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On Sampled-Data Control of Nonlinear Asynchronous Switched Systems

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***Abstract*— In this paper, the sampled-data stabilization problem of nonlinear asynchronous switched systems is studied. In particular, a new methodology for the design of sampled-data controllers is provided for fully nonlinear asynchronous switched systems (i.e. not necessarily affine in the control inputs) described by locally Lipschitz functions. Firstly, the new notion of Steepest Descent Switching Feedback (SDSF) is introduced. Then, it is proved the existence of a suitably fast sampling such that the digital implementation of SDSFs (continuous or not) ensures the semi-global practical stability property with arbitrarily small final target ball of the related sampled-data closed-loop system under any kind of switching with arbitrarily pre-fixed dwell time. The stabilization in the sample-and-hold sense theory is used as a tool to prove the results. Possible discontinuities in the function describing the controller at hand are also managed. The case of aperiodic sampling is included in the theory here developed. The proposed theoretical results are validated through a numerical example.**

I. Introduction

In the last decades, switching control systems have received a great attention by the researchers due to their wide applications in mechanical and chemical engineering, industrial electronics, networked control systems, and so on [1]-[4]. Switching systems consist of a finite number of subsystems (which are also called system modes) and a switching signal that orchestrates switching between these subsystems. Nowadays, due to the growing utilization of digital technologies in many practical engineering applications, a crucial aspect to take into account when we are dealing with the design of controllers is the unavoidable presence of sampling in the devices implementing the proposed control strategy [5]. In the context of switched systems, sampled-data controllers have been proposed in the literature by assuming the perfect matching of the system and controller modes (see, for instance, [6], [7]). On the other hand, in many practical applications, the assumption that the controller updating is synchronized with the system switching is unfeasible. The asynchronicity phenomenon leads to a mismatch of the system and controller modes widely increasing the difficulties in analysis and synthesis of sampled-data control systems. Many approaches have been proposed in the literature concerning the sampled-data control of nonlinear asynchronous switched systems [8]-[11]. On the other hand, all the existing results address only particular classes of nonlinear switched systems (mainly in control-affine form) not considering, moreover: (i) an arbitrary dwell time between switchings; (ii) time-varying sampling periods; (iii) possible discontinuities in the function describing the controller at hand. To our best knowledge, results concerning the sampled-data stabilization of nonlinear asynchronous switched systems, not necessarily in control affine form and described by locally Lipschitz functions, allowing for aperiodic sampling and possible discontinuities in the function describing the controller, have never been provided in the literature.

In this paper, we fill this gap by providing a new methodology for the design of sampled-data stabilizers for fully nonlinear asynchronous switched systems (i.e., not necessarily affine in the control input) described by locally Lipschitz functions and taking also into account arbitrary dwell times between switchings, time-varying sampling periods and possible discontinuities in the function describing the controller. The proposed design procedure is based on the Artstein's approach (see, for instance, [12]-[18]), here exploited, for the first time in the literature, in the context of the sampled-data stabilization of fully nonlinear asynchronous switched systems. Firstly, inspired by the well-known notion of steepest descent feedback [13], the new notion of Steepest Descent Switching Feedback (SDSF), continuous or not, is introduced for the design of the proposed sampled-data controller. Then, the stabilization in the sample-and-hold sense theory [13]-[15], [19], [20] is used to prove the existence of a suitably fast sampling such that the digital implementation of SDSFs (continuous or not) guarantees the semi-global practical stability property of the related sampled-data closed-loop system under any kind of asynchronous switching with arbitrarily pre-fixed dwell time. The case of aperiodic sampling is included in the theory here developed. A numerical example is proposed for the validation of the provided theoretical results.

Notations. $\mathbb{Z}^+$ is the set of nonnegative integer numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^\star$ denotes the extended real line $[-\infty, \infty]$, $\mathbb{R}^+$ denotes the set of nonnegative reals $[0, \infty)$. The symbol $\|\cdot\|$ stands for any $(1, 2, \cdots, \infty)$ norm of a real vector. For a given positive integer n and a given positive real h, the symbol $\mathcal{B}_h^n$ denotes the subset $\{x \in \mathbb{R}^n \mid \|x\| \leq h\}$. Let us here recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is: of class $\mathcal{K}$ if $\gamma(0) = 0$, $\gamma(s) > 0$, $s > 0$, and it is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. For a locally Lipschitz function $f_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and for a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}^+$, the upper right-hand Dini directional derivative $D^+V_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\star$, of the functional V with respect to the function f_s, is defined, for $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $h \in (0, 1)$, as $D^+V_s(x, u) = \limsup_{h \to 0^+} \frac{V(x_{h,u}) - V(x)}{h}$ where $x_{h,u} = x + hf_s(x, u)$.

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II. Design of Sampled-Data Switched Controllers

Let us consider a nonlinear switched system described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), u(t)) \tag{1}$$

where: $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the input; n, m are positive integer; $\sigma : \mathbb{R}^+ \to S = \{1, \cdots, p\}$ is a right-continuous, piece-wise constant function characterizing the switching signal with an arbitrary dwell time $\Delta > 0$; $f_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $s \in S$, are locally Lipschitz functions, with $f_s(0, 0) = 0$. In the following, a new methodology for the design of sampled-data stabilizers for the system (1) is presented. In particular, the proposed design procedure relies on the classical Artstein's approach (see, for instance, [12]-[18]), where candidate Lyapunov functions are used for the design of stabilizers. Such an approach will be here extended, for the first time in the literature, to the context of the sampled-data stabilization of nonlinear asynchronous switched systems with arbitrarily pre-fixed dwell time. To this aim, let $\mathcal{V}$ be the set of candidate common Lyapunov functions $V : \mathbb{R}^n \to \mathbb{R}^+$:

(a) admitting locally Lipschitz first-order partial derivatives;

(b) for which there exist functions $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2$, such that for any $x \in \mathbb{R}^n$, the following inequalities hold

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|). \tag{2}$$

In the following, the notion of SDSF, induced by candidate common Lyapunov functions, is introduced. Such a notion is inspired by the well–known notion of steepest descent feedback [13], [15] here suitably adapted and modified in order to provide a new methodology, based on the Artstein's strategy [12], for the design of sampled-data stabilizers for nonlinear asynchronous switched systems.

