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Modeling Unbalanced Power Flow with Δ-connected Devices

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***Abstract*— In this tutorial we present a simple approach to modeling unbalanced three-phase power flows. We allow general non-ideal models of voltage sources, ZIP loads as well as distribution lines and transformers. The basic idea is to explicitly separate a device/transformer model into an *internal model*, that depends on the characteristics of the single-phase devices or transformers, and a *conversion rule*, that depends on their configuration. This approach provides two benefits. First it facilitates the modeling of secondary distribution circuits where only the end devices are directly controllable, not the currents or powers at the secondary transformers. Second it allows us to exploit common structures across different device/transformer variants and derive their external models that are general and unified. We illustrate these benefits by extending a three-phase backward forward sweep method in the literature to allow secondary circuits and formulating a three-phase optimal power flow problem as a quadratically constrained quadratic program.**

I. Introduction

Motivation. Unbalanced three-phase systems are subtle because currents and voltages in different phases are coupled; see, e.g., [1, Chapter 11] for transmission systems and [2] for distribution systems. We often approximate such a system as balanced because a balanced three-phase system has a single-phase equivalent and can be analyzed using single-phase techniques. This approximation is reasonable for transmission systems but usually not for distribution systems. Modeling unbalanced three-phase systems with Δ configured devices or transformers may seem confusing because the voltages and currents across single-phase devices internal to Δ configuration are observed externally only through a linear map that is not invertible. In practice we can only control the *internal* variables of these end devices, e.g., controlling the charging currents of electric vehicle chargers in Δ configuration. Our control decisions interact with other devices over the network, however, only through *terminal* voltages and currents observable externally of three-phase devices. The interplay between internal and terminal variables sometimes causes confusion, and is the key to the modeling and analysis of unbalanced three-phase systems with both Y and Δ-configured devices and transformers.

We have developed such a modeling approach in [3–5]. It makes transparent the unified structure of three-phase systems and shows that models and properties of single-phase networks have direct extensions to three-phase networks. The basic idea is to explicitly separate a device/transformer model into an *internal model* that specifies the characteristics of the single-phase devices or transformers, and a *conversion rule* that maps internal variables to terminal variables. The internal model depends only on the type of components (non-ideal voltage sources, ZIP loads, or different single-phase transformer models) regardless of their configurations. The conversion rule depends only on their configurations regardless of the type of components.

This separation provides two benefits. First it facilitates the modeling of secondary distribution circuits where usually only the end devices are directly controllable, not the currents or powers at the transformers. Second it allows us to exploit common structures across different device/transformer variants and derive their external models that are general, unifying, and simple. We illustrate our method by applying it to solving power flow through backward forward sweep (BFS) (Section VI) and formulating three-phase optimal power flow (OPF) (Section VII).

This tutorial is a summary of this method developed in [3–5]. In particular [4] introduces the basic framework, focusing on ideal devices (voltage sources and ZIP loads) without transformers, and deriving single-phase analysis for a balanced network; [5] applies the method to modeling three-phase transformers. This tutorial presents models for nonideal devices and illustrates our approach in BFS and OPF (We will focus on systems with only Δ devices due to page limit).

Literature. There is a large literature on three-phase power flow analysis and we only make a few brief remarks. Three-phase load flow solvers have been developed since at least the 1960s, e.g., see [6] for solution in the sequence coordinate and [7, 8] in the phase coordinate. A three-phase network is equivalent to a single-phase circuit where each node in the equivalent circuit is indexed by a (bus, phase) pair [8]. Single-phase power flow algorithms such as Newton Raphson [9] or Fast Decoupled methods [10] can be directly applied to the equivalent circuit. The main difference with a single-phase network is the models of three-phase devices in the equivalent circuit, such as models for three-phase lines [2, 11, 12], transformers and co-generators [8, 13–15][2, Ch 8][16–19], constant-power devices [1, Chapter 11], as well as voltage regulators, and loads [2], etc.

A state-of-the-art algorithm in [1, Chapter 11] expresses currents in terms of voltages for both PQ and PV buses, applies the Newton-Raphson algorithm to the resulting nonlinear current balance equation $I = YV$ in the sequence

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domain. It allows both grounded and ungrounded loads in Y and Δ configurations. For transmission networks, computing in the sequence domain has the advantage that, when most lines in the network are symmetric and thus have decoupled representation in the sequence coordinate, the Jacobian matrix is sparse. Sometimes an approximate solution is computed by ignoring the coupling across zero, positive, and negative-sequence variables and solving the three sequence networks separately as single-phase networks, e.g., [20]. Distribution networks usually does not enjoy such simplification and hence computation is usually done in the phase coordinate. For example, [21] writes the AC power flow equation as a fixed point iteration on voltages whose fixed points are power flow solutions. It derives sufficient conditions for the iteration to be a contraction, ensuring the existence and uniqueness of the solution. Sufficient conditions are proved in [22] for the invertibility of three-phase admittance matrix which then ensures the validity of Z-bus method for computing power flow solutions. Finally recent studies on three-phase AC optimal power flow problems and their semidefinite relaxations include e.g. [23–25].

Paper organization. In Section II we describe the internal models that define the internal behavior of non-ideal voltage sources and ZIP loads, and the conversion rule that maps internal models to external models. In Section III we present a three-wire model of transmission or distribution lines. In Section IV we summarize a unified model of transformers that cover all standard transformers. In Section V we put the component models of Sections II, III, IV together to construct an overall network model. Finally we illustrate our modeling approach in Section VI by extending the three-phase backward forward sweep method of [26] to include Δ-connected devices, and in Section VII by formulating a three-phase optimal power flow problem as a quadratically constrained quadratic program.

Notation. Let $\mathbb{C}$ denote the set of complex numbers. For $a \in \mathbb{C}$, Re a and Im a denote its real and imaginary parts respectively, and $\bar{a}$ or a^{H} denotes its complex conjugate. We use $\mathbf{i}$ to denote $\sqrt{-1}$. A vector $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$ is a column vector. Its componentwise complex conjugate is denoted by $\overline{x}$. For any matrix A, A^{T} and A^{H} denote its transpose and Hermitian transpose respectively. If x is a matrix then $\mathrm{diag}\,(x)$ is the vector whose components are the diagonal entries of x, whereas if x is a vector then $\mathrm{diag}(x)$ is a diagonal matrix with x_i as its diagonal entries. Finally $\mathbf{1} \in \mathbb{C}^3$ is the column vector of size 3 whose entries are all 1s and $\mathbb{I}$ is the identity matrix of size 3.

II. Three-phase device models

A three-phase device consists of three single-phase devices arranged in Y or Δ configuration. The behavior of each single-phase device defines the internal behavior of the three-phase device and is independent of the configuration. The mapping between internal voltages and currents across the single-phase devices and the terminal voltages and currents of the three-phase device is the conversion rule and is independent of the type of devices. The internal behavior and the conversion rule together determine the behavior of the three-phase device that is observable externally. We explain each of these next.

