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Passivity of Linear Singularly Perturbed Systems

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***Abstract*— The passivity of singularly perturbed systems (SPSs) is generally studied without taking advantage of the time-scale separation present in this class of systems. To fill this gap, the objective of this letter is to provide easy-to-verify well-posed conditions characterizing the passivity of a perturbation variable-dependent SPS starting from the passivity of its associated reduced-order system. To achieve this goal, we rely on the connection between positive realness and passivity, as well as the notion of phase for multi-input multi-output (MIMO) systems. We use a benchmark DC motor to illustrate that classical reasoning used for stability analysis of SPSs, which is based on the stability of the reduced-order (slow) and boundary layer (fast) subsystems, cannot be applied to guarantee the passivity of an SPS. On top of that, our methodology explains how the time-scale separation can be used to analyze the passivity of general linear time-invariant (LTI) systems. The approach is illustrated on a numerical example.**

I. Introduction

The notion of passivity has been introduced by Willems [16]. This notion gained a lot of interest due to the useful relationship it provides between the storage function and the energy of the studied system. Thus, it allows to define natural energy-based Lyapunov functions for the system. The concept of passivity was proven to be a natural framework to characterize various phenomena arising in control systems, as detailed in [13] and the two full issues referred therein.

Singularly perturbed systems [9] are systems with dynamics evolving on multiple time scales. The time-scale separation is either intrinsic to the system, e.g. in electromechanical systems, or it can be induced through feedback design, e.g. using low or high gains. The classical stability analysis of SPSs is done using the separation in *reduced-order* (slow) and *boundary layer* (fast) systems (details in [9]). It is noteworthy that, based on Tikhonov's theorem, we can conclude the stability of the SPS based on the stability of the reduced-order and boundary layer systems, avoiding numerical ill-conditioning related to the mix of the two time scales. As shown in Section III-A of this letter, this type of reasoning fails when studying passivity and tailored tools are required.

Passivity analysis of SPS models is not a new topic. It has received a lot of attention during the past decades, see e.g., [2], [5], [12], [17]. To the best of the authors' knowledge, most papers do not take advantage of the time-scale separation present in the SPS framework. Notable exceptions are [1], [15] which address particular classes of linear systems with the right-hand side being perturbation-variable independent, and the latter generalizing to include sector-bound nonlinearities.

To fill this gap, we consider linear SPSs with all system matrices affected by the perturbation variable. Our main contributions are: (i) a passivity analysis of the general case of perturbation variable-dependent linear SPSs based on the passivity of the reduced-order model and the properties of an extended boundary layer which captures the residual dynamics; (ii) a numerical method to determine the upper bound on the perturbation parameter guaranteeing that the passivity of the reduced-order system is preserved by the full SPS. The resulting conditions are solved using methods readily available in the literature and which apply naturally to our problem.

The remainder of the letter is organized as follows: Section II describes the mathematical background, Section III starts with a motivating example, presents the problem statement, and develops the main results on the passivity of the SPS systems. Section IV presents an academic example and Section V gives some conclusions.

Notations: We use the standard notation $\mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{R}^n, \mathbb{Z}_2^n$ to denote the set of real numbers, positive real numbers, complex numbers, the space of n-dimensional vectors with real components and the set of n-dimensional vectors with components in $\{0, 1\}$. For a complex matrix $X \in \mathbb{C}^{m\times p}$, X^H denotes its complex conjugate transpose, $\angle z$ is the phase of $z \in \mathbb{C}$. O and I denote the zero and identity matrices. A descriptor state-space system characterized by matrices (E, A, B, C, D) of appropriate dimensions is written as:

$$\Sigma : \begin{pmatrix} E\dot{x} \\ y \end{pmatrix} = \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right) \begin{pmatrix} x \\ u \end{pmatrix}, \qquad (1)$$

corresponding to the transfer matrix $\Sigma(s){=}C(sE{-}A)^{-1}B+D$. A (descriptor) state-space system denoted by Σ is in regular time scale t, with derivative $\frac{dx(t)}{dt} = \dot{x}(t)$. Notation Σ^τ means that the system is in the fast time scale τ, linked to t through $\varepsilon > 0$ as:

$$\tau = \frac{t - t_0}{\varepsilon} \quad \Leftrightarrow \quad \frac{d\tau}{dt} = \frac{1}{\varepsilon}. \qquad (2)$$

A vector-valued function $f(x, \varepsilon)$ is said to be $\mathcal{O}(\varepsilon)$ on a compact set D_x if there exist constants $k, \varepsilon^\star > 0$ such that:

$$\|f(x, \varepsilon)\|_2 \le k\varepsilon,\ \forall \varepsilon \in [0, \varepsilon^\star],\ \forall x \in D_x. \qquad (3)$$

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II. Mathematical Background

A. Singularly Perturbed Systems

This work focuses on MIMO LTI singularly perturbed systems. For a small singular perturbation parameter $0<\varepsilon\ll 1$ and time range $t\in[t_0,\infty)$, define the state-space dynamics Σ_ε, with conventional slow $x\in\mathbb{R}^n$ and fast $z\in\mathbb{R}^m$ state variables, along with input $u\in\mathbb{R}^r$ and output $y\in\mathbb{R}^r$ signals, with initial conditions $x(t_0)=x_0$, $z(t_0)=z_0$:

$$\Sigma_\varepsilon:\begin{pmatrix}\dot{x}\\ \varepsilon\dot{z}\\ \hline y\end{pmatrix}=\left(\begin{array}{cc|c}A_{11}(\varepsilon) & A_{12}(\varepsilon) & B_1(\varepsilon)\\ A_{21}(\varepsilon) & A_{22}(\varepsilon) & B_2(\varepsilon)\\ \hline C_1 & C_2 & O\end{array}\right)\begin{pmatrix}x\\ z\\ \hline u\end{pmatrix}. \quad (4)$$

Note that (4) is a special case of (1), with $E=\operatorname{diag}(I,\,\varepsilon I)$. The elements of the state and input matrices $A_{ij}\equiv A_{ij}(\varepsilon)$, $B_i\equiv B_i(\varepsilon)$, $i,j=1,2$ are assumed to be locally Lipschitz functions in the variable ε. In what follows, to simplify the notations, the dependency on ε will be omitted. Denote $A^0_{ij}=A_{ij}(0)$, $B^0_i=B_i(0)$, $i,j=1,2$.