Definition 1: Let $V \in \mathcal{V}$. A set of locally bounded functions $K_s : \mathbb{R}^n \to \mathbb{R}^m$, $s = 1, \cdots, p$, (continuous or not) is said to be a SDSF for the system described by (1), induced by V, if there exists a function $\alpha_3 \in \mathcal{K}$, such that for any $x \in \mathbb{R}^n$, the following inequality holds

$$\sup_{s \in S} D^+ V_s(x, K_s(x)) \leq -\alpha_3(|x|). \tag{3}$$

Assumption 1: There exists a SDSF for the system described by (1) (see Definition 1).

Remark 1: Notice that, as in the classical approaches based on the notion of steepest descent feedbacks (see, for instance, [13]-[15] and [20]), in the design methodology here proposed, no kind of stabilization property is required to be known, whether holding or not, for the SDSF at hand, when possibly applied in a continuous-time basis. Since discontinuities in the functions K_s, $s \in S$, are allowed (see Definition 1 and Assumption 1), the Filippov solutions environment (see [21]) and the related Lyapunov theory should be considered for studying the eventual stabilization property in the continuous-time basis. Here this analysis with Filippov solutions is not required. Notice that, there exist systems for which neither a continuous feedback, nor a discontinuous one, exists such that the continuous-time implementation (if possible) yields the closed-loop system to be asymptotically stable in the classic or in the Filippov solutions, respectively. This is because the Brockett's condition fails for those systems (the reader is referred to [23] and, for the case of Filippov's solutions, to [24]). See, for instance, the nonholonomic integrator system deeply discussed in [13]. On the other hand, the nonholonomic integrator system admits a discontinuous feedback by which stabilization in the sample-and-hold sense [13] is guaranteed. As well, in [22], an example is shown for which no continuous stabilizing feedback exists when implemented in continuous time. Again, for that system a discontinuous feedback which guarantees the stabilization in the sample-and-hold sense exists (see [13], [20], [22]). Therefore, the possibility of allowing for discontinuities in the feedback, together with unnecessary stability properties in the Filippov's framework, enlarges the chances of successful designing sampled-data stabilizers. In Section IV, a switched system, where subsystems are as the one proposed in [22] (see also [13], [20]), is studied.

In order to introduce the proposed sampled-data controller, we recall the notion of partition of $[0, +\infty)$ [13], [15].

Definition 2: A partition $\pi = \{t_i, i = 0, 1, \cdots\}$ of $\mathbb{R}^+ = [0, \infty)$ is a countable, strictly increasing sequence t_i, with $t_0 = 0$, such that $t_i \to \infty$ as $i \to \infty$. The diameter $\mathrm{diam}(\pi)$ of π is defined as $\sup_{i \geq 0}(t_{i+1} - t_i)$. The dwell time $\mathrm{dwell}(\pi)$ of π is defined as $\inf_{i \geq 0} t_{i+1} - t_i$. For any positive reals $a \in (0, 1]$ and $\delta > 0$, $\pi_{a,\delta}$ denotes any partition π with $a\delta \leq \mathrm{dwell}(\pi) \leq \mathrm{diam}(\pi) \leq \delta$.

Under Assumption 1, for a given partition $\pi_{a,\delta}$, the proposed sampled-data controller for the system (1) is described by

$$u(t) = K_{\sigma(t_j)}(x(t_j)), \; t_j \leq t < t_{j+1}, \; j \in \mathbb{Z}^+, \tag{4}$$

where K_s, $s \in S$, are the functions in Definition 1.

III. Main Results

In this section, the main results of the paper are provided. In particular, it is shown that there exists a suitably fast sampling δ such that the sampled-data closed-loop system described by (1)-(4) is semi-globally practically stable with arbitrarily small final target ball of the origin.

Theorem 1: Let Assumption 1 hold. Let a be an arbitrary real in $(0, 1]$. Let $\Delta > 0$ be any positive real. Then, for any positive reals R, $r \in (0, R)$, there exist positive reals δ, T, E, such that, for any partition $\pi_{a,\delta} = \{t_j, \; j \in \mathbb{Z}^+\}$ of $[0, \infty)$, for any switching signal $\sigma : \mathbb{R}^+ \to S = \{1, \cdots, p\}$ with dwell time Δ, for any initial condition $x_0 \in \mathcal{B}^n_R$, the corresponding unique locally absolutely continuous solution $x(t)$, of the sampled–data closed–loop system (1)-(4), exists for all $t \geq 0$ and satisfies the inequalities

$$|x(t)| \leq E, \quad \forall\, t \geq 0, \qquad |x(t)| \leq r, \quad \forall\, t \geq T.$$

