that the function $\phi(x)$ is positive definite, since $x^\top Px$ has this property and the definition of the $\phi_{2\nu}(x)$ functions ensures they are nonnegative. Adopt $\phi(x)$ as a trial Lyapunov function for the closed-loop system obtained with the optimal law, which is

$$\dot{x} = Ax - \frac{1}{2}BB^\top \nabla\phi(x) = Ax - BB^\top Px - \frac{1}{2}BB^\top \sum_{\nu=2}^{N} \nabla\phi_{2\nu}(x) \tag{13}$$

It is readily established that along trajectories of the closed loop system, there holds

$$\begin{aligned}
&\frac{d}{dt}\phi(x(t) = -m(x(t)) - \frac{1}{4}[\nabla\phi(x)]^\top BB^\top[\nabla\phi(x)] \\
&= -x^\top(t)Qx(t) - \sum_{\nu=2}^{N} \xi_{2\nu}(x(t)) - \frac{1}{2}[\sum_{\nu=2}^{N} B^\top \nabla\phi_{2\nu}(x(t))]^2
\end{aligned}$$

This expression is clearly nonpositive, and from it, the required global asymptotic stability follows (the Lasalle theorem being used in case Q is not positive definite).

Exponential stability on an arbitrarily large bounded set follows if the origin, which is the only equilibrium point, is locally exponentially stable. This is equivalent to the property that the closed-loop system linearized around the origin is exponentially stable. From (13), this closed-loop system is simply $\dot{x} = (A - BB^\top P)x$, the stability of which follows from standard linear-quadratic theory. For more details refer to Theorem 4.3 in [6].

The BIBS property comes as an immediate consequence of the exponential stability claim. Consider $r_x > 0$ and $r_u > 0$ such that $\{\|x(0)\| \leq r_x\} \in D_x$ and $\{\|v\| \leq r_v\} \in D_v$. Given we have exponential stability at the origin, there exists a Lyapunov function $V(x)$ such that $c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$. Using the fact that B is bounded and Bu is a Lipschitz map, we can directly conclude from Theorem 5.1 in [11] that the system (11) is BIBS stable and (12) holds. ■

Remark 2: We note that this approach imposes a higher penalty than initially intended. While the $\xi_{2\nu}$ term penalizes states with a degree of 2ν, the additional terms introduced (refer to (7)) penalize states with a degree of $(2\nu - 1)^2$, and in a nontransparent manner.

We remark that [1] notes that the analytic solution of (8) is possible. Since both $\xi_{2\nu}(x)$ and $\phi_{2\nu}(x)$ are homogeneous multinomials, (8) results in L equations with $L = \frac{n(n+1)...(n+2\nu-1)}{(2\nu)!}$, which can be solved for L unknowns by using the inversion of a square matrix. However, it was pointed out that even for $n = 6$ and $2\nu = 6$, then $L = 462$. An alternative calculation in [1] was also proposed that requires the knowledge of the eigenvalues and left eigenvectors of $A - BB^\top P$ to construct the eigenfunctions of the operator $((A - BB^\top P)x)^\top \nabla(\cdot)$. This approach includes representing or expanding $\xi_{2\nu}(x)$ in terms of these eigenfunctions and using obtained coefficients and eigenvalues of the operator to construct $\phi_{2\nu}(x)$. This alternative approach is appropriate for simple examples, but it is not scalable and becomes challenging to solve when $A - BB^\top P$ has complex eigenvalues.

IV. Closed-loop NEOC

In this section, we revisit the principles of closed-loop NEOC, which were originally introduced in [6], [7]. The concept of neighboring extremal optimal control involves determining the adjustments needed in an existing optimal control law due to changes in parameters, such as initial conditions, dynamics, or performance index. This concept was initially explored in the 1960s and has since undergone substantial development, particularly focusing on open-loop optimal control laws. The framework of NEOC for closed-loop laws was introduced in [7], where the authors addressed the problem by utilizing the first variation of the Hamilton-Jacobi equation and subsequently solving the resulting linear partial differential equation.