In singular perturbation theory [9], two time scales t and τ are considered to provide complementary perspectives relative to the slow and fast dynamics, respectively, see (2).

When $\varepsilon=0$, $\overline{x}$ and $\overline{z}$ will be the degenerate state variables. This leads to an n^{th} order approximation $\overline{\Sigma}$ of Σ_ε, called the reduced-order (slow) dynamics. In this setup, the differential equation for z degenerates into an algebraic constraint:

$$0=A^0_{21}\overline{x}+A^0_{22}\overline{z}+B^0_2u. \quad (5)$$

We make the following assumption.

Assumption 1: The matrix A^0_{22} is invertible.

Assumption 1 is standard in the literature and ensures that (5) has a unique solution:

$$\overline{z}=h(\overline{x},u)=-\left(A^0_{22}\right)^{-1}\left(A^0_{21}\overline{x}+B^0_2u\right). \quad (6)$$

Replacing (6) in the first state equation of (4) leads to the state-space expression of the reduced-order dynamics:

$$\overline{\Sigma}:\begin{pmatrix}\dot{\overline{x}}\\ \hline y\end{pmatrix}=\left(\begin{array}{c|c}\overline{A} & \overline{B}\\ \hline \overline{C} & \overline{D}\end{array}\right)\begin{pmatrix}\overline{x}\\ \hline u\end{pmatrix},\quad \overline{x}(t_0)=x_0, \quad (7)$$

where $\overline{A}=A^0_{11}-A^0_{12}(A^0_{22})^{-1}A^0_{21}$, $\overline{B}=B^0_1-A^0_{12}(A^0_{22})^{-1}B^0_2$, $\overline{C}=C_1-C_2(A^0_{22})^{-1}A^0_{21}$, $\overline{D}=-C_2(A^0_{22})^{-1}B^0_2$. Let us define:

$$\hat{x}:=x-\overline{x},\quad \hat{z}:=z-\overline{z}. \quad (8)$$

The dynamics of $\hat{x}$ and $\hat{z}$ are obtained by differentiating (8) and rewriting it in the fast time scale τ from (2). On any compact set encompassing the operating domain of the system for $t\in[t_0,\infty)$, we have:

$$\frac{d\hat{x}(\tau)}{d\tau}=\varepsilon\left(\widehat{A}_{11}\overline{x}+A_{11}\hat{x}+A_{12}\hat{z}+\widehat{B}_1u\right)=\mathcal{O}(\varepsilon); \quad (9a)$$

$$\frac{d\hat{z}(\tau)}{d\tau}=\widehat{A}_{21}\overline{x}+A_{21}\hat{x}+A_{22}\hat{z}+\widehat{B}_2u+\varepsilon\frac{\partial h}{\partial u}\dot{u}, \quad (9b)$$

where $\frac{\partial h}{\partial u}=-\left(A^0_{22}\right)^{-1}B^0_2$ appears by differentiating $\hat{z}$ in (8),

$$\widehat{A}_{11}=\Delta A_{11}-\Delta A_{12}(A^0_{22})^{-1}A^0_{21}; \quad (10a)$$

$$\widehat{A}_{21}=\left(A_{21}-A_{22}(A^0_{22})^{-1}A^0_{21}\right)+\varepsilon(A^0_{22})^{-1}A^0_{21}\overline{A}; \quad (10b)$$

$$\widehat{B}_1=\Delta B_1-\Delta A_{12}(A^0_{22})^{-1}B^0_2; \quad (10c)$$

$$\widehat{B}_2=\left(B_2-A_{22}(A^0_{22})^{-1}B^0_2\right)+\varepsilon(A^0_{22})^{-1}A^0_{21}\overline{B}, \quad (10d)$$

and we use the auxiliary notations $\Delta A_{11}=A_{11}(\varepsilon)-A^0_{11}$, $\Delta A_{12}=A_{12}(\varepsilon)-A^0_{12}$, $\Delta B_1=B_1(\varepsilon)-B^0_1$.

Setting $\varepsilon=0$ in (9) we obtain the *boundary layer* system, which is autonomous, as $\hat{x}(\tau)|_{\varepsilon=0}=0$ and $\overline{x}$, $\overline{z}$, u are constants. Its state-space representation is:

$$\left.\frac{d\hat{z}(\tau)}{d\tau}\right|_{\varepsilon=0}=A^0_{22}\;\hat{z}(\tau)|_{\varepsilon=0}. \quad (11)$$

Assumption 2: The matrix A^0_{22} is Hurwitz.

This assumption is the standard SPS condition [9] ensuring that the equilibrium point $\hat{z}(\tau)|_{\varepsilon=0}=0$ of the boundary layer system (11) is globally asymptotically stable, uniformly in $\overline{x}$, $\overline{z}$, u, t_0.

The next theorem (Tikhonov) guarantees a sufficiently good approximation of Σ_ε based on its reduced-order system $\overline{\Sigma}$.

Theorem 1 ([9]): If Assumptions 1 and 2 are satisfied, then:

$$x(t)=\overline{x}(t)+\mathcal{O}(\varepsilon),\quad z(t)=\overline{z}(t)+\hat{z}(\tau)|_{\varepsilon=0}+\mathcal{O}(\varepsilon), \quad (12)$$

are valid for all $t\in[t_0,T]$, $T\geq t_0$, and there exists $t_1\geq t_0$ such that $z(t)=\overline{z}(t)+\mathcal{O}(\varepsilon)$ is valid for all $t\in[t_1,T]$, $T\geq t_1$.