Proof of Theorem 1. The following proof is based on the stabilization in the sample-and-hold sense approach (see, for instance, [13]-[15], [19], [20]). We highlight here that, the reasoning in [13]-[15], [19], [20] concerning the stabilization in the sample-and-hold sense theory cannot be directly applied in the context of nonlinear asynchronous switching systems and many non-trivial new developments are required in order to simultaneously cope with time-varying sampling periods and the presence of possible switching between two consecutive sampling instants. See, for instance, steps **1)-7)**, (10)-(14) and the subsequent reasoning which is completely reformulated with respect to the ones in [13]-[15], [19], [20]. The proof is organized as follows: (I) it is shown that the solution of the closed-loop system (1)-(4) does not blow up in the interval $[0, t_1]$ and, furthermore, belongs to $\mathcal{B}^n_E$ (see (7)); (II) by the use of the result in point (I) and steps **1)-7)**, it is shown that suitably inequalities hold for the function V evaluated on the solution of (1)-(4) in the generic interval $[t_j, t_{j+1}]$, $j = 0, 1, ..., j_{\max} - 1$, where: $j_{\max}$ is the maximal positive integer such that $t_{j_{\max}} \leq l$ and l is the maximum positive real such that $x(t) \in \mathcal{B}^n_E$, $0 \leq t \leq l$ (see the reasoning in (8)-(9), for the first sampling interval $[0, t_1]$ and, (10)-(14) for the generic interval $[t_j, t_{j+1}]$, $j = 0, 1, ..., j_{\max} - 1$); (III) by the use of the inequalities proved in point (II) and by introducing a preliminary result (see Claim 1), in Claim 2, it is shown that the solution of (1)-(4) exists in $[0, +\infty)$ and, moreover, $x(t) \in \mathcal{B}^n_E$, $\forall\, t \geq 0$ (i.e., $l = +\infty$ and $j_{\max} = +\infty$); (IV) by the use of the inequalities proved in point (II), of Claim 2 and by proving a suitable inequality (see Claim 3), it shown that $x(t) \in \mathcal{B}^n_r$, $\forall\, t \geq T$, with T provided in step **7)**.

Let: **1)** r, R be any positive reals, $0 < r < R$; **2)** $a \in (0, 1]$ and $\Delta > 0$ be arbitrarily fixed; **3)** e_1, e_2, E be positive reals satisfying: $0 < e_2 < e_1 < r < R < E$, $\alpha_1(r) > \alpha_2(e_1)$, $\alpha_1(E) > \alpha_2(R)$; **4)** $U > \sup_{s \in S,\, x \in \mathcal{B}^n_E} |K_s(x)|$; **5)** L_V, L_D and M be positive reals such that, for all $x_1, x_2 \in \mathcal{B}^n_E$, $u_1, u_2 \in \mathcal{B}^m_U$, $s \in S$ the following inequalities hold

$$|D^+V_s(x_1, u_1) - D^+V_s(x_2, u_2)| \leq L_D(|x_1 - x_2| + |u_1 - u_2|), \\ \sup_{s \in S} |f_s(x_1, u_1)| \leq M, \quad |V(x_1) - V(x_2)| \leq L_V |x_1 - x_2|; \tag{5}$$

6) $\beta = \alpha_3(e_2)$ and $\bar{n} = \left[\frac{6 L_V M}{\beta a} + \frac{2}{a}\right] + 1$;

7) $T = (\bar{n} + 1)\left(\frac{2\alpha_2(R)}{L_V M} + 1\right)$ and δ be such that

$$0 < \delta \leq 1, \quad \beta > 3 L_D M \delta, \quad e_2 + \delta M < e_1, \\ R + \delta M < E, \quad \alpha_1(r) > \alpha_2(e_1) + \tfrac{4}{3}\beta\delta + 2 L_V M \delta, \\ (\bar{n} + 2)\delta < \Delta, \quad L_V M \delta \leq \alpha_2(R) - \alpha_2(e_1). \tag{6}$$

Let $\sigma(t)$ be any switching signal with dwell time Δ. Let $x_0 \in \mathcal{B}^n_R$. Let us consider a partition $\pi_{a,\delta}$ of $\mathbb{R}^+$. Let $x(t)$ be the solution of the closed-loop system (1)-(4), in a maximal time interval $[0, b)$, $0 < b \leq +\infty$. Notice that, $\Delta > (\bar{n} + 2)\delta$. It follows that in the first interval $[0, t_1]$ no switching occurs. We show first that the solution exists in $[0, t_1]$. Otherwise, by contradiction, if the solution blows up, there exists a time $\tau \in [0, t_1)$ such that $|x(t)| < E$, $t \in [0, \tau)$, and $|x(\tau)| = E$. But, from (5), (6), for $t \in [0, \tau]$, the inequalities hold:

$$|x(t)| \leq |x_0| + \textstyle\int_0^t |f_{\sigma(0)}(x(\tau), K_{\sigma(0)}(x(0)))| d\tau \leq R + \delta M < E. \tag{7}$$