A. Internal and terminal variables

A generic three-phase device that consists of three single-phase devices in Δ configuration is shown in Figure 1. Its internal behavior is described in terms of the *internal*

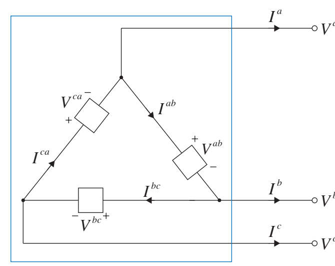


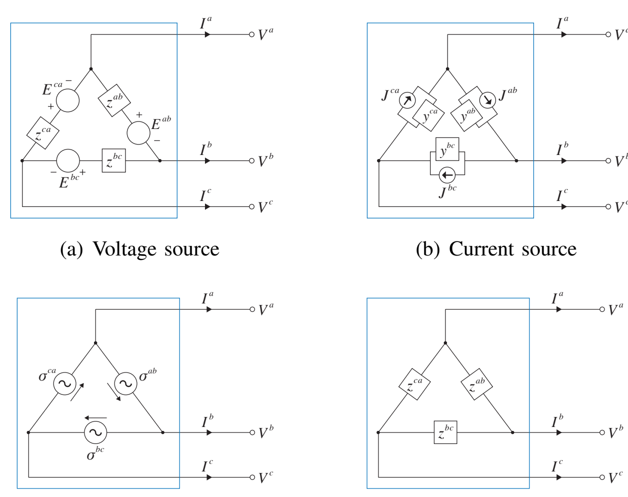
Fig. 1. Internal and external variables of a Δ-configured device.

variables: line-to-line voltages $V^\Delta := \left(V^{ab}, V^{bc}, V^{ca}\right) \in \mathbb{C}^3$, currents $I^\Delta := \left(I^{ab}, I^{bc}, I^{ca}\right) \in \mathbb{C}^3$, and powers $s^\Delta := \left(s^{ab}, s^{bc}, s^{ca}\right) \in \mathbb{C}^3$ across the single-phase devices. By definition $s^\Delta := \mathrm{diag}(V^\Delta I^{\Delta\mathsf{H}})$ in the direction of I^Δ.

The external behavior of the three-phase device is described in terms of its *terminal variables*: terminal voltages $V := \left(V^a, V^b, V^c\right) \in \mathbb{C}^3$ with respect to an arbitrary but common reference point (e.g., the ground), currents $I := \left(I^a, I^b, I^c\right) \in \mathbb{C}^3$ in the direction out of the device, and powers $s := \left(s^a, s^b, s^c\right) \in \mathbb{C}^3$. By definition $s := \mathrm{diag}(VI^{\mathsf{H}})$ are the powers across each of the terminals a, b, c and the common reference point.

B. Internal models

We present the internal models of four *non-ideal* devices in Figure 2, often referred to as voltage sources and ZIP loads (see [4] for ideal devices).



(a) Voltage source

(b) Current source

(c) Power source

(d) Impedance

Fig. 2. Three-phase devices in Δ configuration.

a) Voltage source $\left(E^{\Delta}, z^{\Delta}\right)$*:* A three-phase voltage source in Δ configuration is characterized by a vector $E^{\Delta} := (E^{ab}, E^{bc}, E^{ca}) \in \mathbb{C}^3$ of voltages across three ideal single-phase voltage sources in phases a, b, c in series with impedances in each phase specified by an impedance matrix $z^{\Delta} := \mathrm{diag}(z^{ab}, z^{bc}, z^{ca}) \in \mathbb{C}^{3\times 3}$, as shown in Figure 2(a). Its internal model is defined by relations between its internal voltage V^{Δ}, current I^{Δ} and power s^{Δ}:

$$V^{\Delta} = E^{\Delta} + z^{\Delta} I^{\Delta} \tag{1a}$$

$$s^{\Delta} = \mathrm{diag}\left(E^{\Delta} I^{\Delta \mathsf{H}}\right) + \mathrm{diag}\left(z^{\Delta} I^{\Delta} I^{\Delta \mathsf{H}}\right) \tag{1b}$$

b) Current source $\left(J^{\Delta}, y^{\Delta}\right)$*:* A three-phase current source in Δ configuration is characterized by three single-phase ideal current sources, specified by $J^{\Delta} := (J^{ab}, J^{bc}, J^{ca})$, in parallel with shunt admittances $y^{\Delta} := \mathrm{diag}(y^{ab}, y^{bc}, y^{ca})$; see Figure 2(b). Its internal model is:

$$I^{\Delta} = J^{\Delta} + y^{\Delta} V^{\Delta} \tag{2a}$$

$$s^{\Delta} = \mathrm{diag}\left(V^{\Delta} J^{\Delta \mathsf{H}}\right) + \mathrm{diag}\left(V^{\Delta} V^{\Delta \mathsf{H}} y^{\Delta \mathsf{H}}\right) \tag{2b}$$

c) Power source σ^{Δ}*:* A three-phase power source in Δ configuration is characterized by three single-phase ideal power sources, specified by $\sigma^{\Delta} := (\sigma^{ab}, \sigma^{bc}, \sigma^{ca})$, as shown in Figure 2(c). Each single-phase ideal power source supplies a constant power σ^{ϕ} and hence the internal model of the three-phase power source σ^{Δ} is:

$$s^{\Delta} := \mathrm{diag}\left(V^{\Delta} I^{\Delta \mathsf{H}}\right) = \sigma^{\Delta} \tag{3}$$

d) Impedance z^{Δ}*:* A three-phase impedance in Δ configuration is characterized by three single-phase impedances specified by $z^{\Delta} := \mathrm{diag}(z^{ab}, z^{bc}, z^{ca})$, as shown in Figure 2(d). Its internal model is

$$V^{\Delta} = z^{\Delta} I^{\Delta} \tag{4a}$$

$$s^{\Delta} = \mathrm{diag}\left(z^{\Delta} I^{\Delta} I^{\Delta \mathsf{H}}\right) \tag{4b}$$

C. Conversion rule

A *conversion rule* maps the internal voltage and current $\left(V^{\Delta}, I^{\Delta}\right)$ to the terminal voltage and current (V, I). For devices in Δ configurations, it is (see [3] for the conversion rule for Y-configured devices):

$$V^{\Delta} = \Gamma V, \qquad I = -\Gamma^{\mathsf{T}} I^{\Delta} \tag{5a}$$

where the conversion matrices are

$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \Gamma^{\mathsf{T}} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The spectral properties of $\Gamma, \Gamma^{\mathsf{T}}$ underlie much of the behavior of three-phase systems, balanced or unbalanced [3]. The conversion rule depends only on the configuration (Y or Δ) and not on the type of devices. The conversion between the internal power s^{Δ} and the terminal power s is indirect through the terminal voltage V and the internal current I^{Δ}:

$$s^{\Delta} := \mathrm{diag}\left(V^{\Delta} I^{\Delta \mathsf{H}}\right) = \mathrm{diag}\left(\Gamma V I^{\Delta \mathsf{H}}\right) \tag{5b}$$

$$s := \mathrm{diag}\left(V I^{\mathsf{H}}\right) = -\mathrm{diag}\left(V I^{\Delta \mathsf{H}} \Gamma\right) \tag{5c}$$

D. External models

We now apply the conversion rule (5) to the internal models to derive the external model of each of these devices. All proofs can be found in [3].

