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Convergence of Backward/Forward Sweep for Power Flow Solution in Radial Networks

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Abstract— **Solving power flow is perhaps the most fundamental calculation related to the steady state behavior of alternating-current (AC) power systems. The normally radial (tree) topology of a distribution network induces a spatially recursive structure in power flow equations, which enables a class of efficient solution methods called backward/forward sweep (BFS). In this paper, we revisit BFS from a new perspective, focusing on its convergence. Specifically, we describe a general formulation of BFS, interpret it as a special Gauss-Seidel algorithm, and then illustrate it in a single-phase power flow model. We prove a sufficient condition under which the BFS is a contraction mapping on a closed set of safe voltages and thus converges geometrically to a unique power flow solution. We verify the convergence condition, as well as the accuracy and computational efficiency of BFS, through numerical experiments in IEEE test systems.**

I. Introduction

The steady state of an alternating-current (AC) power system is described by power flow equations. Given the values of some variables (e.g., nodal power injections) in power flow equations, a *power flow* (or load flow) problem is one that solves for other variables (e.g., nodal voltages).

Solving power flow fast and accurately, as the foundation of power system analysis and decision making, has been the focus of extensive studies e.g. [1]. The Newton-Raphson (NR) algorithm for power flow, first proposed in [2] and implemented in [3], works for general networks with cycles. However, it needs to compute Jacobian or solve a linear system in each iteration, a significant computational burden for large networks. The fast decoupled algorithm proposed in [4] is an approximate and greatly simplified version of NR. It works well when line power losses are small, which is a reasonable assumption for high-voltage transmission systems but not for distribution systems.

In the original paper [5] of the single-phase dist-flow model, the solution approach used one-time backward sweep to express all variables in terms of power injections at the feeder head and all branch points, followed by an NR algorithm to solve for these injections. The existence and uniqueness of the solution were studied in [6]. By exploiting the approximate sparsity of the Jacobian matrix in [5], fast decoupled methods were developed and their convergence properties were analyzed in [7]. The fast decoupled methods were extended to three-phase radial networks in [8]. The existence and uniqueness of the solution for three-phase radial networks were analyzed in [9].

A distribution network under normal operation features a radial (tree) topology that induces a spatially recursive structure in power flow equations. This structure can be leveraged to design a class of efficient solution methods called backward/forward sweep (BFS), first proposed in [10] for three-phase distribution systems. It was also called the ladder iterative technique [11, Chapter 10]. A BFS algorithm for a single-phase network was presented in [12] where nodal voltages and line currents are computed iteratively. It was extended in [13] to allow P-V buses (i.e., buses with fixed active power injections and voltage magnitudes) by computing line power flows instead of currents. Both algorithms [12], [13], after modification, are applicable to weakly meshed (transmission) networks as well as radial networks. Another variant of BFS, proposed in [14], calculates voltages in both forward and backward iterations in linear feeders with voltage-dependent loads.

The BFS algorithm in [12] was extended in [15] from single-phase to three-phase networks, and in [16] to four-wire neutral-grounded networks. In [17], three-phase voltages and line currents are calculated with generalized line models that incorporate transformers and constant impedance loads. Transformers of different configurations have been included in BFS through modified augmented nodal analysis [18].

The deployment of distributed generation requires power flow methods to be adapted accordingly. For microgrids with droop-controlled generators, [19] proposes a complex power compensation approach to solve power flows in islanded mode; [20] modifies the BFS method to calculate voltages and line currents; and [21] further incorporates voltage and frequency-dependent loads in BFS. Distributed generators with constant voltage control, modeled as P-V buses, are handled in BFS for three-phase networks, via a detailed modeling of voltage-dependent reactive power control [22], a P-V injection sensitivity matrix [23], a loop analysis method [24], or an NR process to correct the power mismatch at P-V buses and loop breakpoints [25].