Thus, for $t = \tau$, the absurd inequality arises $E < E$. Therefore, $x(t) \in \mathcal{B}^n_E$ for $t \in [0, t_1]$. Let us define $W(t) = V(x(t))$, $t \in [0, t_1]$. Using the mean value theorem for integrals with some $t^\star \in [0, t]$ and taking into account (3), (5), the following holds for any fixed $t \in (0, t_1]$

$$W(t) - W(0) = t\left(\tfrac{1}{t}\textstyle\int_0^t D^+V_{\sigma(0)}(x(\tau), K_{\sigma(0)}(x(0))) d\tau\right) \\ = t D^+V_{\sigma(0)}(x(t^\star), K_{\sigma(0)}(x(0))) - t D^+V_{\sigma(0)}(x(0), K_{\sigma(0)}(x(0))) \\ + t D^+V_{\sigma(0)}(x(0), K_{\sigma(0)}(x(0))) \leq t L_D M \delta - t\alpha_3(|x(0)|) \tag{8}$$

where the inequality: $|x(t^\star) - x(0)| \leq M\delta$ has been used. From (8) and taking into account (6), one gets $W(t) - W(0) \leq -t\alpha_3(|x(0)|) + \frac{\beta}{3}t$. Following [13], in the case $|x_0| > e_2$, taking into account β in step 6), one has $W(t) \leq W(0) - \frac{2}{3}\beta t$, $\forall\, t \in [0, t_1]$. On the other hand, in the case $|x_0| \leq e_2$, using the first inequality of (7) and taking into account (5), (6), $|x(t)| \leq e_2 + \delta M < e_1$, $\forall\, t \in [0, t_1]$. Then, taking into account (2), $W(t) = V(x(t)) \leq \alpha_2(|x(t)|) < \alpha_2(e_1)$, $t \in [0, t_1]$. These two cases can be written as

$$W(t) \leq \left(W(0) - \tfrac{2}{3}\beta t\right) H(|x_0| - e_2) + \alpha_2(e_1) H_0(e_2 - |x_0|), \tag{9}$$

$t \in [0, t_1]$, where H_0, H are the Heaviside functions, defined as $H_0(c) = \begin{cases} 1 & c \geq 0 \\ 0 & c < 0 \end{cases}$ $H(c) = \begin{cases} 1 & c > 0 \\ 0 & c \leq 0 \end{cases}$ with $c \in \mathbb{R}$. Let l be the maximal positive real such that, for the solution $x(t)$ of the closed-loop system (1)-(4), the relation $x(t) \in \mathcal{B}^n_E$, $0 \leq t \leq l$, holds. We allow l to be $+\infty$. Let $W(t) = V(x(t))$, $0 \leq t \leq l$. Let $j_{\max}$ be the maximal positive integer such that $t_{j_{\max}} \leq l$. We allow $j_{\max}$ and $t_{j_{\max}}$ to be $+\infty$ (when l is $+\infty$). Then, in any interval $[t_j,\, t_{j+1}]$, $j = 0, 1, ..., j_{\max} - 1$, we can have two possible cases:
(a) no switching occurs in the sampling interval $[t_j,\, t_{j+1})$ or the switching time is equal to the sampling instant t_j;
(b) a switching occurs in the sampling interval $[t_j,\, t_{j+1})$ and the switching time $t^* \in (t_j, t_{j+1})$. In the case **(a)**, by the same reasoning used in the interval $[0, t_1]$ (see (8)-(9)), for $t \in [t_j,\, t_{j+1}]$,

$$W(t) \leq \left(W(t_j) - \tfrac{2}{3}\beta(t - t_j)\right) H(|x(t_j)| - e_2) \\ + \alpha_2(e_1) H_0(e_2 - |x(t_j)|). \tag{10}$$

In particular, for $t = t_{j+1}$,

$$W(t_{j+1}) \leq \left(W(t_j) - \tfrac{2}{3}\beta(t_{j+1} - t_j)\right) H(|x(t_j)| - e_2) \\ + \alpha_2(e_1) H_0(e_2 - |x(t_j)|). \tag{11}$$

In the case **(b)**, two possible cases can occur: **(b.1)** $|x(t_j)| \leq e_2$; **(b.2)** $|x(t_j)| > e_2$. In the case **(b.1)**, by repeating the same reasoning in (7) and taking into account (2), (5), (6), $W(t) \leq \alpha_2(e_1)$. In the case **(b.2)**, taking into account (5), for $t \in [t_j,\, t_{j+1}]$, we have that

$$W(t) - W(t_j) \leq |W(t) - W(t_j)| = |V(x(t)) - V(x(t_j))| \\ \leq L_V \textstyle\int_{t_j}^t |f_{\sigma(\tau)}(x(\tau), K_{\sigma(t_j)}(x(t_j)))| d\tau \leq L_V M \delta. \tag{12}$$

Let $t^*_1,\, t^*_2, \cdots$ be the sequence of switching times in $[0, +\infty)$. Notice that this sequence is allowed to be bounded or unbounded. If it is bounded, no switching will occur from a certain time on. In the following, for $j = 0, 1, \cdots j_{\max} - 1$, we will denote with i_j the sequence defined recursively as $i_0 = 1$ and, for $j \geq 0$, $i_{j+1} = i_j$, if $t_{j+1} < t^*_{i_j}$ and $i_{j+1} = i_j + 1$ otherwise. Taking into account both cases **(a)** and **(b)**, for $j = 0, 1, ..., j_{\max} - 1$, for $t \in [t_j,\, t_{j+1}]$,

$$W(t) \leq \left(W(t_j) - \tfrac{2}{3}\beta(t - t_j)\right) H(|x(t_j)| - e_2) \\ + \alpha_2(e_1) H_0(e_2 - |x(t_j)|) + \left(\tfrac{2}{3}\beta(t - t_j) \\ + L_V M \delta\right) H(|x(t_j)| - e_2) H(t_{j+1} - t^*_{i_j}) H(t^*_{i_j} - t_j). \tag{13}$$

In particular, for $j = 0, 1, ..., j_{\max} - 1$,

$$W(t_{j+1}) \leq \left(W(t_j) - \tfrac{2}{3}\beta(t_{j+1} - t_j)\right) H(|x(t_j)| - e_2) \\ + \alpha_2(e_1) H_0(e_2 - |x(t_j)|) + \left(\tfrac{2}{3}\beta(t_{j+1} - t_j) \\ + L_V M \delta\right) H(|x(t_j)| - e_2) H(t_{j+1} - t^*_{i_j}) H(t^*_{i_j} - t_j). \tag{14}$$

Notice that, in the case $t_{j_{\max}} < l < +\infty$, the inequality (13) holds also for $t \in [t_{j_{\max}}, l]$. Let us now introduce and prove the following claim.