B. Passivity of Descriptor Systems

Next, we discuss some properties of descriptor systems required to develop our main result. Let us recall the MIMO descriptor state-space (DSS) model from (1). The DSS Σ is called regular [4] (in the implicit case of square matrices) if $\det(sE-A)\not\equiv 0$. Otherwise, it is singular.

Classical DSS system operations can be found in [11], of which the inverse of a model Σ and the series connection will be useful for this work. The inverse of system Σ from (1) is:

$$\Sigma^{-1}:\begin{pmatrix}E\dot{x}_i\\ O\dot{x}_y\\ \hline y\end{pmatrix}=\left(\begin{array}{cc|c}A & B & O\\ -C & -D & I\\ \hline O & I & O\end{array}\right)\begin{pmatrix}x_i\\ x_y\\ \hline u\end{pmatrix}, \quad (13)$$

where $\left(x_i^\top,x_y^\top\right)^\top$ is an extended state vector. The series interconnection of two DSS models Σ_1 (output) and Σ_2 (input) is defined as $\mathcal{S}(\Sigma_1,\Sigma_2)=\Sigma_1\cdot\Sigma_2$:

$$\mathcal{S}(\Sigma_1,\Sigma_2):\begin{pmatrix}E_1\dot{x}_1\\ E_2\dot{x}_2\\ \hline y\end{pmatrix}=\left(\begin{array}{cc|c}A_1 & B_1C_2 & B_1D_2\\ O & A_2 & B_2\\ \hline C_1 & D_1C_2 & D_1D_2\end{array}\right)\begin{pmatrix}x_1\\ x_2\\ \hline u\end{pmatrix}. \quad (14)$$

On the other hand, passivity of LTI systems is tightly coupled with the concepts of phase, relative degree, and positive realness, as stated in [8]. Intuitively, a single-input single-output (SISO) system can only be passive if its relative degree is less than or equal to 1. We denote the (vector) relative degree of the system Σ by $\rho(\Sigma)$, as in [7].

To introduce the *phases* of a MIMO LTI system Σ, we briefly present the mathematical background described in [14]. Let Ω be the set of frequencies for which $j\Omega$ is the set of poles of Σ on the imaginary axis. For any $\omega\in[-\infty,\infty]\setminus\Omega$, the numeric range of the matrix $\Sigma(j\omega)\in\mathbb{C}^{r\times r}$ is:

$$W(\Sigma(j\omega))=\left\{a^H\cdot\Sigma(j\omega)\cdot a,\; a\in\mathbb{C}^r,\;\|a\|=1\right\}, \quad (15)$$

which is a compact and convex subset of $\mathbb{C}$. A matrix $\Sigma(j\omega)$ is called *sectorial* if $0 \notin W(\Sigma(j\omega))$. Each sectorial matrix admits a sectorial decomposition [18], i.e. there exist $T_\omega \in \mathbb{C}^{r\times r}$ invertible and $Z_\omega \in \mathbb{C}^{r\times r}$ diagonal unitary matrices such that $\Sigma(j\omega) = T_\omega^H Z_\omega T_\omega$. The matrix Z_ω is unique up to a permutation of its diagonal elements.

Definition 1: The phases ϕ_i of a sectorial matrix $\Sigma(j\omega)$ are the phases of the eigenvalues of matrix Z_ω and:

$$\underline{\phi}(\Sigma(j\omega)) \leq \phi_1(\Sigma(j\omega)) \leq \cdots \leq \phi_n(\Sigma(j\omega)) \leq \overline{\phi}(\Sigma(j\omega)).$$

The *phase center* $\gamma(\Sigma(j\omega))$ is selected as:

$$\gamma(\Sigma(j\omega)) = \frac{1}{2}\left(\underline{\phi}(\Sigma(j\omega)) + \overline{\phi}(\Sigma(j\omega))\right). \tag{16}$$

In this letter, we use the following definition.

Definition 2: For a given frequency $\omega \in [-\infty,\infty]\setminus\Omega$, the phases of a system Σ at ω are the interval:

$$\varphi(\Sigma(j\omega)) = \left[\underline{\phi}(\Sigma(j\omega)), \overline{\phi}(\Sigma(j\omega))\right], \tag{17}$$

while the phases of the system are $\varphi(\Sigma)=[\underline{\phi}(\Sigma), \overline{\phi}(\Sigma)]$, with:

$$\underline{\phi}(\Sigma) = \inf_{\omega\in[-\infty,\infty]\setminus\Omega} \underline{\phi}(\Sigma(j\omega)), \tag{18a}$$

$$\overline{\phi}(\Sigma) = \sup_{\omega\in[-\infty,\infty]\setminus\Omega} \overline{\phi}(\Sigma(j\omega)). \tag{18b}$$

For the series interconnection of two $r \times r$ systems Σ_1 and Σ_2 having the poles on the imaginary axis $j\Omega_1$ and $j\Omega_2$, respectively, the phases of the resulting system are bounded by the direct sum of the individual phases of Σ_1 and Σ_2.