In contrast to the design and testing of BFS algorithms, their convergence properties have received little attention. An exception is [26], which proves convergence in a network of constant impedance loads, under a sufficient condition on network structure and parameters. The key facilitating factor of the proof is that the constant impedance loads

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make BFS equivalent to a linear iterative update of voltages. However, for constant power loads (or injections) that are more common in power flow problems, the voltage update functions become nonlinear, to which the analysis in [26] is not applicable.

This paper studies the convergence of BFS with constant power injections, filling the gap mentioned above. We describe a general formulation of BFS, interpret it as a special Gauss-Seidel algorithm, and then illustrate it in a single-phase distribution network. We then prove a sufficient condition for a BFS algorithm to converge to a unique solution within voltage limits. This condition, related to the constant power injections, the network topology and parameters, and the voltage limits, guarantees that BFS is a contraction mapping. We illustrate the BFS algorithm and its convergence condition in a small network. We compare BFS and NR methods (the latter implemented in Matpower) via numerical experiments, in single-phase networks adapted from IEEE 13, 37, 123-bus and 8,500-node distribution test systems. The results verify the convergence condition we found, as well as the accuracy and computational efficiency of BFS.

The rest of the paper is organized as follows. Section II interprets the general BFS method as a Gauss-Seidel algorithm. Section III illustrates BFS in a single-phase distribution network and proves its convergence. Section IV presents numerical results. Section V concludes the paper. We omit proofs in this tutorial paper.

II. General Backward/Forward Sweep

In this section we interpret the general BFS method as a special Gauss-Seidel algorithm. The power flow variables are partitioned into two vectors $x \in \mathbb{A}_1^{n_1}$ and $y \in \mathbb{A}_2^{n_2}$, where $\mathbb{A}_1$, $\mathbb{A}_2$ can be the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or tuples. By choosing the variables and their partition (x, y) appropriately, the power flow equations can be written in a fixed-point form:

$$(x, y) \quad = \quad (f(x, y),\ g(x, y)) \tag{1}$$

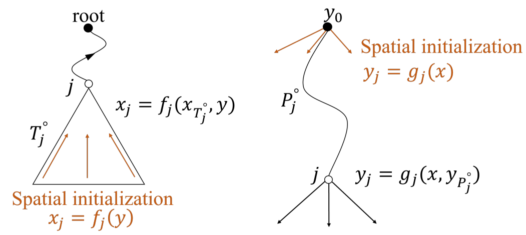
where $f : \mathbb{A}_1^{n_1} \times \mathbb{A}_2^{n_2} \to \mathbb{A}_1^{n_1}$ and $g : \mathbb{A}_1^{n_1} \times \mathbb{A}_2^{n_2} \to \mathbb{A}_2^{n_2}$ are vector-valued continuous mappings with components:

$$x_j \quad = \quad f_j(x, y), \quad \forall j = n_1, \ldots, 1, \tag{2a}$$
$$y_j \quad = \quad g_j(x, y), \quad \forall j = 1, \ldots, n_2. \tag{2b}$$

In (2a), we arrange index j for x_j in a descending order for a reason that will be clear later. A Gauss-Seidel (GS) algorithm is an iterative algorithm to find a fixed point (x, y) of (1). In each iteration t, it updates components of (x, y) in turn, using the latest available component values:

$$x_j(t) = f_j\big(x_{n_1}(t), \ldots, x_{j+1}(t),\ x_j(t-1), \ldots, x_1(t-1), y(t-1)\big), \quad \forall j = n_1, \ldots, 1, \tag{3a}$$
$$y_j(t) = g_j\big(x(t),\ y_1(t), \ldots, y_{j-1}(t),\ y_j(t-1), \ldots, y_{n_2}(t-1)\big), \quad \forall j = 1, \ldots, n_2. \tag{3b}$$

In a tree network, the power flow equations (2) have a spatially recursive structure as follows. Let N denote both



(a) Backward sweep. (b) Forward sweep.