Recall that our objective is to find the optimal control law that minimizes (6). Since directly solving the new non-quadratic optimal control problem is challenging, we adopt an iterative approach by gradually transitioning the solution from the LQR case (2) to the new problem (6). We introduce a scalar parameter α that serves as the means for this transition. To utilize the iterative framework of NEOC and in preparation for introducing a homotopy in the next section, we modify the performance index (6) with the scalar parameter α as

$$V\left(x_0, u(\cdot), \alpha\right) = \lim_{T\to\infty} \int_0^T [\|u\|^2 + x^\top Q x + \alpha \sum_{\nu=2}^{N} \xi_{2\nu}(x)] dt, \quad (14)$$

where $\alpha \in [0,1]$. We note that $\alpha = 0$ corresponds to the originally designed LQR cost functional (2), whereas $\alpha = 1$ corresponds to the desired performance index (6). For a specific value of α, the minimum performance index $\phi(x,\alpha)$ can be defined as the minimum value of the cost function at the optimal u, expressed as:

$$\phi(x,\alpha) = \min_u V(x, u(\cdot), \alpha). \quad (15)$$

Observe that one can rewrite (4) in parametrized form:

$$[\nabla\phi(x,\alpha)]^\top Ax + m(x,\alpha) - \frac{1}{4}[\nabla\phi(x,\alpha)]^\top BB^\top \nabla\phi(x,\alpha) = 0, \quad (16)$$

where $m(x,\alpha) = x^\top Q x + \alpha \sum_{\nu=2}^{N} \xi_{2\nu}(x)$, and the optimal control law is given by $u^* = -\frac{1}{2}B^\top \nabla\phi(x,\alpha)$.

To study the consequence of small perturbation in the parameter, we define a vector function $\xi(x,\alpha)$ by

$$\xi(x,\alpha) = \frac{\partial\phi(x,\alpha)}{\partial\alpha}, \quad (17)$$

which also means that $\nabla\xi(x,\alpha) = \frac{\partial\nabla\phi(x,\alpha)}{\partial\alpha}$. Differentiating the parameterized steady-state Hamilton-Jacobi equation (16), we obtain:

$$\nabla\xi(x,\alpha)^\top \left[Ax - \frac{1}{2}BB^\top \nabla\phi(x,\alpha)\right] = -\frac{\partial m(x,\alpha)}{\partial\alpha} \quad (18)$$

This means that formally there holds $\xi(x,\alpha) = \int_0^\infty \frac{\partial m(y,\alpha)}{\partial\alpha} dt$, with $y(\cdot)$ defined[2] by $\dot{y} = Ay - \frac{1}{2}BB^\top \nabla\phi(y,\alpha); y(0,\alpha) = x$.

The variation in optimal performance resulting from a small adjustment $\delta\alpha$ away from the initial value α is represented by $\xi(x,\alpha)^\top \delta\alpha$ and the change in optimal control law is given by

$$\delta u(x,\alpha,\delta\alpha) = -\frac{1}{2}B^\top \nabla\xi(x,\alpha)\delta\alpha. \quad (19)$$

[2]The stabilizing property of the control law is crucial here.

The NEOC law is derived by incorporating this adjustment into the original feedback law, resulting in:

$$u_{NE}(x,\hat{\alpha}) = -B^\top Px - \frac{1}{2}B^\top \nabla\xi(x,\alpha)\delta\alpha. \quad (20)$$

Here $\hat{\alpha} = \alpha + \delta\alpha$ is the perturbed system parameter.

V. Numerical Algorithm

In this section, we first introduce a numerical algorithm to obtain the NEOC solution for the modified closed-loop optimal control problem (14) for a specific value of α. This involves determining how the optimal control law changes when the value of α is adjusted by $\delta\alpha$. Then, through the use of a homotopy, we gradually vary α from 0 to 1, transitioning from LQR to solve the original optimal control problem (6).