Theorem 1 (Voltage source)**.** *For a three-phase voltage source* (E^{Δ}, z^{Δ}) *in* Δ *configuration, the following are equivalent:*

1) *Internal model:* $V^{\Delta} = E^{\Delta} + z^{\Delta} I^{\Delta}$ *and* $\boldsymbol{1}^{\mathsf{T}} V^{\Delta} = 0$.
2) *External model:* $I = \left(\Gamma^{\mathsf{T}} y^{\Delta}\right) E^{\Delta} - Y^{\Delta} V$ *where* $y^{\Delta} := (z^{\Delta})^{-1}$ *and* $Y^{\Delta} := \Gamma^{\mathsf{T}} y^{\Delta} \Gamma$.
3) *External model:* $V = \hat{\Gamma} E^{\Delta} - Z^{\Delta} I + \gamma \boldsymbol{1}$, $\boldsymbol{1}^{\mathsf{T}} I = 0$ *for some* $\gamma \in \mathbb{C}$ *where*

$$\hat{\Gamma} := \frac{1}{3} \Gamma^{\mathsf{T}} \left(\mathbb{I} - \frac{1}{\zeta} \tilde{z}^{\Delta} \boldsymbol{1}^{\mathsf{T}}\right)$$

$$Z^{\Delta} := \frac{1}{9} \Gamma^{\mathsf{T}} z^{\Delta} \left(\mathbb{I} - \frac{1}{\zeta} \boldsymbol{1}\, \tilde{z}^{\Delta \mathsf{T}}\right) \Gamma$$

$\tilde{z}^{\Delta} := z^{\Delta} \boldsymbol{1} = \left(z^{ab}, z^{bc}, z^{ca}\right)$ *and* $\zeta := z^{ab} + z^{bc} + z^{ca}$. □

For an ideal voltage source where $z^{\Delta} = 0$ we have $\hat{\Gamma} := \frac{1}{3} \Gamma^{\mathsf{T}}$ and $Z^{\Delta} = 0$. Its external model is ($\mathbf{1}^{\mathsf{T}} E^{\Delta} = 0$)

$$V = \frac{1}{3} \Gamma^{\mathsf{T}} E^{\Delta} + \gamma \mathbf{1}, \qquad \mathbf{1}^{\mathsf{T}} I = 0$$

where γ is fixed by a reference voltage. The terminal power is then $s = \frac{1}{3} \mathrm{diag}\left(\Gamma^{\mathsf{T}} E^{\Delta} I^{\mathsf{H}}\right) + \gamma \bar{I}$.

Unlike a voltage source that specifies its internal voltage E^{Δ}, a current source (J^{Δ}, y^{Δ}) specifies its internal current J^{Δ} which then uniquely determines its terminal current I through the conversion rule (5a). Its external model is

$$I = -\left(\Gamma^{\mathsf{T}} J^{\Delta} + Y^{\Delta} V\right) \tag{6a}$$

where $Y^{\Delta} := \Gamma^{\mathsf{T}} y^{\Delta} \Gamma$. The terminal power injection is

$$s = -\mathrm{diag}\left(V J^{\Delta \mathsf{H}} \Gamma + V V^{\mathsf{H}} Y^{\Delta \mathsf{H}}\right) \tag{6b}$$

For an ideal current source where $y^{\Delta} = 0$ we have $I = -\Gamma^{\mathsf{T}} J^{\Delta}$ and $s = -\mathrm{diag}(V J^{\Delta \mathsf{H}} \Gamma)$.

For a power source σ^{Δ}, application of the conversion rule (5a) to the internal model (3) leads to the external model

$$s = -\mathrm{diag}\left(V I^{\Delta \mathsf{H}} \Gamma\right), \quad \sigma^{\Delta} = \mathrm{diag}\left(\Gamma V I^{\Delta \mathsf{H}}\right) \tag{7}$$

The internal power σ^{Δ} and the terminal power s can only be indirectly related through (V, I^{Δ}). Since σ^{Δ} is the power delivered to the single-phase devices while s is the power injected from the three-phase power source to the network it is connected to, (7) implies that (the negative of) its total internal power is equal to its total terminal power, i.e., $\mathbf{1}^{\mathsf{T}} s = -\mathbf{1}^{\mathsf{T}} \sigma^{\Delta}$. In particular the total terminal power $\mathbf{1}^{\mathsf{T}} s$

is independent of the loop-flow $\beta := \frac{1}{3}\mathbf{1}^{\mathsf{T}} I^{\Delta}$ and the zero-sequence terminal voltage $\gamma := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V$, even when s does.

Finally we have the external model of an impedance.

Theorem 2 (Impedance). *For a three-phase impedance z^{Δ} in Δ configuration, the following are equivalent:*

1) *Internal model: $V^{\Delta} = z^{\Delta} I^{\Delta}$ and $\mathbf{1}^{\mathsf{T}} z^{\Delta} I^{\Delta} = 0$.*
2) *External model: $I = -Y^{\Delta} V$.*
3) *External model: $V = -Z^{\Delta} I + \gamma\mathbf{1}$, $\mathbf{1}^{\mathsf{T}} I = 0$ for some $\gamma \in \mathbb{C}$.*

Here Y^{Δ} and Z^{Δ} are defined in Theorem 1.

The theorem allows us to relate the terminal power injection s to V or to I as:

$$\begin{aligned} s &= \operatorname{diag}\left(VI^{\mathsf{H}}\right) &= -\operatorname{diag}\left(VV^{\mathsf{H}} Y^{\Delta\mathsf{H}}\right) \\ s &= \operatorname{diag}\left(VI^{\mathsf{H}}\right) &= -\operatorname{diag}\left(Z^{\Delta} II^{\mathsf{H}}\right) + \gamma \overline{I} \end{aligned}$$

for some $\gamma \in \mathbb{C}$ determined by a reference voltage.

III. Three-phase line model

A three-phase transmission or distribution line (j,k) is characterized by 3×3 series and shunt *admittance matrices* $\left(y^s_{jk}, y^m_{jk}\right) \in \mathbb{C}^{6\times 3}$ associated with bus j and 3×3 series and shunt admittance matrices $\left(y^s_{kj}, y^m_{kj}\right) \in \mathbb{C}^{6\times 3}$ associated with bus k. In general $y^s_{jk} = y^s_{kj}$, but y^m_{jk} and y^m_{kj} may not be equal. The terminal voltages (V_j, V_k) and the sending-end currents (I_{jk}, I_{kj}) are related according to

$$I_{jk} = y^s_{jk}\left(V_j - V_k\right) + y^m_{jk} V_j \tag{8a}$$

$$I_{kj} = y^s_{kj}\left(V_k - V_j\right) + y^m_{kj} V_k \tag{8b}$$

Note that the voltages (V_j, V_k) and currents (I_{jk}, I_{kj}) are terminal voltages and currents regardless of whether the three-phase devices connected to terminals j and k are in Y or Δ configuration.

To describe the relationship between the sending-end line power and the voltages (V_j, V_k), define the matrices $S_{jk}, S_{kj} \in \mathbb{C}^{3\times 3}$ by

$$S_{jk} := V_j\left(I_{jk}\right)^{\mathsf{H}}, \quad S_{kj} := V_k\left(I_{kj}\right)^{\mathsf{H}} \tag{8c}$$

The three-phase sending-end line power from terminals j to k along the line is the vector $\operatorname{diag}(S_{jk})$ of diagonal entries and that in the opposite direction is the vector $\operatorname{diag}(S_{kj})$. The off-diagonal entries of these matrices represent electromagnetic coupling between phases.