Fig. 1. General backward/forward sweep.

the set and the number of buses (i.e., nodes) in the network,[1] *excluding* the root as bus 0. With $N = n_1 = n_2$, we label the buses by index j, starting from $j = 1$ which is the first child of the root, propagating towards the leaves in a breadth-first search. Each variable (x_j, y_j) is associated with bus j; if x_j or y_j is a line variable (e.g. line current), then bus j is the end of the line that is *farther away* from the root.

Let T_j° denote the set of buses in the subtree rooted at bus j, excluding j. We have $\mathsf{T}_j^\circ \subseteq \{N, ..., j+1\}$ due to the order in which the buses are labeled. Define $x_{\mathsf{T}_j^\circ} := \big(x_k, \forall k \in \mathsf{T}_j^\circ\big)$. We say that x in (2a) satisfies a *spatially recursive structure* if, given y, each x_j depends on x only through $x_{\mathsf{T}_j^\circ}$:

$$x_j \quad = \quad f_j(x_{\mathsf{T}_j^\circ},\ y), \quad \forall j \in N. \tag{4}$$

The boundary condition for (4) is that, if j is a leaf bus, then $\mathsf{T}_j^\circ = \emptyset$ and $x_j = f_j(\emptyset, y) =: f_j(y)$. This relation starts (4) as a *backward sweep* to recursively update x, working from the leaves towards the root, as illustrated in Figure 1(a).

Let P_j° denote the set of buses in the unique path from bus 0 to bus j, including neither 0 nor j. We have $\mathsf{P}_j^\circ \subseteq \{1, ..., j-1\}$. Define $y_{\mathsf{P}_j^\circ} := \big(y_k, \forall k \in \mathsf{P}_j^\circ\big)$. We say that the variable y in (2b) satisfies a *spatially recursive structure* if given x, each y_j depends on y only through $y_{\mathsf{P}_j^\circ}$:

$$y_j \quad = \quad g_j(x,\ y_{\mathsf{P}_j^\circ}), \quad \forall j \in N. \tag{5}$$

The boundary condition for (5) is that, if j is a child of the root bus 0, then $\mathsf{P}_j^\circ = \emptyset$ and $y_j = g_j(x, \emptyset) =: g_j(x)$. It is often the case that y_j also depends on y_0 at bus 0, which is a given constant instead of a variable and thus does not appear as input to function g_j. This relation starts (5) as a *forward sweep* to recursively update y, working from the root towards the leaves, as illustrated in Figure 1(b).

If (x, y) in (2) satisfies the spatially recursive structure, the general BFS algorithm, Algorithm 1, is essentially a special GS algorithm to compute a fixed point of (1), equivalently (4) and (5). In particular, the GS algorithm (3) is reduced to lines 2-b) and 2-c) of Algorithm 1, since x_j depends only on $x_{\mathsf{T}_j^\circ}$ (given y) and y_j only on $y_{\mathsf{P}_j^\circ}$ (given x).

Under the general BFS framework, different algorithms may differ in their choices of variables (x, y) and the associated power flow equations, based in part on what information is given in a power flow problem. These choices are not unique and may result in different convergence properties.

[1] We slightly abuse notation for convenience when no confusion is caused.

Algorithm 1: General backward/forward sweep

Input: $(f_j, g_j, \mathsf{T}^\circ_j, \mathsf{P}^\circ_j,\ \forall j \in N)$, and $(y_0, y(0)) \in \mathbb{A}_2^{N+1}$ with $y(0) = (y_j(0), \forall j \in N)$.

Output: a solution (x, y) of (4) and (5).

1) **Initialization**:
   - $\mathsf{T}^\circ_j = \emptyset$ for all leaf buses j;
   - $\mathsf{P}^\circ_j = \emptyset$ for all children j of the root;
   - $t = 0$.
2) **while** stopping criterion is not met **do**
   a) $t \leftarrow t + 1$;
   b) *Backward sweep*: **for** j from the leaves towards the root **do**
   $$x_j(t) \leftarrow f_j(x_{\mathsf{T}^\circ_j}(t), y(t-1)),\ \forall j \in N;$$
   c) *Forward sweep*: **for** j from the root towards the leaves **do**
   $$y_j(t) \leftarrow g_j(x(t), y_{\mathsf{P}^\circ_j}(t)),\ \forall j \in N.$$
3) **Return**: $x = x(t)$, $y = y(t)$.