We model the minimum performance index of (14), $\phi(x,\alpha)$, as a sum of an infinite series. This series consists of smoothly differentiable, linearly independent weighted basis functions $\{\psi_i(x)\}_{i=1}^\infty$, each multiplied by their respective coefficients $\{b_i(\alpha)\}_{i=1}^\infty$, which vary with the parameter α. That means there exists some coefficients $\{b_i(\alpha)\}_{i=1}^\infty$ such that $\phi(x,\alpha) = \sum_{i=1}^\infty b_i(\alpha)\psi_i(x)$. Note that these basis functions are chosen to ensure that $\phi(x,\alpha)$ belongs to the Hilbert space $L^2(\Omega)$, ensuring square integrability.

For any fixed α, we aim to find an appropriate choice of the coefficient vector $\{w_i(\alpha)\}_{i=1}^r$ that is a least squares approximation over the whole set Ω for a given choice of basis function of the following equation[3]

$$\begin{aligned} &w(\alpha)^\top \nabla\Psi(x)Ax + m(x,\alpha) \\ &- \frac{1}{4}w(\alpha)^\top \nabla\Psi(x)BB^\top \nabla\Psi(x)^\top w(\alpha) \approx 0, \end{aligned} \quad (21)$$

where $w(\alpha) = [w_1(\alpha), ..., w_r(\alpha)]^\top \in \mathbb{R}^r$, $\Psi(x) = [\psi_1(x), ..., \psi_r(x)]^\top \in \mathbb{R}^r$, and $\nabla\Psi(x) \in \mathbb{R}^{r\times n}$. The best least squares approximate solution is obtained by choosing the coefficient vector $w(\alpha)$ to ensure that the error between the left and right sides is orthogonal to the basis functions [12]. Hence there holds

$$\begin{aligned} &\left\langle w(\alpha)^\top \nabla\Psi(x)Ax, \psi_i(x)\right\rangle_\Omega + \langle m(x,\alpha), \psi_i(x)\rangle_\Omega \\ &- \frac{1}{4}\left\langle \left\|B^\top \nabla\Psi(x)^\top w(\alpha)\right\|^2, \psi_i(x)\right\rangle_\Omega = 0, \end{aligned} \quad (22)$$

for each $i = 1, 2, \ldots, r$, where the inner product between two continuous functions is defined as the integral of the product of the two functions over the entire space Ω. These equations constitute r linear equations in r unknowns, which are the entries of $w(\alpha)$.[4] The associated optimal control law approximation is obtained as $u(x,\alpha) = -\frac{1}{2}B^\top \nabla\Psi(x)^\top w(\alpha)$.

To determine the sensitivity of the optimal performance index and its corresponding control law (17),

[3]To make the computation manageable, we truncate the infinite series to a finite number of r terms.

[4]In [13], it is confirmed that the equation set is nonsingular, ensuring that $w(\alpha)$ is well-defined.

we employ a calculation similar to (18). By differentiating equation (22) with respect to α, we incorporate derivatives of the weighting coefficients with respect to the parameters, yielding r linear equations as follows:

$$\left(\frac{\partial w(\alpha)}{\partial \alpha}\right)^{\top} E_i = F_i, \tag{23}$$

for $i = 1, 2, \dots, r$, where E_i and F_i are defined as

$$\begin{aligned} E_i &= \left\langle \nabla\Psi(x)Ax, \psi_i(x) \right\rangle_{\Omega} \\ &\quad - \frac{1}{2}\left\langle \nabla\Psi(x)BB^{\top}\nabla\Psi(x)^{\top} w(\alpha), \psi_i(x) \right\rangle_{\Omega}, \\ F_i &= -\left\langle \sum_{\nu=2}^{N} \xi_{2\nu}(x), \psi_i(x) \right\rangle_{\Omega}. \end{aligned}$$

We have used here the definition of $m(x, \alpha) = x^{\top}Qx + \alpha \sum_{\nu=2}^{N} \xi_{2\nu}(x)$. The variation in optimal control law can be obtained using (19) under small perturbation $\delta\alpha$:

$$\delta u(x, \alpha, \delta\alpha) = -\frac{1}{2} B^{\top} \nabla\Psi(x)^{\top} \frac{\partial w(\alpha)}{\partial \alpha} \delta\alpha.$$