Lemma 1 ([14]): Let $\omega \in [-\infty,\infty] \setminus (\Omega_1 \cup \Omega_2)$ such that matrices $\Sigma_1(j\omega)$, $\Sigma_2(j\omega) \in \mathbb{C}^{r\times r}$ are two sectorial matrices with phase centers $\gamma(\Sigma_i(j\omega))$. Then, the matrix $\Sigma_1(j\omega)\Sigma_2(j\omega)$ has $\operatorname{rank}\left((\Sigma_1(j\omega)\Sigma_2(j\omega))^2\right)$ nonzero eigenvalues $\lambda_i(\Sigma_1(j\omega)\Sigma_2(j\omega))$ such that the phase $\angle\lambda_i(\Sigma_1(j\omega)\Sigma_2(j\omega))$ can take values between $\gamma(\Sigma_1(j\omega)) + \gamma(\Sigma_2(j\omega)) - \pi$ and $\gamma(\Sigma_1(j\omega)) + \gamma(\Sigma_2(j\omega)) + \pi$. Moreover,

$$\underline{\phi}(\Sigma_1(j\omega)) + \underline{\phi}(\Sigma_2(j\omega)) \leq \angle\lambda_i(\Sigma_1(j\omega)\Sigma_2(j\omega)) \leq \\ \leq \overline{\phi}(\Sigma_1(j\omega)) + \overline{\phi}(\Sigma_2(j\omega)). \tag{19}$$

The necessary and sufficient conditions for a system to be strictly positive real can be expressed in terms of MIMO LTI phases, as follows.

Lemma 2 ([14]): The $r \times r$ transfer matrix Σ for which $\Sigma(s) + \Sigma(-s)^\top \not\equiv 0$ is strictly positive real if and only if $\Sigma(s)$ is Hurwitz and $\varphi(\Sigma) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $\varphi(\Sigma) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then the system is positive real only.

The passivity of system Σ can thus be studied based on the phase concept, according to Lemma 5.4 from [8].

Lemma 3 ([8]): The MIMO LTI system Σ is strictly passive if it is strictly positive real and is passive if it is positive real.

Remark 1: The equivalence between (strongly) positive real systems and asymptotically stable (strictly-input) passive systems is proved in [10].

III. Passivity Characterization

A. Motivating Example

The typical procedure involving SPSs is to study the reduced-order and boundary layer systems and, based on Theorem 1, to conclude the desired outcome for a small $\varepsilon > 0$. This paradigm does not work for passivity analysis, as illustrated by the following example.

Consider a simple DC motor LTI model [9], with angular speed ω_r, inductor current i_r and command voltage u_r, scaled to relative units, in the form (4):

$$\Sigma_\varepsilon^{\text{DCM}} : \begin{pmatrix} \dot{\omega}_r \\ \varepsilon \dot{i}_r \\ \hline \omega_r \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right) \begin{pmatrix} \omega_r \\ i_r \\ \hline u_r \end{pmatrix}, \tag{20}$$

$\omega_r(t_0) = \omega_{r,0}$, $i_r(t_0) = i_{r,0}$, where the perturbation variable is the ratio between electrical and mechanical time constants, i.e. $\varepsilon = \frac{T_e}{T_m}$, with $T_m \gg T_e$. This formulation leads to a physically meaningful dimensionless parameter $0 < \varepsilon \ll 1$.

The reduced-order dynamics are:

$$\overline{\Sigma}^{\text{DCM}} : \begin{pmatrix} \dot{\overline{\omega}}_r \\ \hline \overline{\omega}_r \end{pmatrix} = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right) \begin{pmatrix} \overline{\omega}_r \\ \hline u_r \end{pmatrix}, \ \overline{\omega}_r(t_0) = \omega_{r,0}. \tag{21}$$

The transfer functions of the reduced-order system (21) and full-order one (20), respectively, are:

$$\overline{\Sigma}(s) = \frac{1}{s+1}, \quad \Sigma_\varepsilon(s) = \frac{1}{(s+1)(\varepsilon s+1)}. \tag{22}$$

Although both the reduced-order $\overline{\Sigma}$ and the boundary layer subsystems are passive, the full system Σ_ε is not passive irrespective of $\varepsilon > 0$, as Lemma 2 is not satisfied. Furthermore, Theorem 1 from [1] is not applicable, as A_{11}^0 is not Hurwitz. In fact, passivity properties change when ε changes from 0 to $\varepsilon > 0$ due to changes in the relative degree.

B. Problem Statement

As shown by the previous example, the passivity of the reduced-order and the quotient boundary layer systems does not ensure the passivity of the full system. Therefore, our goal is to provide conditions for Σ_ε to remain passive for positive values of ε. Thus we address the next two problems.

Problem 1: Given the SPS Σ_ε in (4) with passive reduced-order dynamics $\overline{\Sigma}$ from (7), under which conditions there exists a value $\varepsilon^\star > 0$ such that $\Sigma_{\varepsilon^\star}$ is passive?

The existence of such an $\varepsilon^\star > 0$ leads to the so-called ε-bound computation [3] problem.

Problem 2: Determine a bound $\varepsilon^\star > 0$ for the perturbation variable such that system Σ_ε maintains passivity for all $\varepsilon \in [0, \varepsilon^\star]$, i.e. solve the optimization problem:

$$\max_{\varepsilon^\star > 0} \ \varepsilon^\star \quad \text{s.t.} \quad \text{System (4) is passive } \forall \varepsilon \in [0, \varepsilon^\star]. \tag{23}$$

Starting from Problem 2, in order to have $\varepsilon^\star > 0$, the passivity of Σ_ε has be ensured for $\varepsilon = 0$. This leads to the next assumption.

Assumption 3: The reduced-order system $\overline{\Sigma}$ is passive.