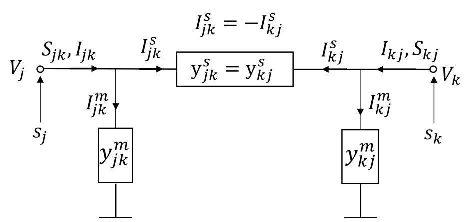


Fig. 2. Π-equivalent circuit of a distribution line.

Usually the voltage at the substation bus (the root of the tree) is specified as y_0 and the line currents or powers downstream of the leaves are zero. These two boundary conditions determine that most BFS algorithms compute line currents or powers as x in the backward sweep and bus voltages as y in the forward sweep. Next, we will look at such an example and analyze its convergence properties.

III. BFS in Single-Phase Distribution Networks

We now present the BFS algorithm of [12] and analyze its convergence.

A. Network model and BFS algorithm

In a single-phase radial distribution network, suppose the complex voltage V_0 at the root bus 0 and complex power injections $(s_j, \forall j \in N)$ at all non-root buses are given. The network is considered as a directed tree graph, where every line points *away* from the root. A line $j \to k$ is modeled by the Π-equivalent circuit in Figure 2, where nonzero constant parameter $y^s_{jk} = y^s_{kj}$ is the series admittance,[2] with its reciprocal $z^s_{jk} = z^s_{kj} := (y^s_{jk})^{-1}$ the series impedance, and y^m_{jk} and y^m_{kj} are the shunt admittances of the line at buses j and k, respectively. Let I_{jk} denote the sending-end current from buses j to k and I^s_{jk} the current across the series admittance y^s_{jk}, i.e., $I_{jk} = I^s_{jk} + y^m_{jk}V_j$ where V_j is the complex voltage at bus j. For each non-root bus $j \in N$, let $l := l(j)$ denote its unique parent bus between bus 0 and bus j. In this way, currents I^s_{lj} across all lines $l \to j$ are identified by $j \in N$.

We will compute currents $I^s := (I^s_{lj}, \forall j \in N)$ and voltages $V := (V_j, \forall j \in N)$. All other variables, such as sending-end currents I_{lj} and powers S_{lj}, can then be determined. Taking I^s as x and V as y, the recursive structure (4)(5) is the power flow equations in [12]:

$$I^s_{lj} = \sum_{k:j\to k} I^s_{jk} - \left(\frac{s_j}{V_j}\right)^* + y^m_{jj}V_j,\ \forall j \in N, \quad (6a)$$

$$V_j = V_l - z^s_{lj} I^s_{lj},\ \forall j \in N \quad (6b)$$

where $y^m_{jj} := y^m_{jl} + \sum_{k:j\to k} y^m_{jk}$ is the total shunt admittance incident on bus j. Equation (6a) is the Kirchhoff's current law, i.e., the sum of currents flowing into a bus equals that flowing out. In particular, the power injection $s_j = V_j I^*_j$, where I_j is the complex current injection to bus j and I^*_j is its conjugate. Equation (6b) is the Ohm's law.

Add iteration indices to (6), by changing I^s_{lj}, I^s_{jk} to $I^s_{lj}(t)$, $I^s_{jk}(t)$ and V_j to $V_j(t-1)$ in (6a), and changing V_j, V_l, I^s_{lj} to $V_j(t)$, $V_l(t)$, $I^s_{lj}(t)$ in (6b). This specifies the general BFS, Algorithm 1, as a BFS algorithm in the single-phase radial distribution network. The boundary conditions are:

- $I^s_{jk}(t) = 0$ for all leaf buses j and all $t \geq 1$;
- V_0 at the root bus 0 is given and fixed;
- $V(0) = (V_j(0), \forall j \in N)$ is given, e.g., all as V_0.