Claim 1: For any $j = 0, 1, ..., j_{\max}$, the inequality $W(t_j) \leq \alpha_2(R)$ holds.
Proof. Claim 1 follows by an induction reasoning from (14) and taking into account that the time elapsed between two consecutive switching is greater than $(\bar{n}+2)\delta$, (see (6)). □
In the following, it is proved that $l = +\infty$.

Claim 2: The solution $x(t)$ of the closed–loop system (1)-(4) exists in $[0, +\infty)$ and, moreover, $x(t) \in \mathcal{B}_E^n$, $\forall\, t \geq 0$.
Proof. By contradiction, let $l < +\infty$. Let $\tau \in [t_1, l]$ be the first time such that $|x(t)| = E$ (recall that $|x(t)| = E$ cannot hold for $t \in [0, t_1]$). Then, we have $W(\tau) \geq \alpha_1(E)$. Let i be the largest positive integer such that $t_i \leq \tau$. For $j = 0, 1, ..., i-1$, from Claim 1, $W(t_j) \leq \alpha_2(R)$ holds for $j = 0, 1, ..., i$. In particular, the inequality holds $W(t_i) \leq \alpha_2(R)$. Now, for $t \in [t_i, \tau]$, the inequality (13) holds (recall that $\tau \leq l$). Therefore, taking into account step 3) and (13), from the same reasoning used in the proof of Claim 1, the inequalities hold, for $t \in [t_i, \tau]$, $W(t) \leq \alpha_2(R) < \alpha_1(E)$, and, in particular, the inequalities hold $W(\tau) \leq \alpha_2(R) < \alpha_1(E)$. Then, taking into account step 3), we obtain $E = |x(\tau)| \leq \alpha_1^{-1}(\alpha_2(R)) < \alpha_1^{-1}(\alpha_1(E)) = E$. The absurd inequality arises $E < E$. We conclude that Claim 2 is true. □
Since Claim 2 holds true (i.e., $l = +\infty$), let $W(t) = V(x(t))$, $t \in \mathbb{R}^+$. Let, for $j = 0, 1, ...$,

$$\chi(j) = \max\{W(t_j), \alpha_2(e_1)\} > 0, \quad j \in \mathbb{Z}^+. \tag{15}$$

From (11), one works out

$$\begin{aligned}&\chi(j+1) \leq \max\left\{\chi(j) - \tfrac{2}{3}\beta(t_{j+1} - t_j), \alpha_2(e_1)\right\} H(|x(t_j)| - e_2)\\ &+ \chi(j) H_0(e_2 - |x(t_j)|) + (\tfrac{2}{3}\beta(t_{j+1} - t_j) + L_V M\delta) \times\\ &H(|x(t_j)| - e_2) H(t_{j+1} - t^*_{i_j}) H(t^*_{i_j} - t_j)\end{aligned} \tag{16}$$

Claim 3: If $j_1 \in \mathbb{Z}^+$ is the first nonnegative integer such that $\chi(j_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$, (we allow that $j_1 = +\infty$) then for all $j \geq j_1$, $\chi(j+1) \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$.
Proof. Let $j_1 \in \mathbb{Z}^+$ be the first nonnegative integer such that $\chi(j_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$. We have two cases: **(a)** no switching occurs for $t \geq t_{j_1}$ (i.e., the next switching instant $t^*_{i_{j_1}} = +\infty$); **(b)** switching occur for $t \geq t_{j_1}$ (i.e., the next switching instant $t^*_{i_{j_1}} \neq +\infty$). In the case **(a)**, from (16), it follows that for all $j \geq j_1$, $\chi(j+1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$. In the case **(b)**, we have two subcases: **(b.1)** $H(t_{j_1+1} - t^*_{i_{j_1}}) H(t^*_{i_{j_1}} - t_{j_1}) = 0$; **(b.2)** $H(t_{j_1+1} - t^*_{i_{j_1}}) H(t^*_{i_{j_1}} - t_{j_1}) = 1$. In the case **(b.1)**, from (16),

$$\begin{aligned}&\chi(j_1+1) \leq \max\left\{\chi(j_1) - \tfrac{2}{3}\beta(t_{j_1+1} - t_{j_1}),\right.\\ &\left.\alpha_2(e_1)\right\} \leq \alpha_2(e_1) + \tfrac{2}{3}\beta\delta + L_V M\delta.\end{aligned} \tag{17}$$

From (17), we can re-start, iteratively, the reasoning from **(b)** by taking $j_1 + 1 = j_1$. In the case **(b.2)**, if $|x(t_{j_1})| \leq e_2$, from (16), it follows that $\chi(j_1+1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$. On the other hand, if $|x(t_{j_1})| > e_2$, from (16),

$$\begin{aligned}&\chi(j_1+1) \leq \max\left\{\chi(j_1) - \tfrac{2}{3}\beta(t_{j_1+1} - t_{j_1}), \alpha_2(e_1)\right\}\\ &+ \tfrac{2}{3}\beta(t_{j+1} - t_j) + L_V M\delta \leq \alpha_2(e_1) + \tfrac{4}{3}\beta\delta + 2L_V M\delta.\end{aligned} \tag{18}$$