IV. Three-phase transformer models

A three-phase transformer consists of three transformer windings on the primary side in Y or Δ configuration and three transformer windings on the secondary side in Y or Δ configuration. We therefore have 4 standard transformer configurations, YY, $\Delta\Delta$, ΔY, and $Y\Delta$. For Y configuration, there may or may not have a neutral line, the neutral may or may not be grounded with zero or nonzero grounding impedance. Transformer modeling is quite involved and we only summarize a main result from [3, 5] in this section.

Let the turns ratios and the leakage admittances of the three single-phase transformers be specified by diagonal matrices $a := \operatorname{diag}(a^a, a^b, a^c) \in \mathbb{R}^{3\times 3}$ and $y := \operatorname{diag}(y^a, y^b, y^c) \in \mathbb{C}^{3\times 3}$ respectively. Define a 6×6 admittance matrix Y_{YY} and a column vector $\gamma \in \mathbb{C}^6$:

$$Y_{YY} := \begin{bmatrix} y & -ay \\ -ay & a^2 y \end{bmatrix}, \qquad \gamma := \left(V^n_j \mathbf{1}, V^n_k \mathbf{1}\right)$$

where (V^n_j, V^n_k) are the neutral voltages at terminals j and k of the transformer if they are in Y configuration.

Let $V := (V_j, V_k) \in \mathbb{C}^6$ and $I := (I_{jk}, I_{kj}) \in \mathbb{C}^6$ denote its terminal voltages and currents. The external model of a three-phase transformer is defined by the following relation between V and I.

Theorem 3 (Transformers [3, 5]). *The terminal voltage V and current I of a three-phase transformers in YY, $\Delta\Delta$, ΔY or $Y\Delta$ configuration satisfy:*

$$I = D^{\mathsf{T}} Y_{YY} D\,(V - \gamma)$$

where

$$YY: \qquad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix}$$

$$\Delta\Delta: \qquad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}$$

$$\Delta Y: \qquad D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix}$$

$$Y\Delta: \qquad D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}$$

where $\mathbb{I}$ is the identity matrix of size 3. □

Since we focus on Δ-configured devices, we assume the neutral of any Y configuration of a transformer is directly grounded so $V^n_j = V^n_k := 0$. Then Theorem 3 implies that a three-phase transformer can be modeled as a line where the line currents (I_{jk}, I_{kj}), line power flows (S_{jk}, S_{kj}), and nodal voltages (V_j, V_k) satisfy (8) with

$$\begin{aligned} y^s_{jk} &:= -[D^{\mathsf{T}} Y_{YY} D]_{jk}, \qquad y^s_{kj} := -[D^{\mathsf{T}} Y_{YY} D]_{kj} \\ y^m_{jk} &:= [D^{\mathsf{T}} Y_{YY} D]_{jj} + [D^{\mathsf{T}} Y_{YY} D]_{jk} \\ y^m_{kj} &:= [D^{\mathsf{T}} Y_{YY} D]_{kk} + [D^{\mathsf{T}} Y_{YY} D]_{kj} \end{aligned}$$

In general y^s_{jk} and y^s_{kj} may be different for transformers in ΔY or $Y\Delta$ configuration.

V. Network model

In this section we put the components models in Sections II, III, IV together to construct an overall network model. Consider a network with $N{+}1$ three-phase devices connected by three-phase lines represented as an undirected graph $G := (\overline{N}, E)$ where all buses $j \in \overline{N} := \{0, 1, \ldots, N\}$ and all lines $(j,k) \in E$ have 3 phases. A bus is where the terminals of three-phase devices (Section II) are connected. A line may model a transmission or distribution line (Section III), a transformer (Section IV), or a combination. Line (j,k) is characterized by 3×3 series and shunt admittance matrices $\left(y^s_{jk}, y^m_{jk}\right)$ and $\left(y^s_{kj}, y^m_{kj}\right)$.

Associated with each bus $j \in \overline{N}$ are three-phase nodal voltage $V_j \in \mathbb{C}^3$, current injection $I_j \in \mathbb{C}^3$ and power injection $s_j \in \mathbb{C}^3$. Let $V := (V_j, j \in \overline{N})$, $I := (I_j, j \in \overline{N})$, and $s := (s_j, j \in \overline{N})$. Associated with each line $(j,k) \in E$ are the sending-end current and power flow $(I_{jk}, S_{jk}) \in \mathbb{C}^6$ from j to k, and the sending-end current and power flow $(I_{kj}, S_{kj}) \in \mathbb{C}^6$ in the opposite direction. We now use the component models of the previous subsection to construct an overall network model, in two steps.

A. Nodal current or power balance

The nodal current balance at bus $j \in \overline{N}$ is expressed as $I_j = \sum_{k:j\sim k} I_{jk}$. Substituting (8a), we have, for $j \in \overline{N}$,

$$I_j \;=\; \sum_{k:j\sim k} \left(y^s_{jk}(V_j - V_k) \,+\, y^m_{jk} V_j \right) \tag{9}$$

The nodal power balance at buses $j \in \overline{N}$ is expressed as $s_j = \sum_{k:j\sim k} \operatorname{diag}(S_{jk})$. Substituting (8a) into $S_{jk} := V_j I^{\mathsf{H}}_{jk}$, we have, for $j \in \overline{N}$,

$$s_j \;=\; \sum_{k:j\sim k} \operatorname{diag}\left(V_j \left(V_j - V_k\right)^{\mathsf{H}} y^{s\mathsf{H}}_{jk} + V_j V_j^{\mathsf{H}} y^{m\mathsf{H}}_{jk} \right) \tag{10}$$

B. Overall model

The overall network model has two components:

1) A *nodal balance equation* that relates the terminal variables (V, I, s):
   - Either the (linear) current balance equation (9);
   - Or the (quadratic) power balance equation (10).
2) A *device model* for each three-phase device j:
   - Either its internal model (1), (2), (3), (4) and the conversion rule (5);
   - Or its external model in Theorem 1, (6), (7), Theorem 2.

If only voltage sources, current sources and impedances are involved then the overall model is linear, consisting of the nodal current balance equation (9) and (linear) device models. If power sources are also involved then, even though (9) can still be used, the overall model will be nonlinear because of nonlinear power source model (3).

VI. Application 1: Backward-forward sweep

The key idea of our approach to modeling three-phase networks is to separate internal and terminal variables and connect them through conversion rules. We now illustrate its benefit by extending the three-phase backward-forward sweep (BFS) method of [26] to include Δ-connected devices.

Consider a radial network modeled as a directed graph G, rooted at bus 0 and with each line pointing *away* from the root bus 0. Each line is characterized by 3×3 admittance matrices $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. Suppose without loss of generality that there is exactly one three-phase power source at each bus j either in Y or Δ configuration. At every non-root bus $j \in N := \{1, \ldots, N\}$, the internal power $\sigma^{Y/\Delta}_j \in \mathbb{C}^3$ of the power source is given and its terminal voltage and current (V_j, I_j) are to be determined. At bus 0, $V_0 \in \mathbb{C}^3$ is given and the current injection I_0 and the internal power injection $s^{Y/\Delta}_0$ are to be determined.

We assume for simplicity that all neutrals are directly grounded at buses $j \in \overline{N}$ with Y-configured power sources so that $V^n_j = 0$.