A stopping criterion can be based on the discrepancy between the given power injections s_j and the injections $s_j(t)$ implied by $I^s(t)$ and $V(t)$ at the end of iteration t. Specifically let

$$s_j(t) := V_j(t)\left(\sum_{k:j\to k} I^s_{jk}(t) - I^s_{lj}(t)\right)^* + \left(y^m_{jj}\right)^* |V_j(t)|^2$$

for all $j \in N$ and $t \geq 1$, and a stopping criterion can be:

$$\|s(t) - s\|_2 := \sqrt{\sum_{j\in N} |s_j(t) - s_j|^2} < \delta$$

for a given tolerance $\delta > 0$.

B. Convergence analysis

The BFS update functions (6) can be represented compactly using the bus-by-line incidence matrix $C \in \{-1, 0, 1\}^{(N+1)\times N}$ of the radial network, defined by:

$$C_{je} = \begin{cases} 1 & \text{if } e = j \to k \text{ for some bus } k, \\ -1 & \text{if } e = l \to j \text{ for some bus } l, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix C is of rank N and singular. Decompose C into the $N \times N$ non-singular reduced incidence matrix $\hat{C}$ and its first row c_0^T corresponding to the root bus 0:

$$C = \begin{bmatrix} c_0^\mathsf{T} \\ \hat{C} \end{bmatrix}.$$

[2]Here we use y as admittance by convention, which is different from y as a variable in the general BFS method.

Define $N \times N$ diagonal matrices:

$$\hat{s} := \text{diag}\left(s_j, \forall j \in N\right), \quad \hat{y}^m := \text{diag}\left(y^m_{jj}, \forall j \in N\right),$$
$$\hat{y}^s := \text{diag}\left(y^s_{lj}, \forall j \in N\right).$$

The BFS algorithm (6) (with appropriate iteration index t and $(t-1)$ as explained) is equivalent to the following nonlinear dynamical system:

$$\hat{C} I^s(t) \;=\; \hat{s}^* V^{-*}(t-1) - \hat{y}^m V(t-1), \quad \text{(7a)}$$
$$c_0 V_0 + \hat{C}^\mathsf{T} V(t) \;=\; (\hat{y}^s)^{-1} I^s(t) \quad \text{(7b)}$$

where the j-th element of $\hat{C} I^s$ is $\sum_{k: j\to k} I^s_{jk} - I^s_{lj}$, i.e., the net current flowing out of bus $j \in N$; column vector $V^{-*} := (1/V_j^*, \forall j \in N)$; and $\hat{s}^*$ takes the element-wise conjugate of diagonal matrix $\hat{s}$. The j-th element of $(c_0 V_0 + \hat{C}^\mathsf{T} V)$ is the voltage difference $(V_l - V_j)$ across line $l \to j$. Eliminate $\hat{C} I^s(t)$ in (7) to get a dynamical system in terms of V only:

$$\begin{aligned} V(t) \;&=\; \hat{L}^{-1}\left[\hat{s}^* V^{-*}(t-1) - \hat{y}^m V(t-1) - \hat{C}\hat{y}^s c_0 V_0\right] \\ &=: \; F(V(t-1)) \end{aligned} \quad (8)$$

where the invertible, reduced Laplacian matrix $\hat{L} := \hat{C}\hat{y}^s\hat{C}^\mathsf{T}$ encodes the network topology and series admittances.