Now, if no switching occurs for $t \geq t_{j_1+1}$, then, from (16), for any $j \geq j_1 + 1$, $\chi(j+1) \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$. On the other hand, if a new switch occurs for $t \geq t_{j_1+1}$, taking into account that $\Delta > (\bar{n}+2)\delta$, let $\bar{j}$ be the positive integer such that $t^*_{i_{j_1+1}} \in [t_{\bar{j}}, t_{\bar{j}+1}]$ (i.e., the interval in which is situated the new switching instant after $t^*_{i_{j_1}}$). Two cases can occur: **(b.2.1)** There exists an integer $k \in [j_1+1, \bar{j}]$, such that $|x(t_k)| \leq e_2$; **(b.2.2)** an integer $k \geq 0$, such that $|x(t_k)| \leq e_2$, does not exist in $[j_1+1, \bar{j}]$. In the case **(b.2.1)**, taking into account (18), we have that for $j \in [j_1+1, k-1]$, $\chi(j) \leq \chi(j_1+1) - \frac{2}{3}\sum_{i=j_1+1}^{j-1} \beta a\delta \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$, and for $j \in [k, \bar{j}]$, $\chi(j) \leq \alpha_2(e_1)$. In the case **(b.2.2)**, taking into account (18), we have that for $j \in [j_1+1, \bar{j}-1]$, $\chi(j) \leq \chi(j_1+1) - \frac{2}{3}\sum_{i=j_1+1}^{j-1} \beta a\delta \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$, and for $j \in [k, \bar{j}]$, taking into account (18) and step 6), $\chi(\bar{j}) \leq \chi(j_1+1) - \frac{2}{3}\bar{n}\beta a\delta \leq \alpha_2(e_1) - 2L_V M\delta \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$. Then, taking into account both cases **(b.2.1)** and **(b.2.2)**, we can repeat iteratively the same reasoning restarting from **(b.2)** taking $j_1 = \bar{j}$. Thus, Claim 3 is true. □
Let $\bar{k} = [\frac{2\alpha_2(R)}{L_V M\delta}] + 1$. Let $j_2 = (\bar{n}+1)\bar{k}$. Notice that, taking into account step 6), 7), in the interval $[0, t_{j_2}]$, a maximum of $\bar{k}$ switchings can occur. Indeed: $t_{j_2} \leq j_2\delta \leq (\bar{n}+1)\bar{k}\delta \leq \bar{k}\Delta$. Moreover, in the interval $[0, t_{j_2}]$, we have maximum of $\bar{k}$ sampling intervals in which a switching has occurred and at least $(\bar{n}+1)\bar{k} - \bar{k} = \bar{n}\bar{k}$ sampling intervals in which no switching occurs. We claim $j_1 \leq j_2$. By contradiction, if $j_1 > j_2$ then two cases can occur

i. There exists an integer $\ell \in [0, j_2]$, so that $|x(t_\ell)| \leq e_2$;
ii. an integer $\ell \geq 0$, such that $|x(t_\ell)| \leq e_2$, does not exist in $[0, j_2]$.

In case (ii), for any integer $\ell \in [0, j_2]$, we have $|x(t_\ell)| > e_2$. Thus, by (15), (16), taking into account step 6), we obtain

$$\begin{aligned}&\chi(j_2) \leq \chi(0) - \tfrac{2}{3}\bar{n}\bar{k}\beta a\delta + \bar{k}(\tfrac{2}{3}\beta\delta + L_V M\delta) \leq\\ &\leq \alpha_2(R) - \bar{k}L_V M\delta \leq -\alpha_2(R) < 0,\end{aligned} \tag{19}$$

which is absurd, since $\chi(j)$ is nonnegative for any $j \in \mathbb{Z}^+$. Therefore, the above ℓ must exist in $[0, j_2]$. Then, we have $|x(t_\ell)| \leq e_2$, and $\bar{W}(\ell) \leq \alpha_2(e_1)$. This is still absurd, because $\ell \leq j_2$, and $j_1 > j_2$ was supposed to be the first nonnegative integer such that $\chi(j_1) \leq \alpha_2(e_1) + \frac{2}{3}\beta\delta + L_V M\delta$. Therefore, it must hold $j_1 \leq j_2$. We conclude that, for any integer $j \geq j_2$, from Claim 3, $\chi(j) \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$, which implies $W(t) \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$, $t \geq t_{j_2}$. Taking into account steps 6) and 7), $T \geq t_{j_2}$. Then, for $t \geq T$, from (3), we obtain $\alpha_1(|x(t)|) \leq W(t) \leq \alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta$. Then, from (6), we have $|x(t)| \leq \alpha_1^{-1}(\alpha_2(e_1) + \frac{4}{3}\beta\delta + 2L_V M\delta) < \alpha_1^{-1}(\alpha_1(r)) = r$. It follows that, $x(t) \in \mathcal{B}_r^n$ for any $t \geq T$. The proof is complete. □

Remark 2: We highlight here that, the results provided in Theorem 1 still hold in the case of suitably small actuation disturbances affecting the sampled-data controller (4). Moreover, if the SDSF at hand is described by locally Lipschitz functions, then the results provided in Theorem 1 still hold in the case of suitably small measurement noises which can be properly reduced to the case of actuation disturbances. This feature is inherited by the robustness property of stabilizers in the sample-and-hold sense (see, [13], for a detailed discussion in the case of no switching systems). On the other hand, at this stage, in the case of discontinuous feedbacks, it is not known whether the proposed technique is robust with respect to measurement errors, even if suitably small. The robustification of sampled-data controllers (continuous or not) for nonlinear asynchronous switched systems with respect to any bounded actuation disturbances and any bounded observation errors is a very interesting topic which is left for future investigations (see, for instance, [27] for the case without switching).