Finally, we introduce the homotopy approach to decompose the parameter change ($\alpha : 0 \to 1$) into K distinct equal steps, each corresponding to a small adjustment. To initialize the algorithm, we first determine the coefficient vector w_0 that best approximates the equation $w_0^{\top}\Psi(x) = x^{\top}Px$. This approximation will be accurate if the basis functions are multinomials. The coefficient vector updates at each step as follows:

$$w_{k+1} = w_k + \left(\frac{\partial w(\alpha)}{\partial \alpha}\right) \frac{1}{K}.$$

After completing K iterations, the algorithm yields w_K, which provides an approximation of the minimum performance index $\phi(x)$ of the original optimal control problem (6). It is worth noting that the choice of K depends on the complexity of the non-quadratic terms.

VI. Simulations

In this section, we provide two examples to illustrate the design of nonlinear controllers using closed-loop NEOC and homotopy. Specifically, we demonstrate their application in reducing overshoot without significantly increasing the rise/decay time.

Example 1: We first consider the simple example from [1], where the system is governed by dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

where x_1, x_2, and x_3 represent the position, velocity, and acceleration, respectively. We consider a regulator problem where the task is to drive the position to zero (from some initial value $x_1(0)$). Compared to an LQR controller, the objective is to design a non-linear feedback controller that reduces the overshoot of x_2 while ensuring it does not significantly increase the

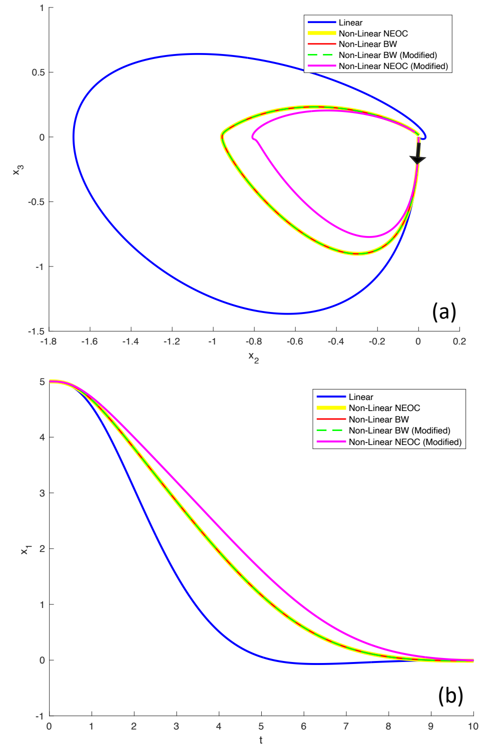

Fig. 1: Example 1. Comparison of the overshoots in (a) velocity, acceleration, and (b) position for linear and non-linear feedback controllers with the initial state $[x_1(0), x_2(0), x_3(0)] = [5, 0, 0]$.

decay time of x_1. Overshoot of x_3 is not explicitly specified as a design objective, but the designs also result in its improvement.

For the benchmark, we employ an LQR with $Q = I_3$. To mitigate the overshoots in x_2, we introduce the term $x_1^2 x_2^2$ in the performance index (6). This term penalizes high values of x_2 (velocity) especially when the position error x_1 is large, while also balancing the decay time of x_1. For comparative analysis, we examine trajectories from three controllers: the LQR, the Bass and Webber controller, and the NEOC control law (with basis functions of even-degree multinomials, up to degree 4). Both non-linear controllers employ the same additional non-quadratic term. Trajectories of the states are illustrated in Fig. 1a and Fig. 1b. Notably, both non-linear feedback controllers yield similar trajectories despite employing different methodologies. They effectively reduce overshoots in both x_2 and x_3 while extending the decay time of x_1. It is noteworthy that both non-linear controllers are cubic in the state. Subsequently, we adjust the non-quadratic term in the performance index to $x_1^2 x_2^4$ for both the non-linear controllers. For NEOC, we restrict the choice of basis functions to degree 4. As illustrated in Fig. 1a, this adjustment further reduces the overshoot without significantly affecting the decay time for NEOC, while no substantial change is observed for the Bass and Webber approach. Note that the NEOC controller