We focus on the compact form (8) of BFS to prove its convergence. Without loss of generality, use the per unit system with the nominal voltage 1 per unit (pu). Consider the following set of voltages that satisfy the safety limit:

$$\mathbb{V} := \left\{ V \in \mathbb{C}^N \mid 1-\epsilon \le |V_j| \le 1+\epsilon, \ \forall j \in N \right\} \quad (9)$$

with a given constant $\epsilon \in (0,1)$, e.g., $\epsilon = 0.05$. The following lemma provides a sufficient condition to make (8) a mapping from $\mathbb{V}$ onto $\mathbb{V}$. This kind of condition, which was missing in previous work such as [1], not only facilitates our convergence proof of BFS, but also guarantees voltage safety of the obtained solution.

Lemma 1. *Suppose $|V_0| = 1$ for easy exposition. Define column vectors of nonnegative real numbers:*

$$|s| := [|s_1|, \ldots, |s_N|]^\mathsf{T}, \quad |y^m| := [|y^m_{11}|, \ldots, |y^m_{NN}|]^\mathsf{T},$$
$$|\bar{I}| := \frac{|s|}{1-\epsilon} \; + \; (1+\epsilon)|y^m|$$

and $N \times N$ real matrices:

$$|\hat{y}^s| := \text{diag}\left(|y^s_{lj}|, \forall j \in N\right), \quad |\hat{L}| := \hat{C}|\hat{y}^s|\hat{C}^\mathsf{T}.$$

Suppose the following condition is satisfied:

$$\frac{1}{\epsilon}\left\| |\hat{L}|^{-1}|\bar{I}| \right\|_\infty \;\le\; 1 \quad (10)$$

where $\|u\|_\infty := \max_i |u_i|$. If $V \in \mathbb{V}$ then $F(V) \in \mathbb{V}$.

The following result provides a sufficient condition for the convergence of the BFS algorithm (8), by proving F to be a contraction mapping from $\mathbb{V}$ onto $\mathbb{V}$.

Theorem 1. *Suppose $|V_0| = 1$ and (10) in Lemma 1 are satisfied, and*

$$\rho := \frac{1}{(1-\epsilon)^2}\left\|\hat{L}^{-1}\hat{s}^*\right\|_2 + \left\|\hat{L}^{-1}\hat{y}^m\right\|_2 \;<\; 1 \quad (11)$$

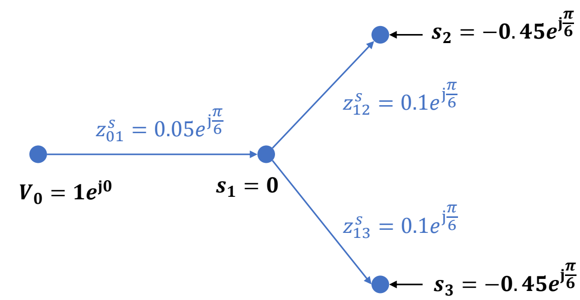


Fig. 3. A small example of power flow. Network parameters and input variables are given in per unit. All shunt admittances are set as zero.

where $\|\cdot\|_2$ takes the spectral norm of a complex square matrix. Then, on the set $\mathbb{V}$:

1) *There is a unique fixed point (i.e., power flow solution) V of (8);*
2) *Starting from any $V(0) \in \mathbb{V}$, the sequence $(V(t), \forall t \ge 1)$ produced by (8) converges geometrically to the fixed point V, i.e., $\|V(t) - V\|_2 \le \rho^t \|V(0) - V\|_2$.*

The proofs of Lemma 1 and Theorem 1 are skipped. We remark on the conditions in Theorem 1. Parameter ϵ that controls the voltage limits in $\mathbb{V}$ determines a trade-off between satisfying the condition (10) in Lemma 1 for voltage limits and (11) in Theorem 1 on the rate ρ of contraction. Specifically, condition (10) contains two factors $\frac{1}{\epsilon(1-\epsilon)}$ and $\frac{1}{\epsilon}$ on its left-hand side, both of which increase as ϵ decreases within $(0, 0.5]$, making it harder to satisfy (10), i.e., harder to enforce self-mapping on a smaller set $\mathbb{V}$. Meanwhile, the modulus ρ of mapping F in (11) decreases as ϵ decreases, making it easier to be a contraction mapping on a smaller set $\mathbb{V}$. Other factors that may help satisfy (10) and (11) include:

- Smaller bus current injections that come from power injections s_j and/or shunt admittances y^m_{jj}, depending on whether their currents reinforce or cancel each other;
- Smaller elements of matrices $|\hat{L}|^{-1}$ and $\hat{L}^{-1}$, which may be realized by larger series admittances y^s_{lj} and shorter common paths between pairs of buses to the root; the latter is often easier for smaller-depth trees.