Let $\left(I^s_{jk},\ j \to k \in E\right)$ be the branch current through the *series* admittance matrix $y^s_{jk} \in \mathbb{C}^{3\times 3}$ so that the receiving current at bus j from its parent $i := i(j)$ is $\left(I^s_{ij} - y^m_{ji} V_j\right) \in \mathbb{C}^3$. Figure 3 summarizes the variables in the three-phase BFS. The current balance equation and the Ohm's law are:

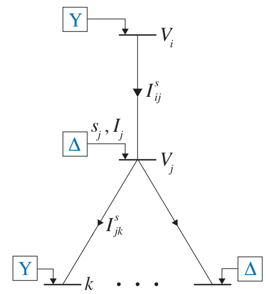


Fig. 3. Notation for BFS on unbalanced three-phase radial networks.

$$\begin{aligned} I_j \;+\; \left(I^s_{ij} - y^m_{ji} V_j\right) \;&=\; \sum_{k:j\to k} \left(I^s_{jk} + y^m_{jk} V_j\right), \quad j \in N \\ V_i - V_j \;&=\; z^s_{ij} I^s_{ij}, \qquad\qquad\qquad j \in N \end{aligned}$$

where $z^s_{ij} := \left(y^s_{ij}\right)^{-1}$ are the series impedances. Eliminating I_j by substituting $I_j = \left(\operatorname{diag} \bar{V}_j\right)^{-1} \bar{s}_j$ and rearranging, we obtain the following power flow model in terms of branch variables $\left(I^s_{jk}, j \to k \in E\right)$ and nodal variables $\left(V_j,\ j \in \overline{N}\right)$:

$$I^s_{ij} \;=\; \sum_{k:j\to k} I^s_{jk} \;-\; \left(\left(\operatorname{diag}\ \overline{V}_j\right)^{-1} \overline{s}_j \;-\; y^m_{jj} V_j \right), \quad j \in N \tag{11a}$$

$$V_j \;=\; V_i \;-\; z^s_{ij} I^s_{ij}, \qquad j \in N \tag{11b}$$

where $i := i(j)$ denotes the parent of j, $y^m_{jj} := y^m_{ji} + \sum_{k:j\to k} y^m_{jk}$ are the total shunt admittances incident on j, and for any vector x, $\overline{x}$ denotes its component-wise complex conjugate. This power flow model is proposed in [26]. It leads naturally to a three-phase BFS algorithm for solving for a fixed point $(V_j, j \in \overline{N}, I^s_{ij}, i \to j \in E)$ (power flow solution) of (11), which is a three-phase extension of the single-phase BFS algorithm in [27].

The paper [26] only considers primary distribution circuits and therefore assumes that the terminal power s_j are fixed and given. We now show how to extend the model to allow secondary distribution circuits where it is not the terminal powers s_j or currents I_j that are directly controllable, but the end devices. We hence assume only their internal powers $s^{Y/\Delta}_j := \sigma^{Y/\Delta}_j$ are fixed and given, not s_j.

Identify lines $j \to k \in E$ by the non-root buses $k \in N$. Given V_0 and $\sigma := \left(\sigma^{Y/\Delta}_j,\ j \in N\right)$, the BFS will compute the following branch and nodal variables respectively:

$$x \;:=\; \left(I^s_{ij},\; j \in N\right), \quad y \;:=\; \Big(V_j, I_j, I_j^{Y/\Delta},\; j \in N\Big)$$

All other variables, such as injections $I_0, s_0, s_0^{Y/\Delta}, s_j \in \mathbb{C}^3$, branch flow matrices $S_{jk} \in \mathbb{C}^{3\times 3}$, and $(\gamma_j, \beta_j) \in \mathbb{C}^2$ of power sources σ_j^{Δ}, can be computed once (x, y) are determined. The forward sweep to update y iteratively is based on device models as well as the Ohm's law:

$$I_j^Y \;=\; \left(\text{diag } \bar{V}_j\right)^{-1} \bar{\sigma}_j^Y, \qquad I_j \;=\; -I_j^Y, \qquad j \in N \tag{12a}$$

$$I_j^\Delta \;=\; \left(\text{diag}\left(\Gamma \bar{V}_j\right)\right)^{-1} \bar{\sigma}_j^\Delta, \quad I_j \;=\; -\Gamma^{\mathsf{T}} I_j^\Delta, \quad j \in N \tag{12b}$$

$$V_j \;=\; V_i \;-\; z^s_{ij} I^s_{ij}, \qquad\qquad j \in N \tag{12c}$$

where $\bar{v}$ denotes the componentwise complex conjugate of a vector v. The update equations for the internal currents $I_j^{Y/\Delta}$ use the device model $\sigma_j^Y = \text{diag}\left(V_j I_j^{Y\mathsf{H}}\right)$ for power sources in Y configuration (since $V_j^n := 0$ by assumption) and $\sigma_j^\Delta = \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right)$ in Δ configuration. Here, we have used, for vectors $v, w \in \mathbb{C}^n$, $\text{diag}(vw^{\mathsf{H}}) = \text{diag}(v)\bar{w} = \text{diag}(\bar{w})v \in \mathbb{C}^n$ where $\text{diag}(v)$ is the diagonal matrix whose diagonal is the vector v. The backward sweep to update x iteratively is based on the current balance equation above:

$$I^s_{ij} \;=\; \sum_{k: j \to k} I^s_{jk} \;-\; \left(I_j \;-\; y^m_{jj} V_j\right), \qquad j \in N \tag{13}$$

Hence, for a non-root bus $j \in N$, the given internal power $\sigma_j^{Y/\Delta}$ determines, through its internal current $I_j^{Y/\Delta}$, its terminal voltage and current (V_j, I_j) according to (12a)(12b). These terminal variables interact across the network according to the network equations (12c)(13).

BFS defined by (12)(13) proceeds as follows.

0) *Input*: voltage V_0 pu and internal power $\left(\sigma_j^{Y/\Delta}, j \in N\right)$.
1) *Initialization.*
   - $I^s_{jk}(t) := 0$ for all leaf nodes j for iterations $t = 1, 2, \ldots$.
   - $V_0(t) := V_0$ for all iterations $t = 0, 1, \ldots$.
   - $V_j(0) := V_0$ at all buses $j \in N$.
   - For all devices $j \in N$ in Y configuration:

$$\begin{aligned} I_j^Y(0) \;&=\; \left(\text{diag } \bar{V}_j(0)\right)^{-1} \bar{\sigma}_j^Y \\ I_j(0) \;&=\; -I_j^Y(0) \end{aligned}$$

   For all devices $j \in N$ in Δ configuration:

$$\begin{aligned} I_j^\Delta(0) \;&=\; \left(\text{diag}\left(\Gamma \bar{V}_j(0)\right)\right)^{-1} \bar{\sigma}_j^\Delta \\ I_j(0) \;&=\; -\Gamma^{\mathsf{T}} I_j^\Delta(0) \end{aligned}$$

2) *Backward forward sweep.* Iterate for $t = 1, 2, \ldots$ until a stopping criterion (see below) is satisfied:
   a) *Backward sweep.* Starting from the leaf nodes and iterating towards bus 0, compute for $i \to j \in N$

$$\begin{aligned} I^s_{ij}(t) \;\;\leftarrow\;\; &\sum_{k: j \to k} I^s_{jk}(t) \\ &-\; \left(I_j(t-1) - y^m_{jj}\, V_j(t-1)\right) \end{aligned}$$

   where $y^m_{jj} := y^m_{ji} + \sum_{k: j \sim k} y^m_{jk}$.

   b) *Forward sweep.* Starting from bus 0 and iterating towards the leaf nodes, compute for $j \in N$

$$\begin{aligned} & V_j(t) \;\leftarrow\; V_i(t) \;-\; z^s_{ij}\, I^s_{ij}(t) \\ Y: \quad & I_j^Y(t) \;\leftarrow\; \left(\text{diag } \bar{V}_j(t)\right)^{-1} \bar{\sigma}_j^Y \\ & I_j(t) \;\leftarrow\; -I_j^Y(t) \\ \Delta: \quad & I_j^\Delta(t) \;\leftarrow\; \left(\text{diag}\left(\Gamma \bar{V}_j(t)\right)\right)^{-1} \bar{\sigma}_j^\Delta \\ & I_j(t) \;\leftarrow\; -\Gamma^{\mathsf{T}} I_j^\Delta(t) \end{aligned}$$

   where $z^s_{ij} := \left(y^s_{ij}\right)^{-1}$.