The above assumption is not conservative because the passivity of the SPS model Σ_ε requires the passivity of the slow subsystem $\overline{\Sigma}$. Furthermore, $\overline{\Sigma}$ is Hurwitz, of minimum phase, and as $\overline{\Sigma}$ is passive, its vector relative degree can only take the following values:

$$\overline{\rho} \equiv \rho(\overline{\Sigma}) = (b_1, \ldots, b_r) \in \mathbb{Z}_2^r. \tag{24}$$

C. Proposed Solution

Consider a multiplicative factorization of system Σ_ε based on the known reduced-order model $\overline{\Sigma}$. For non-commutative state-space multiplications there are two possible factorizations [9], $\Sigma_\varepsilon{=}\overline{\Sigma}\cdot\widetilde{\Sigma}_\varepsilon^{(R)}$ and $\Sigma_\varepsilon{=}\widetilde{\Sigma}_\varepsilon^{(L)}\cdot\overline{\Sigma}$, called *actuator* and *sensor* forms. From now on, we proceed with the actuator form and denote $\widetilde{\Sigma}_\varepsilon\equiv\widetilde{\Sigma}_\varepsilon^{(R)}$. Then, $\widetilde{\Sigma}_\varepsilon$ can be expressed as:

$$\widetilde{\Sigma}_\varepsilon=\left(\overline{\Sigma}\right)^{-1}\cdot\Sigma_\varepsilon, \tag{25}$$

Even though $\overline{\Sigma}$ is a regular state-space model, its inverse is an improper system if $\overline{D}=O$ and can be written as a state-space model only if $\operatorname{rank}\overline{D}=r$. To eliminate this constraint, we proceed with the inverse DSS model (13). Thus, the passivity of Σ_ε can be analyzed in terms of $\widetilde{\Sigma}_\varepsilon$ from (25) and the, by Assumption 3, already passive system $\overline{\Sigma}$. From the SPS definition, we have:

$$\overline{\Sigma}=\Sigma_0 \overset{(25)}{\Rightarrow} \widetilde{\Sigma}_0=I_r. \tag{26}$$

Remark 2: The dynamic systems $\widetilde{\Sigma}_\varepsilon$, for $\varepsilon\to 0^+$, and I_r are not isomorphic due to the change from a differential to an algebraic equation, i.e.:

$$\lim_{\varepsilon\to 0^+}\widetilde{\Sigma}_\varepsilon\neq\widetilde{\Sigma}_0=I_r. \tag{27}$$

A possible way to simplify the dynamics of $\overline{\Sigma}$ from the eigenstructure of Σ_ε in (25) is to use a state-space realization of Σ_ε which emphasizes the coupling between the reduced-order states $\overline{x}$ and perturbations $\hat{x}$, $\hat{z}$. This can be obtained from system (9) in the regular time scale t:

$$\Sigma_\varepsilon:\begin{pmatrix}\dot{\overline{x}}\\ \dot{\hat{x}}\\ \varepsilon\dot{\hat{z}}\\ \hline y\end{pmatrix}=\left(\begin{array}{ccc|c}\overline{A}&O&O&\overline{B}\\ \widehat{A}_{11}&A_{11}&A_{12}&\widehat{B}_1\\ \widehat{A}_{21}&A_{21}&A_{22}&\widehat{B}_2\\ \hline \overline{C}&C_1&C_2&\overline{D}\end{array}\right)\begin{pmatrix}\overline{x}\\ \hat{x}\\ \hat{z}\\ \hline u\end{pmatrix}+\begin{pmatrix}0\\0\\ \frac{\partial h}{\partial u}\dot{u}\\ 0\end{pmatrix}.$$

The above expression is not in the standard form (1), as $\varepsilon\dot{\hat{z}}$ depends on $\dot{u}$, see (9b). A change of variables to overcome the dependency on $\frac{\partial h}{\partial u}\dot{u}$ is:

$$\hat{\zeta}=\hat{z}-(A_{22}^0)^{-1}B_2^0u \;\Rightarrow\; \dot{\hat{\zeta}}=\dot{\hat{z}}-(A_{22}^0)^{-1}B_2^0\dot{u}. \tag{28}$$

With this step, a full characterization of Σ_ε becomes:

$$\Sigma_\varepsilon:\begin{pmatrix}\dot{\overline{x}}\\ \dot{\hat{x}}\\ \varepsilon\dot{\hat{\zeta}}\\ \hline y\end{pmatrix}=\left(\begin{array}{ccc|c}\overline{A}&O&O&\overline{B}\\ \widehat{A}_{11}&A_{11}&A_{12}&\widetilde{B}_1\\ \widehat{A}_{21}&A_{21}&A_{22}&\widetilde{B}_2\\ \hline \overline{C}&C_1&C_2&O\end{array}\right)\begin{pmatrix}\overline{x}\\ \hat{x}\\ \hat{\zeta}\\ \hline u\end{pmatrix}, \tag{29}$$

with $\widetilde{B}_1{=}\Delta B_1{+}A_{12}^0(A_{22}^0)^{-1}B_2^0$, $\widetilde{B}_2{=}B_2+\varepsilon(A_{22}^0)^{-1}A_{21}^0\overline{B}$.

Remark 3: If, instead of the output y, the fast output $\hat{y}{=}C_1\hat{x}{+}C_2\hat{z}$ is desired, then the output and feedforward matrices in (29) are replaced by $\widehat{C}{=}(O\ C_1\ C_2)$ and $\widehat{D}{=}{-}\overline{D}$.

Applying the inverse (13) to $\overline{\Sigma}$ and series connection (14) to (25), we obtain a DSS formulation of the quotient system:

$$\widetilde{\Sigma}_\varepsilon:\begin{pmatrix}\widetilde{E}_\varepsilon\dot{\xi}\\ \hline y\end{pmatrix}=\left(\begin{array}{ccccc|c}\overline{A}&O&O&O&\overline{B}&O\\ \widehat{A}_{11}&A_{11}&A_{12}&O&\widetilde{B}_1&O\\ \widehat{A}_{21}&A_{21}&A_{22}&O&\widetilde{B}_2&O\\ O&O&O&\overline{A}&\overline{B}&O\\ O&O&O&-\overline{C}&-\overline{D}&I_r\\ \hline \overline{C}&C_1&C_2&O&O&O\end{array}\right)\begin{pmatrix}\xi\\ \hline u\end{pmatrix}, \tag{30}$$

with block diagonal matrix $\widetilde{E}_\varepsilon{=}\operatorname{diag}(I_n,I_n,\varepsilon I_m,I_n,O_r)$, and state $\xi=\left(\xi_k^\top\right)^\top=\begin{pmatrix}\overline{x}^\top & \hat{x}^\top & \hat{\zeta}^\top & \overline{x}_i^\top & \overline{x}_y^\top\end{pmatrix}^\top$, $k=\overline{1,5}$.