We also remark that conditions (10) and (11) are sufficient for the convergence of BFS (8) (equivalently (6)), but may not be necessary. They may be conservative as the numerical experiments below will show.

C. Illustration in a small network

We illustrate the BFS algorithm and its convergence condition using the small example in Figure 3. The line series impedances, root bus voltage, and non-root bus power injections are given in the figure, all in per unit. All shunt admittances are set as zero. Initialized as $V_j(0) = V_0$,

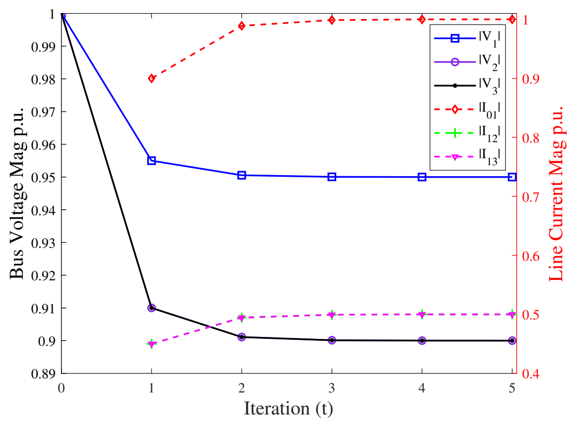


Fig. 4. The magnitudes of bus voltages and line currents over the BFS iterations to solve the case in Figure 3.

$j = 1, 2, 3$, the BFS algorithm (6) for iterations $t \geq 1$ is:

$$\begin{aligned}
\text{Backward sweep:} \quad I_{12}^s(t) &= -\frac{s_2^*}{V_2^*(t-1)} \\
I_{13}^s(t) &= -\frac{s_3^*}{V_3^*(t-1)} \\
I_{01}^s(t) &= I_{12}^s(t) + I_{13}^s(t) \\
\text{Forward sweep:} \quad V_1(t) &= V_0 - z_{01}^s I_{01}^s(t) \\
V_2(t) &= V_1(t) - z_{12}^s I_{12}^s(t) \\
V_3(t) &= V_1(t) - z_{13}^s I_{13}^s(t).
\end{aligned}$$

This BFS process can be expressed in the compact form (8), with $y_{lj}^s = (z_{lj}^s)^{-1}$ for all lines $l \to j$ and incidence matrix:

$$C = \begin{bmatrix} c_0^\mathsf{T} \\ \hat{C} \end{bmatrix}, \; c_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \hat{C} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

to construct the reduced Laplacian matrix $\hat{L} = \hat{C}\hat{y}^s\hat{C}^\mathsf{T}$.

Figure 4 shows the magnitudes of bus voltages $V = (V_1, V_2, V_3)$ and line currents $I^s = (I_{01}^s, I_{12}^s, I_{13}^s)$ across the BFS iterations. The plots for V_2 and V_3 overlap, so do those for I_{12}^s and I_{13}^s, due to the symmetry of the network structure and parameters. The BFS converges quickly to the actual power flow solution $V = (0.95, 0.9, 0.9)e^{\mathrm{j}0}$ whose angles happen to be all zero. Using $\epsilon = 0.1$ pu, the left-hand side of condition (10) for Lemma 1 is exactly 1, which barely guarantees safe voltages at/above 0.9 pu. The rate of contraction in condition (11) for Theorem 1 is $\rho = 0.12$, sufficiently small to lead to the quick convergence.