3) *Output*: branch variable $x := \left(I^s_{ij}(t),\; j \in N\right)$ and nodal variable $y := \Big(V_j(t), I_j(t), I_j^{Y/\Delta(t)},\; j \in N\Big)$.

The stopping criterion in [26] is based on the discrepancy between the given internal powers $\sigma_j^{Y/\Delta}$ and those implied by the nodal variable $\Big(V_j(t), I_j(t), I_j^{Y/\Delta(t)},\; j \in N\Big)$ in each iteration t. Specifically, from the device model in (12), let

$$\hat{\sigma}_j(t) \;:=\; \begin{cases} \text{diag}\left(V_j(t) I_j^{Y\mathsf{H}}(t)\right) & \text{for } Y \text{ configuration} \\ \text{diag}\left(\Gamma V_j(t) I_j^{\Delta\mathsf{H}}(t)\right) & \text{for } \Delta \text{ configuration} \end{cases}$$

Then the stopping criterion in [26] is

$$\|\hat{\sigma}(t) - \sigma^{Y/\Delta}\|_2^2 \;\;:=\;\; \sum_{j \in N} \Big(\hat{\sigma}_j(t) - \sigma_j^{Y/\Delta}\Big)^2 \;\;<\;\; \epsilon$$

for a given tolerance $\epsilon > 0$.

VII. Application 2: Optimal Power Flow

In this section we illustrate the network model of Section V by formulating a three-phase optimal power flow (OPF) problem. We assume for simplicity that all devices are in Δ configuration (see [3] for devices in Y configuration.

We assume without loss of generality that there is exactly one device at each bus. Our formulation includes, as optimization variables, some internal variables u_j of the devices at buses j and the terminal voltages and power injections (V_j, s_j) at buses j. The internal variables u_j represent quantities that can be controlled, e.g., the charging current of electric vehicle chargers. Their values affect the terminal variables (V_j, s_j) through the conversion rule (5). This is described by the external models of Section II-D.

A. Network constraints

The power balance equation (10) imposes equality constraints between the terminal variables (V_j, s_j). The operational constraints on (V, s) are: for all $j \in \overline{N}$, $(j,k) \in E$,

$$\begin{aligned} &\text{injection limits:} && s_j^{\min} \;\leq\; s_j \;\leq\; s_j^{\max} && \text{(15a)} \\ &\text{voltage limits:} && v_j^{\min} \;\leq\; \text{diag}\left(V_j V_j^{\mathsf{H}}\right) \;\leq\; v_j^{\max} && \text{(15b)} \\ &\text{line limits:} && \text{diag}\left(I_{jk}(V)\, I_{jk}^{\mathsf{H}}(V)\right) \;\leq\; \ell_{jk}^{\max} && \text{(15c)} \\ & && \text{diag}\left(I_{kj}(V)\, I_{kj}^{\mathsf{H}}(V)\right) \;\leq\; \ell_{kj}^{\max} && \text{(15d)} \end{aligned}$$

where $(I_{jk}(V), I_{kj}(V))$ are given by (8a). The constraint (15a) can be due to limits on the busbar to which the three-phase device is connected. The constraints (15a)(15b) are local at each bus j but (15c)(15d) are global.

B. Device constraints

There are two types of device constraints, both being local at each bus j. The conversion between internal variables u_j and terminal variables (V_j, s_j) imposes equality constraints. Operational limits impose inequality constraints on u_j.

1) *Voltage source* $(E_j^{\Delta}, z_j^{\Delta})$: $u_j := E_j^{\Delta} \in \mathbb{C}^3$. Applying Theorem 1 to $s_j = \text{diag}(V_j I_j^{\mathsf{H}})$ yields

$$s_j \ = \ \text{diag}\left(V_j u_j^{\mathsf{H}}\left(\bar{y}_j^{\Delta}\Gamma\right) - V_j V_j^{\mathsf{H}} Y_j^{\Delta\mathsf{H}}\right) \tag{16a}$$

The operational constraint is:

$$v_j^{\Delta\min} \ \le \ \text{diag}\left(u_j u_j^{\mathsf{H}}\right) \ \le \ v_j^{\Delta\max} \tag{16b}$$

2) *Current source* $(J_j^{\Delta}, y_j^{\Delta})$: $u_j \ := \ J_j^{\Delta} \ \in \ \mathbb{C}^3$. The relation between internal variable u_j and the terminal variable (V_j, s_j) is, from (6b),

$$s_j \ = \ -\,\text{diag}\left(V_j u_j^{\mathsf{H}}\,\Gamma \ + \ V_j V_j^{\mathsf{H}}\, Y_j^{\Delta\mathsf{H}}\right) \tag{16c}$$

The operational constraint is:

$$\text{diag}\left(u_j u_j^{\mathsf{H}}\right) \ \le \ \ell_j^{\Delta\max} \tag{16d}$$

3) *Power source* σ_j^{Δ}: The internal variables are the internal power $u_j := \sigma_j^{\Delta} \in \mathbb{C}^3$ and current $\tilde{u}_j := I_j^{\Delta} \in \mathbb{C}^3$ across the single-phase devices. The relation between $(u_j, \tilde{u}_j)$ and (V_j, s_j) is, from (7),

$$s_j \ = \ -\,\text{diag}\left(V_j \tilde{u}_j^{\mathsf{H}} \Gamma\right) \tag{16e}$$

$$u_j \ = \ \text{diag}\left(\Gamma V_j \tilde{u}_j^{\mathsf{H}}\right) \tag{16f}$$

The operational constraints are:

$$s_j^{\Delta\min} \ \le \ u_j \ \le \ s_j^{\Delta\max} \tag{16g}$$

$$\text{diag}\left(\tilde{u}_j\, \tilde{u}_j^{\mathsf{H}}\right) \ \le \ \ell_j^{\Delta\max} \tag{16h}$$

4) *Impedance* z_j^{Δ}: We assume z_j^{Δ} is not adjustable but it imposes a constraint on the terminal voltage V_j and current I_j according to Theorem 2:

$$s_j \ := \ \text{diag}\left(V_j I_j^{\mathsf{H}}\right) \ = \ -\,\text{diag}\left(V_j V_j^{\mathsf{H}} Y_j^{\Delta\mathsf{H}}\right) \tag{16i}$$

C. OPF

Let $C_0(u, V, s)$ be the cost function. It may represent generation cost, real power loss, estimation error, voltage deviations, or user disutility, depending on applications. Then a simple OPF formulation in the three-phase setting is

$$\min_{(u,V,s)} \quad C_0(u, V, s) \qquad \text{s.t.} \qquad (10)(15)(16) \tag{17}$$

D. OPF as QCQP

We now express OPF as a standard form QCQP:

$$\begin{aligned} \min_{x} \ \ & C(x) \\ \text{s.t.} \ \ & x^{\mathsf{H}} A_k x + a_k^{\mathsf{H}} x + x^{\mathsf{H}} a_k \ \le \ b_k, \quad k = 1, \ldots, K \end{aligned}$$

by eliminating s using (10) and then writing (15)(16) as quadratic forms in $x := (u, V)$.