Transitioning from the regular time scale t to the fast time scale τ, the state derivatives become:

$$\dot{\xi}_k=\frac{d\xi_k(t)}{dt}=\frac{d\xi_k(\tau)}{\varepsilon d\tau}\;\Rightarrow\;\frac{d\xi_k(\tau)}{d\tau}=\mathcal{O}(\varepsilon),\ k\in\{1,2,4\}, \tag{31}$$

which, in the time base τ, will cause a multiplication by ε to all terms of the input and state matrices of system $\widetilde{\Sigma}_\varepsilon$ from (30). Furthermore, row $k=3$ cancels ε from matrix $\widetilde{E}_\varepsilon$, and row $k=5$ lacks the derivative, so ε can be simplified. Thus, the previous system written in time base τ becomes $\widetilde{\Sigma}_\varepsilon^\tau$:

$$\begin{pmatrix}\widetilde{E}_\varepsilon^\tau\dot{\xi}\\ \hline y\end{pmatrix}=\left(\begin{array}{ccccc|c}\varepsilon\overline{A}&O&O&O&\varepsilon\overline{B}&O\\ \varepsilon\widehat{A}_{11}&\varepsilon A_{11}&\varepsilon A_{12}&O&\varepsilon\widetilde{B}_1&O\\ \widehat{A}_{21}&A_{21}&A_{22}&O&\widetilde{B}_2&O\\ O&O&O&\varepsilon\overline{A}&\varepsilon\overline{B}&O\\ O&O&O&-\overline{C}&-\overline{D}&I_r\\ \hline \overline{C}&C_1&C_2&O&O&O\end{array}\right)\begin{pmatrix}\xi\\ \hline u\end{pmatrix}, \tag{32}$$

where $\widetilde{E}_\varepsilon^\tau=\operatorname{diag}(I_n,I_n,I_m,I_n,O_r)$. By assumption, $A_{ij}(\varepsilon)$, $B_i(\varepsilon)$ are Lipschitz in ε, therefore

$$\lim_{\varepsilon\to 0^+}\widetilde{\Sigma}_\varepsilon^\tau=\widetilde{\Sigma}_{0+}^\tau=\widetilde{\Sigma}_0^\tau, \tag{33}$$

which then leads to the DSS:

$$\widetilde{\Sigma}_0^\tau:\begin{pmatrix}\widetilde{E}_0^\tau\dot{\xi}\\ \hline y\end{pmatrix}=\left(\begin{array}{ccccc|c}O&O&O&O&O&O\\ O&O&O&O&O&O\\ \widehat{A}_{21}&A_{21}&A_{22}&O&\widetilde{B}_2&O\\ O&O&O&O&O&O\\ O&O&O&-\overline{C}&-\overline{D}&I_r\\ \hline \overline{C}&C_1&C_2&O&O&O\end{array}\right)\begin{pmatrix}\xi\\ \hline u\end{pmatrix}, \tag{34}$$

where $\widetilde{E}_0^\tau=\operatorname{diag}(I_n,I_n,I_m,I_n,O_r)$. In contrast to (27), system $\widetilde{\Sigma}_{0+}^\tau$, i.e. the limit for $\varepsilon\to 0^+$, is identical to the *quotient boundary layer* system $\widetilde{\Sigma}_0^\tau$.

The passivity of the system is not affected by the time scale t or τ, since it requires the same positive realness property. As such, using Definition 2 and Lemma 1, considering $j\Omega$ as the set of imaginary axis poles, we have:

$$\varphi(\Sigma_\varepsilon)\subseteq\varphi(\overline{\Sigma})+\varphi(\widetilde{\Sigma}_\varepsilon)\;\Leftrightarrow\;\varphi(\Sigma_\varepsilon^\tau)\subseteq\varphi(\overline{\Sigma}^\tau)+\varphi(\widetilde{\Sigma}_\varepsilon^\tau). \tag{35}$$

Based on Assumptions 1–3, the solution to Problem 1 is summarized in the following theorem, considering all admissible configurations of the binary word $\rho(\overline{\Sigma})$ from (24).

Theorem 2: There exists $\varepsilon^\star>0$ such that the system $\Sigma_{\varepsilon^\star}$ from (4) is passive, if $\widetilde{\Sigma}_{0+}^\tau$ is Hurwitz and one of the following cases apply:

A) $\widetilde{\Sigma}_{0+}^\tau$ is passive, if $\overline{\rho}=(0,\ldots,0)\equiv\overline{0}$;
B) $\varphi(\widetilde{\Sigma}_{0+}^\tau)\in[0,\pi]$, if $\overline{\rho}=(1,\ldots,1)\equiv\overline{1}$;
C) $\varphi(\widetilde{\Sigma}_{0+}^\tau)\in\left[0,\frac{\pi}{2}\right]$, if $\overline{\rho}\in\mathbb{Z}_2^r\setminus\{\overline{0},\overline{1}\}$.

Proof: We consider the system $\lim\limits_{\varepsilon\to 0^+}\Sigma_\varepsilon^\tau$. We first provide sufficient conditions for (i) Σ_{0+}^τ to be Hurwitz and (ii) $\varphi\left(\Sigma_{0+}^\tau\right)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, i.e., Lemma 2. Note that if the conditions of Lemma 2 are satisfied, then, based on Lemma 3, system $\Sigma_{\varepsilon^\star}$ is passive.