Remark 3: As common in the control strategies based on the Arstein's approach [12], the methodology here proposed for the design of sampled-data switched controllers can be summarized as follows: (i) choose a candidate Lyapunov function belonging to $\mathcal{V}$; (ii) exploit the Lyapunov function in point (i) in order to try and find functions K_s, $s \in S$, satisfying the condition in (3) (i.e., try and find a SDSF); (iii) implement the SDSF in point (ii) according to (4).

In the following, we show an example on how the well-known Sontag's universal formula [17], [18] can be exploited to find SDSFs for nonlinear asynchronous switched systems in control-affine form by means of candidate Lyapunov functions and according to Definition 1. Let us consider a nonlinear switched system described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))u(t) \tag{20}$$

where: $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the input; n, m are positive integer; $\sigma : \mathbb{R}^+ \to S$ (see (1)); $f_s : \mathbb{R}^n \to \mathbb{R}^n$ and $g_s : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $s \in S$, are locally Lipschitz functions. Let $V \in \mathcal{V}$. Let $a_s : \mathbb{R}^n \to \mathbb{R}$ and $b_s : \mathbb{R}^n \to \mathbb{R}^m$, $s \in S$, be the functions defined, for $x \in \mathbb{R}^n$, as

$$a_s(x) = \frac{\partial V(x)}{\partial x} f_s(x),\ b_s(x) = \frac{\partial V(x)}{\partial x} g_s(x). \tag{21}$$

Let $K_s : \mathbb{R}^n \to \mathbb{R}^m$, $s \in S$, be the functions defined, for $x \in \mathbb{R}^n$, as (see [17], [18])

$$K_s(x) = \begin{cases} -\frac{a_s(x)+\sqrt{a_s^2(x)+|b_s(x)|^4}}{|b_s(x)|^2} b_s^T(x), & b_s(x) \neq 0, \\ 0, & b_s(x) = 0. \end{cases} \tag{22}$$

Proposition 1: Assume that for any $x \in \mathbb{R}^n$ and $s \in S$, the following implication holds

$$b_s(x) = 0 \Rightarrow a_s(x) < 0. \tag{23}$$

Moreover, assume that there exists a function α_3 of class $\mathcal{K}$ such that $\sqrt{a_s^2(x) + |b_s(x)|^4} \geq \alpha_3(|x|)$, $\forall x \in \mathbb{R}^n$ and $s \in S$. Then, the functions K_s, $s \in S$, defined in (22) are a SDSF for the system (20).

Proof of Proposition 1. The proof readily follows from the reasoning in [17], [18] applied to each mode $s \in S$. □

We highlight that, here the well-known small control property of the feedback K_s in (22) (see, for instance, [17], [18]) is not required because discontinuities in the functions describing the SDSF at hand are allowed (see Definition 1 and Remark 1).

Remark 4: Notice that, in Theorem 1, any sequence of switching times, which we denote here with $\{t_1^*, t_2^*, \cdots\}$, is allowed provided that, $t_1^* \geq \Delta$ and $t_{i+1}^* - t_i^* \geq \Delta$, $i = 1, 2, \cdots$, with $\Delta > 0$ an arbitrarily fixed positive real. Moreover, the sequence of switching times are not necessarily synchronized with the sampling instants. Which means that the sequence of switching times $\{t_1^*, t_2^*, \cdots\}$ are not necessarily a subset of the partition $\pi_{a,\delta}$.

In the following corollary, it is shown that if a continuous-time global stabilizer (see, for instance, [25], [26]) is available for the system described by (1), then there exists a suitably fast sampling δ such that: the digital implementation of the continuous-time stabilizer at hand ensures the semi-global practical stability property of the related sampled-data closed-loop system.

Corollary 1: Assume that there exist locally Lipschitz functions $\tilde{K}_s : \mathbb{R}^n \to \mathbb{R}^m$, $s = 1, \cdots, p$, such that the related continuous-time closed-loop system

$$\dot{x}(t) = f_{\sigma(t)}(x(t), \tilde{K}_{\sigma(t)}(x(t))), \tag{24}$$

is globally asymptotically stable. Let a be an arbitrary real in $(0, 1]$. Let $\Delta > 0$ be any positive real. Then, for any positive reals R, $r \in (0, R)$, there exist positive reals δ, T, E, such that, for any partition $\pi_{a,\delta} = \{t_j,\ j \in \mathbb{Z}^+\}$ of $[0, \infty)$, for any switching signal $\sigma : \mathbb{R}^+ \to S = \{1, \cdots, p\}$ with dwell time Δ, for any initial condition $x_0 \in \mathcal{B}_R^n$, the corresponding unique locally absolutely continuous solution $x(t)$, of the sampled–data closed–loop system (1) with $u(t) = \tilde{K}_{\sigma(t_j)}(x(t_j))$, $t_j \leq t < t_{j+1}$, $j \in \mathbb{Z}^+$, exists for all $t \geq 0$ and satisfies the inequalities

$$|x(t)| \leq E, \quad \forall\, t \geq 0, \qquad |x(t)| \leq r, \quad \forall\, t \geq T.$$