Let

$$\begin{aligned} & e^a := (1,0,0), \qquad e^b := (0,1,0), \qquad e^c := (0,0,1) \\ & e_j \in \{0,1\}^{N+1}, \quad e_j^{\phi} \in \{0,1\}^{3(N+1)}, \quad \phi \in \{a,b,c\} \end{aligned}$$

where e_j has a single 1 in the jth position and e_j^{ϕ} has a single 1 in the $j\phi$th position. Let $\Xi^{\phi} := e^{\phi} e^{\phi\mathsf{H}} \in \mathbb{C}^{3\times 3}$ and $\Xi_j^{\phi} := e_j^{\phi} e_j^{\phi\mathsf{H}} \in \mathbb{C}^{3(N+1)\times 3(N+1)}$. Then $V_j \in \mathbb{C}^3$ and $V_j^{\phi} \in \mathbb{C}$ can be written in terms of $V \in \mathbb{C}^{3(N+1)}$ as:

$$V_j \ = \ \left(e_j^{\mathsf{H}} \otimes \mathbb{I}\right) V, \quad V_j^{\phi} \ = \ e_j^{\phi\mathsf{H}} V, \quad \phi \in \{a,b,c\}$$

where $\mathbb{I}$ is the identity matrix of size 3. Similarly for other quantities such as (I_j, I_j^{ϕ}).

Let $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ denote the admittance matrix where its 3×3 submatrices $Y_{jk} \in \mathbb{C}^{3\times 3}$ are given by

$$Y_{jk} \ := \ \begin{cases} -y_{jk}^s, & j \sim k \ \ (j \neq k) \\ \sum_{l:j\sim l} y_{jl}^s \ + \ y_{jj}^m, & j = k \\ 0 & \text{otherwise} \end{cases}$$

Note that Y_{jk} and Y_{kj} may not be equal, e.g., if (j,k) models a three-phase transformer. For each bus $j \in \overline{N}$, define the matrix $Y_j^{\phi} := \Xi_j^{\phi} Y$. Define the Hermitian and skew Hermitian components of $Y_j^{\phi\mathsf{H}}$:

$$\Phi_j^{\phi} \ := \ \frac{1}{2}\left(Y_j^{\phi\mathsf{H}} + Y_j^{\phi}\right) \tag{18a}$$

$$\Psi_j^{\phi} \ := \ \frac{1}{2\mathbf{i}}\left(Y_j^{\phi\mathsf{H}} - Y_j^{\phi}\right) \tag{18b}$$

Then the following quadratic forms are equivalent expressions of (10):

$$p_j^{\phi} \ := \ \text{Re}\left(s_j^{\phi}\right) \ = \ V^{\mathsf{H}} \Phi_j^{\phi} V \tag{18c}$$

$$q_j^{\phi} \ := \ \text{Im}(s_j^{\phi}) \ = \ V^{\mathsf{H}} \Psi_j^{\phi} V \tag{18d}$$

We will abbreviate this by $s_j^{\phi}(V) := V^{\mathsf{H}}\left(\Phi_j^{\phi} + \mathbf{i}\Psi_j^{\phi}\right) V$. We next use (18) to write (15)(16) as quadratic forms in (u, V).

For (15):

$$\begin{array}{lrcccl} \text{injection limits:} & p_j^{\phi\min} & \le & V^{\mathsf{H}} \Phi_j^{\phi} V & \le & p_j^{\phi\max} \\ & q_j^{\phi\min} & \le & V^{\mathsf{H}} \Psi_j^{\phi} V & \le & q_j^{\phi\max} \\ \text{voltage limits:} & v_j^{\phi\min} & \le & V^{\mathsf{H}} \Xi_j^{\phi} V & \le & v_j^{\phi\max} \\ \text{line limits:} & \left|I_{jk}^{\phi}\right|^2 & := & V^{\mathsf{H}}\, \hat{Y}_{jk}^{\phi} V & \le & \ell_{jk}^{\phi\max} \end{array}$$

where $\hat{Y}_{jk}^{\phi} := \tilde{Y}_{jk}^{\mathsf{H}}\, \Xi^{\phi}\, \tilde{Y}_{jk}$ is a $3(N+1)\times 3(N+1)$ matrix and $\tilde{Y}_{jk}$ is a $3 \times 3(N+1)$ matrix given by

$$\tilde{Y}_{jk} \quad := \quad \left((e_j - e_k)^{\mathsf{T}} \otimes y_{jk}^s \ + \ e_j^{\mathsf{T}} \otimes y_{jk}^m\right)$$

Similarly for $\left|I_{kj}^{\phi}\right|^2$.

To write (16) as quadratic forms in (u, V), define the $3(N+1)\times 3(N+1)$ matrix $A_j^{\phi}(\Sigma)$ as a function of matrix $\Sigma \in \mathbb{C}^{3\times 3}$:

$$A_j^{\phi}(\Sigma) \ := \ (e_j \otimes \mathbb{I})\, \Sigma \left(e^{\phi} e_j^{\phi\mathsf{H}}\right)$$

We can express (16) as quadratic forms in (u, V):

1) *Voltage source* (E_j^Δ, z_j^Δ): $u_j := E_j^\Delta$. Applying Theorem 1 to $s_j^\phi(V) = V_j^\phi I_j^{\phi\mathsf{H}}$ yields [1]

$$s_j^\phi(V) \;=\; u^\mathsf{H} A_j^\phi \left(\bar{y}_j^\Delta \Gamma\right) V \;-\; V^\mathsf{H} A_j^\phi \left(Y_j^{\Delta\mathsf{H}}\right) V$$
$$v_j^{\phi\min} \;\le\; u^\mathsf{H} \hat{\Xi}_j^\phi u \;\le\; v_j^{\phi\max}$$

2) *Current source* (J_j^Δ, y_j^Δ): $u_j := J_j^\Delta$. Applying (6a) to $s_j^\phi(V) = V_j^\phi I_j^{\phi\mathsf{H}}$ yields

$$s_j^\phi(V) \;=\; -u^\mathsf{H} A_j^\phi(\Gamma) V \;-\; V^\mathsf{H} A_j^\phi \left(Y_j^{\Delta\mathsf{H}}\right) V$$
$$u^\mathsf{H} \hat{\Xi}_j^\phi u \;\le\; \ell_j^{\phi\max}$$