As Σ_{0+}^{τ} is the series connection between $\lim\limits_{\varepsilon\to 0^+}\overline{\Sigma}^{\tau}$ and $\widetilde{\Sigma}_{0+}^{\tau}$, and according to Assumption 3, $\lim\limits_{\varepsilon\to 0^+}\overline{\Sigma}^{\tau}$ is Hurwitz, system $\widetilde{\Sigma}_{0+}^{\tau}$ is also Hurwitz, so the proof for (i) is complete.

We now focus on the phases of the subsystems. Consider (35) for $\varepsilon \to 0^+$ in the τ time scale:

$$\lim_{\varepsilon\to 0^+}\varphi(\Sigma_{\varepsilon}^{\tau}) \subseteq \lim_{\varepsilon\to 0^+}\varphi(\overline{\Sigma}^{\tau}) + \lim_{\varepsilon\to 0^+}\varphi(\widetilde{\Sigma}_{\varepsilon}^{\tau}). \tag{36}$$

According to (24), we have $\overline{\rho} \in \mathbb{Z}_2^r$. Three possible cases for the phases of $\overline{\Sigma}^{\tau}$ are distinguished, illustrated in Figure 1, each case being equivalent to a set of conditions A)–C). If $\overline{\rho} = (0,\dots,0)$, then $\lim\limits_{\varepsilon\to 0^+}\varphi(\overline{\Sigma}^{\tau})=[0,0]$, so (36) becomes:

$$\lim_{\varepsilon\to 0^+}\varphi(\Sigma_{\varepsilon}^{\tau}) \subseteq \varphi(\widetilde{\Sigma}_{0}^{\tau}). \tag{37}$$

A sufficient condition for (ii) is $\varphi(\widetilde{\Sigma}_0^{\tau}) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which, together with (i), implies – based on Lemmas 2 and 3 – the passivity of $\widetilde{\Sigma}_{0+}^{\tau}$.

If $\overline{\rho} = (1,\dots,1)$, then $\lim\limits_{\varepsilon\to 0^+}\varphi(\overline{\Sigma}^{\tau}) = \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right]$, so (36) can be written as:

$$\lim_{\varepsilon\to 0^+}\varphi(\Sigma_{\varepsilon}^{\tau}) \subseteq \left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] + \varphi(\widetilde{\Sigma}_{0}^{\tau}), \tag{38}$$

therefore, a sufficient condition for (ii) is $\varphi(\widetilde{\Sigma}_0^{\tau}) \in [0, \pi]$.

If $\overline{\rho} \in \mathbb{Z}_2^r \setminus \{\overline{0}, \overline{1}\}$, then $\lim\limits_{\varepsilon\to 0^+}\varphi(\overline{\Sigma}^{\tau}) = \left[-\dfrac{\pi}{2}, 0\right]$. Using (36), a sufficient condition for (ii) is $\varphi(\widetilde{\Sigma}_0^{\tau}) \in \left[0, \frac{\pi}{2}\right]$, which concludes the proof. ∎

Remark 4: Case C) of Theorem 2 is only applicable to MIMO systems. In the particular case of SISO systems, conditions given in Theorem 2–A) and B) are necessary and sufficient because the set inclusion from (35) is verified with equality. Moreover, all conditions from Theorem 2 can be formulated to ensure strict passivity by working with open intervals for phases and with frequencies $\omega \in (0,\infty) \setminus \Omega$ in Definition 2.

Remark 5: The DC motor example from Section III-A falls into the case of Theorem 2–B), but $\varphi(\widetilde{\Sigma}_{0+}^{\tau}) \notin [0, \pi]$, as the fast dynamics introduce an excess pole.

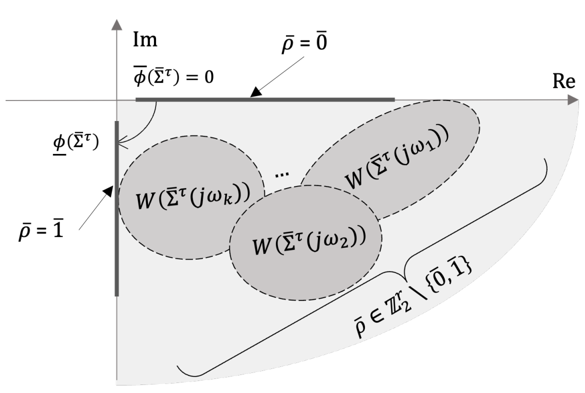

Fig. 1. Three different conditions apply to the passivity analysis of Σ_{ε} depending on the relative degree of the reduced-order subsystem $\overline{\Sigma}$, as stated in Theorem 2: $\overline{\rho} = \overline{0}$, $\overline{\rho} = \overline{1}$, and $\overline{\rho} \in \mathbb{Z}_2^r \setminus \{\overline{0}, \overline{1}\}$, respectively.

The solution to Problem 2 can be elegantly obtained using a line search algorithm or any gradient-free method after an adequate reformulation. Define the function $\chi : \mathbb{R} \to \mathbb{R}$:

$$\chi(\theta_0) = \min_{\varepsilon\in\mathbb{R}^+}\max_{0\le i\le 2}\left\{\frac{f_i(\varepsilon)}{\theta_i}\right\}, \tag{39}$$

with implicit objective f_0, nonconvex constraints f_1, f_2, for $\omega\in[-\infty,\infty]\setminus\Omega$, based on (23) and the phase condition (35):

$$f_0(\varepsilon) := \frac{1}{\varepsilon}; \tag{40a}$$

$$f_1(\varepsilon) := -\underline{\phi}(\overline{\Sigma}(j\omega)) - \underline{\phi}(\widetilde{\Sigma}_{\varepsilon}(j\omega)) \le \theta_1 := \frac{\pi}{2}; \tag{40b}$$

$$f_2(\varepsilon) := +\overline{\phi}(\overline{\Sigma}(j\omega)) + \overline{\phi}(\widetilde{\Sigma}_{\varepsilon}(j\omega)) \le \theta_2 := \frac{\pi}{2}. \tag{40c}$$

According to Theorem 2 from [6], function χ is a continuous and monotone nonincreasing function of θ_0, $\theta_0 < f_0(\varepsilon^\star) \Rightarrow \chi(\theta_0) > 1$, $\theta_0 > f_0(\varepsilon^\star) \Rightarrow \chi(\theta_0) < 1$, and $\theta_0 = f_0(\varepsilon^\star)$ if and only if $\chi(\theta_0) = 1$, where $\varepsilon^\star$ is a solution to Problem 2.