IV. Numerical Experiments

We program the presented BFS algorithm in MATLAB (R2021b) and compare it with the Newton-Raphson (NR) algorithm implemented in Matpower, to verify convergence and performance of BFS. The programs are run on a desktop computer with 11th Gen Intel(R) Core(TM) i7-11700 processors at 2.50 GHz and 16GB RAM.

We use the programs to solve AC power flow in single-phase radial network models modified from the originally three-phase IEEE 13, 37, 123-bus and 8,500-node distribution test systems. Single-phase constant power loads (or injections) and line parameters are obtained by averaging three phases of the original IEEE data. In particular, we remove all capacitors and split-phase transformers from the 8,500-node system (that has $4,876$ buses, each with 1, 2, or 3 phases, where a phase is a node). The $2,354$ buses on the secondary side of the split-phase transformers are consolidated into the primary side, thus reducing the system to $2,522$ buses. We also experiment on a light-load version of the modified 8,500-node system, referred to as 8,500-node-$\frac{1}{3}$, by decreasing the active and reactive power loads at all buses to $1/3$ of the original values.

TABLE I

Convergence condition and solution error of BFS

Test case	(10)	(11)	Max (pu)	RMSE (pu)
13-bus	0.34	0.08	1.6×10^{-5}	1.2×10^{-5}
37-bus	0.61	0.28	2.5×10^{-7}	1.5×10^{-7}
123-bus	1.83	1.20	2.3×10^{-6}	1.6×10^{-6}
8,500-node-$\frac{1}{3}$	1.44	6.11	4.8×10^{-7}	3.5×10^{-7}
8,500-node	4.32	18.32	-	-

TABLE II

Computational efficiency of BFS and Matpower NR

Test case	Iterations		Convergence time (sec)	
	BFS	Matpower	BFS	Matpower
13-bus	3	3	5.3×10^{-5}	1.5×10^{-3}
37-bus	4	3	1.7×10^{-4}	1.7×10^{-3}
123-bus	5	4	7.6×10^{-4}	3.3×10^{-3}
8,500-node-$\frac{1}{3}$	6	3	3.9×10^{-1}	2.0×10^{-2}

We set $\epsilon = 0.1$ pu to calculate the left-hand side of condition (10) for Lemma 1 and (11) for Theorem 1, as shown in Table I. For all test cases except the 8,500-node system, the BFS algorithm converged, i.e., our stopping criterion: $\max_{j \in N} |V_j(t) - V_j(t-1)| < 10^{-4}$ pu was met in these cases. Not all the convergent cases, however, satisfied conditions (10) and (11). This means that the sufficient condition in Theorem 1 is generally conservative. For the 8,500-node system, conditions (10) and (11) were violated so much that BFS did not converge. Indeed, for this case the Matpower NR algorithm also failed to converge.

For the test cases in which the BFS algorithm converged, we compare its voltage solution to that obtained by the Matpower NR algorithm, and show in Table I the maximum error (Max) and root-mean-square error (RMSE) between them across all buses. We observe that BFS is quite accurate in solving power flow, if we take Matpower as a well accepted benchmark. As shown in Table II, the computational efficiency of BFS is comparable with, sometimes better than, the Matpower NR algorithm, in terms of the number of iterations and time taken until convergence.

V. Conclusion

We have interpreted a general BFS method as a special Gauss-Seidel algorithm, and illustrated it in single-phase radial distribution networks. We have proved a sufficient condition for the BFS algorithm of [12] to be a contraction mapping, hence guaranteeing its convergence to a unique power flow solution over the set that satisfies voltage limits. We discussed the implication of the convergence condition in terms of bus power injections, network topology and parameters, and the size of the voltage limits. Numerical experiments in multiple IEEE test cases showed that the BFS algorithm achieves satisfactory convergence, solution accuracy, and computational efficiency, comparable with or better than the Newton-Raphson algorithm in Matpower.

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