3) *Power source* σ_j^Δ: $u_j := \sigma_j^\Delta$ and $\tilde{u}_j := I_j^\Delta$. Applying (5a) to $s_j^\phi(V) = V_j^\phi I_j^{\phi\mathsf{H}}$ and $\sigma_j^\phi = V_j^\phi (I_j^\Delta)^{\phi\mathsf{H}}$ yields

$$s_j^\phi(V) \;=\; -\tilde{u}^\mathsf{H} A_j^\phi(\Gamma) V$$
$$e_j^{\phi\mathsf{H}} u \;=\; \tilde{u}^\mathsf{H} \left(A_j^\phi(\Gamma^\mathsf{T})\right)^\mathsf{T} V$$
$$s_j^{\Delta\min} \le u_j \le s_j^{\Delta\max}, \quad \tilde{u}^\mathsf{H} \hat{\Xi}_j^\phi \tilde{u} \le \ell_j^{\phi\max}$$

4) *Impedance* z_j^Δ: Applying Theorem 2 to $s_j^\phi(V) = V_j^\phi I_j^{\phi\mathsf{H}}$ yields

$$s_j^\phi(V) \;=\; -V^\mathsf{H} A_j^\phi \left(Y_j^{\Delta\mathsf{H}}\right) V$$

VIII. Conclusion

In this tutorial we have summarized some results derived in [3–5] on modeling unbalanced three-phase networks. The key idea is to explicitly separate a device/transformer model into an *internal model* that specifies the characteristics of the single-phase devices or transformers, and a *conversion rule* that maps internal variables to terminal variables based only on its configuration. This separation provides two benefits. First it facilitates the modeling of secondary distribution circuits where only the end devices are directly controllable, not terminal variables. Second it exploits common structures across different device/transformer variants, leading to external models that are general, unifying, and simple.

References

[1] Antonio Gómez-Expósito, Antonio J. Conjeo, and Claudio Cañizares, editors. *Electric Energy Systems: analysis and operation*. CRC Press, 2 edition, 2018.

[2] W. H. Kersting. *Distribution systems modeling and analysis*. CRC, 2002.

[3] S. H. Low. *Power system analysis: a mathematical approach*. 2022. Lecture Notes, Caltech. Draft available at http://netlab.caltech.edu/book/.

[4] S. H. Low. A three-phase power flow model and balanced network analysis. In *Proc. of 11TH Bulk Power Systems Dynamics and Control Symposium (IREP)*, Banff, Canada, July 2022.

[5] S. H. Low. Three-phase transformer modeling. In *Proc. 59th Allerton Conf. on Communication, Control and Computing (Allerton)*, Moticello, IL, September 2023.

[6] A. H. El-Abiad and D. C. Tarsi. Load flow study of untransposed EHV networks. In *In Proceedings of the IEEE Power Industry Computer Application (PICA) Conference*, pages 337–384, Pittsburgh, PA, 1967.

[7] Jr R. Berg, E. S. Hawkins, and W. W. Pleines. Mechanized calculation of unbalanced load flow on radial distribution circuits. *IEEE Transactions on Power Apparatus and Systems*, PAS–86(4):451–421, April 1967.

[1] $\hat{\Xi}_j^\phi = \Xi_j^\phi$ if the internal variables u_j of all devices are in $\mathbb{C}^3$; otherwise $\hat{\Xi}_j^\phi$ has the same structure as Ξ_j^ϕ but has a different size.

[8] M. Laughton. Analysis of unbalanced polyphase networks by the method of phase co-ordinates. Part 1: System representation in phase frame of reference. *Proc. Inst. Electr. Eng.*, 115(8), 1968.

[9] K.A.Birt, J.J. Graff, J.D. McDonald, and A.H. El-Abiad. Three phase load flow program. *EEE Trans. on Power Apparatus and Systems*, 95(1):59–65, January/February 1976.

[10] J. Arrillaga and C. P. Arnold. Fast-decoupled three phase load flow. *Proc. IEE*, 125(8):734–740, 1978.

[11] Tsai-Hsiang Chen, Mo-Shing Chen, Kab-Ju Hwang, Paul Kotas, and Elie A. Chebli. Distribution system power flow analysis – a rigid approach. *EEE Transactions on Power Delivery*, 6(3), July 1991.

[12] R. C. Dugan, R. Gabrick, J. C. Wright, and K. W. Patten. Validated techniques for modeling shell-form EHV transformers. *IEEE Trans. on Power Delivery*, 4(2):1070–1078, April 1989.

[13] Mo-Shing Chen and W.E. Dillon. Power system modeling. *Proc. IEEE*, 62(7):901–915, July 1974.

[14] Tsai-Hsiang Chen, Mo-Shing Chen, Toshio Inoue, Paul Kotas, and Elie A. Chebli. Three-phase cogenerator and transformer models for distribution system analysis. *EEE Transactions on Power Delivery*, 6(4):1671–1681, October 1991.

[15] W. H. Kersting, W. H. Phillips, and W. Carr. A new approach to modeling three-phase transformer connection. *IEEE Trans. on Industry Applications*, 35:168–175, Jan/Feb 1999.

[16] S. S. Moorthy and D. Hoadley. A new phase-coordinate transformer model for Ybus analysis. *IEEE Trans. on Power Systems*, 17(4):951–956, November 2002.

[17] R. C. Dugan. A perspective on transformer modeling for distribution system analysis. In *IEEE Power Energy Society General Meeting*, Toronto, Ont. Canada, 2003.

[18] Massimiliano Coppo, Fabio Bignucolo, and Roberto Turri. Generalised transformer modelling for power flow calculation in multi-phase unbalanced networks. *IET Generation, Transmission & Distribution*, 11(15):3843–3852, 2017.

[19] Sander Claeys, Geert Deconinck, and Frederik Geth. Decomposition of n-winding transformers for unbalanced optimal power flow. *IET Generation, Transmission & Distribution*, 14(24):5961–5969, 2020.

[20] Xiao-Ping Zhang. Fast three phase load flow methods. *EEE Transactions on Power Systems*, 11(3):1547–1554, August 1996.

[21] A. Bernstein, C. Wang, E. Dall'Anese, J.Y. Le Boudec, and C. Zhao. Load flow in multiphase distribution networks: Existence, uniqueness, non-singularity and linear models. *IEEE Transactions on Power Systems*, 33(6):5832–5843, 2018.

[22] Mohammadhafez Bazrafshan and Nikolaos Gatsis. Convergence of the Z-bus method for three-phase distribution load-flow with ZIP loads. *IEEE Transactions on Power Systems*, 33(1):153–165, January 2018.

[23] E. Dall'Anese, H. Zhu, and G. Giannakis. Distributed optimal power flow for smart microgrids. *IEEE Trans. on Smart Grid*, 4(3):1464–1475, Sep. 2013.

[24] Lingwen Gan and Steven H. Low. Convex relaxations and linear approximation for optimal power flow in multiphase radial networks. In *Proc. of the 18th Power Systems Computation Conference (PSCC)*, Wroclaw, Poland, August 2014.

[25] Fengyu Zhou, Yue Chen, and Steven H. Low. Sufficient conditions for exact semi-definite relaxation of optimal power flow in unbalanced multiphase radial networks. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 6227–6233, 2019.

[26] Carol S. Cheng and D. Shirmohammadi. A three-phase power flow method for real-time distribution system analysis. *IEEE Transactions on Power Systems*, 10(2):671–679, May 1995.

[27] D. Shirmohammadi, H. W. Hong, A. Semlyen, and G. X. Luo. A compensation-based power flow method for weakly meshed distribution and transmission networks. *IEEE Transactions on Power Systems*, 3(2):753–762, May 1988.