Proof of Corollary 1. Taking into account the global asymptotic stability property of the system (24), the proof follows from the converse Lyapunov theorems for nonlinear switched systems [2], [28], and the results in Theorem 1. □

IV. Numerical example

Inspired by the examples proposed in [13], [20], [22], in order to show the advantages of the proposed design methodology, let us consider the nonlinear switched system (1) where $S = \{1, 2\}$ and the functions $f_s : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $s = 1, 2$, are defined for $x \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$ as follows:

$$\begin{aligned} f_1(x, u) &= \begin{bmatrix} u_1u_2 & u_2u_3 & u_1u_3 \end{bmatrix}^T, \\ f_2(x, u) &= \begin{bmatrix} u_1u_3 & u_1u_2 & u_2u_3 \end{bmatrix}^T. \end{aligned} \tag{25}$$

Notice that, in [20], [22], it is shown that no continuous feedback exists for the system described by $\dot{x}(t) = f_1(x(t), u(t))$, such that global asymptotic stability is guaranteed for a related classical solution because Brockett's covering condition fails (see Remark 1). In the following, by exploiting the results in Theorem 1, it is shown that the sampled-data stabilization of the nonlinear asynchronous switched system described by (1)-(25) is possible by discontinuous feedbacks. According to the proposed design methodology, let $V : \mathbb{R}^3 \to \mathbb{R}^+$ be the candidate common Lyapunov function defined for $x \in \mathbb{R}^3$ as $V(x) = x_1^2 + x_2^2 + x_3^2$. Notice that, $V \in \mathcal{V}$. Taking into account (25), let $K_s : \mathbb{R}^3 \to \mathbb{R}^3$, $s = 1, 2$, be any functions such that, for any $x \in \mathbb{R}^3$,

$$\begin{aligned} K_1(x) &\in \operatorname{argmin}_{i,v,w\in\{-1,0,1\}} \begin{bmatrix} iv & vw & iw \end{bmatrix} x \\ K_2(x) &\in \operatorname{argmin}_{i,v,w\in\{-1,0,1\}} \begin{bmatrix} iw & iv & vw \end{bmatrix} x. \end{aligned} \tag{26}$$

Notice that, according to Definition 1, the functions K_s in (26) is a SDSF for the system (1), (25) with $\alpha_3(s) = \frac{2}{\sqrt{3}} s$. Indeed, taking into account (25), (26), for any $x \in \mathbb{R}^3$, with entries $x_k \in \mathbb{R}$, $k = 1, 2, 3$, the following equality/inequalities hold (see [20], [22])

$$D^+V_s(x, K_s(x)) \leq -2\max_{k=1,2,3}\{|x_k|\} \leq -\tfrac{2}{\sqrt{3}}|x|. \tag{27}$$

Then, Assumption 1 is here satisfied. From an implementation point of view, at each sampling time, by exploiting the sampled-data measurement $x(t_j)$, it is just required to evaluate 27 possible cases in order to find the input points $i, v, w \in \{-1, 0, 1\}$ at which the function output value $\begin{bmatrix} iv & vw & iw \end{bmatrix} x(t_j)$ (or $\begin{bmatrix} iw & iv & vw \end{bmatrix} x(t_j)$) is minimized. Notice that, an algorithm providing only one input (in the case of multiple minimizers), for any $x \in \mathbb{R}^n$, can easily be used. In the performed simulations: the initial state of the system (1), (25) has been chosen equal to $[1\ 2\ 3]^T$; the dwell time of switching Δ equal to 0.05[s]; suitably small random actuation disturbances with amplitudes not greater than 10^{-3} have been considered (see Remark 3). In Fig. 1, simulations are reported in the case of uniform sampling period $\delta = 0.02$. In particular, Fig. 1 shows the evolutions of the system state variables $x_i(t)$, $i = 1, 2, 3$, of the control inputs $u_i(t)$, $i = 1, 2, 3$ and of switching signal $\sigma(t)$. From Fig. 1, the good and robust performances of the proposed controller can be assessed.

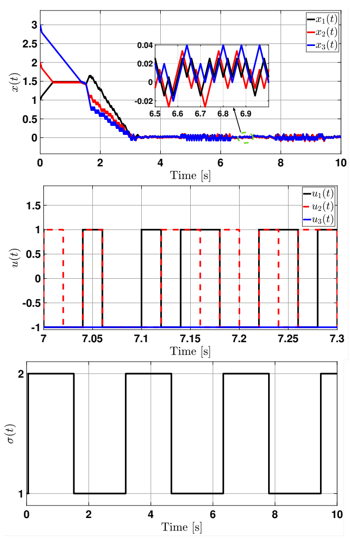

Fig. 1. Panel 1) - Evolution of the system variables $x(t)$. Panel 2) - Input signal $u(t)$. Panel 3) - Switching signal $\sigma(t)$.

V. Conclusions

In this paper, a methodology for the design of sampled-data stabilizers has been provided for nonlinear asynchronous switched systems. By properly revising the well-known notion of Steepest Descent Feedback, the new notion of SDSF has been introduced. Then, the stabilization in the sample-and-hold sense theory has been used as tool to prove the existence of a suitably fast sampling such that the digital implementation of SDSF ensures the semi-global practical stability property with arbitrarily small final target ball of the related sampled-data closed-loop system under any kind of switching with arbitrarily pre-fixed dwell time. Possible discontinuities in the functions describing the controller at hand have been also managed. The case of aperiodic sampling has been taken into account. A numerical example has been presented in order to validate the results.

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