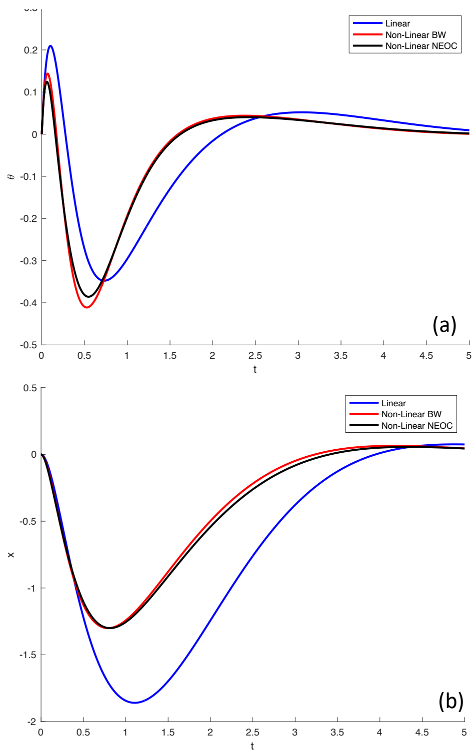

Fig. 2: Example 2. Comparison of the overshoot in pendulum angle and cart position for linear and non-linear feedback controllers with the initial state $[x(0), \dot{x}(0), \theta(0), \dot{\theta}(0)] = [0, 0, 0, 5]$.

retains a cubic degree in states. Conversely, employing Bass and Webber's method for this non-quadratic term necessitates additional computations for a 6th-degree multinomial, resulting in a feedback controller of polynomial degree 5 in the state.

Example 2: Next, we consider [11, Chap. 1] the linearized dynamics of an inverted pendulum mounted on a motorized cart given by

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{p} & \frac{m^2gl^2}{p} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{p} & \frac{mgl(M+m)}{p} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{p} \\ 0 \\ \frac{ml}{p} \end{bmatrix} u,$$

where x and θ denote the cart position and pendulum angle from the vertically upward position, respectively. The masses of the cart and the pendulum are $M = 0.5$ and $m = 0.2$, respectively. The coefficient of friction for the cart is $b = 0.1$. The length of the pendulum is $l = 0.3$, and the mass moment of inertia of the pendulum is $I = 0.006$. For the LQR design, the performance index used is $x^2 + \theta^2$. To reduce the overshoot in position and pendulum angle, we introduce the non-quadratic term of $x^4 + \theta^4$ in the performance index. For NEOC, the choice of basis functions consists of even-degree multinomials, up to degree 4. The resulting trajectories with all three controllers are depicted in Fig. 2a and Fig. 2b, respectively. We note that regarding the cart position (Fig. 2a), the non-linear controllers effectively decrease the overshoot while also reducing the decay time. Conversely, concerning the pendulum angle, they lead to an increase in overshoot but a decrease in decay time. Interestingly, in this particular example, the NEOC method achieves the same decay time as Bass and Webber's approach, yet it yields a lower overshoot.

VII. Conclusions

In this work, we investigate the design of nonlinear control strategies by applying the principles of closed-loop NEOC. This entails incorporating a non-quadratic term into the performance index, tailored to address the specific problem at hand. We propose a numerical approach to tackle this challenge by iteratively solving an approximation of the Hamilton-Jacobi equation, using a combination of homotopy and NEOC methodologies. We introduce a parameter associated with the non-quadratic term in the performance index, which is adjusted from 0 to 1 in a series of steps. To contextualize our work, we compare our methodology with that of Bass and Webber, who utilized a Lyapunov-based approach for specific types of non-quadratic terms. Looking ahead, our research interests include expanding the application of nonlinear controllers for non-quadratic performance indices in set-point control problems.

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