If Problem 1 has a solution and, furthermore, $\widetilde{\Sigma}_0^{\tau}$ is Hurwitz and of minimum phase, as required by Theorem 2, then finding the solution to Problem 2 is equivalent to finding the unique root of a monotone decreasing scalar function in the variable $\varepsilon > 0$:

$$\text{Find } \theta_0^\star = f_0(\varepsilon^\star) \text{ such that } \chi(\theta_0^\star) = 1. \tag{41}$$

In practice, each iteration of χ requires solving an unconstrained minimax problem, which itself is a nontrivial problem, but χ is a nearly linear function in the (already small) neighborhood of the solution $\varepsilon^\star$.

For the effective verification of the conditions developed in this section, the existence of $\varepsilon^\star > 0$ such that system (4) is passive can be easily checked by verifying the passivity of (7) and (34), while the computation of $\varepsilon^\star$ can be performed using the `fmincon` and `fminimax` routines in MATLAB (or other similar software tools), for (40) or (41).

Remark 6: A numerical difficulty arises in the passivity test for $\widetilde{\Sigma}_0^{\tau}$ if $\overline{D} = O$, as the DSS becomes singular. This can be bypassed using $\widetilde{\Sigma}_{0+}^{\tau}$, which is regular and well-posed, with a value $\varepsilon > 0$ sufficiently small that its singularities are well separated from those of $\overline{\Sigma}^{\tau}$ in the complex plane. In contrast to the forms of Remark 2, in the time scale τ, $\lim\limits_{\varepsilon\to 0^+}\widetilde{\Sigma}_{\varepsilon}^{\tau} = \widetilde{\Sigma}_0^{\tau}$.

IV. Numerical Example

Consider an SPS model described by the transfer function with emphasized reduced-order and residual dynamics:

$$\Sigma_{\varepsilon}(s) = \underbrace{\frac{(s+1)(2s+1)}{(5s+1)(6s+1)}}_{\overline{\Sigma}} \cdot \underbrace{\frac{20\varepsilon s+1}{(10\varepsilon s+1)(50\varepsilon s+1)}}_{\widetilde{\Sigma}_{\varepsilon}}. \tag{42}$$

A possible state-space representation complying to (4) is given by $\overline{\Sigma} = \left(I, \overline{A}, \overline{B}, \overline{C}, \overline{D}\right)$, $\widetilde{\Sigma}_{\varepsilon} = \left(\varepsilon I, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\right)$:

$$\overline{\Sigma} = \left(\begin{array}{cc|c} -\frac{11}{30} & -\frac{1}{30} & \frac{1}{30} \\ 1 & 0 & 0 \\ \hline \frac{34}{15} & \frac{14}{15} & \frac{1}{15} \end{array}\right),\ \widetilde{\Sigma}_{\varepsilon} = \left(\begin{array}{cc|c} -\frac{1}{10} & 0 & \frac{1}{40} \\ 0 & -\frac{1}{50} & \frac{3}{200} \\ \hline 1 & 1 & 0 \end{array}\right).$$

The full SPS expression can be obtained based on (14) as $\Sigma_\varepsilon = \mathcal{S}(\overline{\Sigma}, \widetilde{\Sigma}_\varepsilon)$, not written explicitly due to space constraints.

Remark 7: The nominal workflow starts from a given Σ_ε and recover $\overline{\Sigma}$, $\widetilde{\Sigma}_\varepsilon$ using (7) and (30), respectively. As such, $\widetilde{\Sigma}_\varepsilon$ will not be in minimal form, but the use of the frequency response framework implicitly removes the need for it.

The resulting DSS in time scale τ from (34) is regular, and is described by the transfer function: $\widetilde{\Sigma}_0^\tau(s) = \frac{20s+1}{(10s+1)(50s+1)}$.

This example falls into the category from Theorem 2–A), in which both $\overline{\Sigma}^\tau$ and $\widetilde{\Sigma}_{0+}^\tau = \widetilde{\Sigma}_0^\tau$ are passive. The quotient transfer function $\widetilde{\Sigma}_0^\tau$ can be intuitively seen here as a frequency normalization to εs, followed by $\varepsilon \to 0$ in (42). Solving Problem 2, i.e., finding the solution of $\chi(\theta_0) = 1$ from (41) leads to $\varepsilon^\star = 0.02066$. The behaviour of χ with respect to the implicit variable θ_0 is illustrated in Figure 2, which shows the anticipated decreasing nature.

Figures 3 and 4 depict the phase responses of the full SPS (42) for two different values of ε. Figure 3 considers an extreme value of $\varepsilon = 10^{-9}$, which emphasizes the phase behaviour from Theorem 2–A), where, from the point of view of subsystem $\widetilde{\Sigma}_0^\tau$, the residual phase appearing from $\overline{\Sigma}^\tau$ is practically null.

Figure 4 shows the phase at the boundary of the constraint (40b), i.e. the optimal value $\varepsilon^\star$ of the perturbation variable.

V. Conclusions

This letter proposed phase conditions to guarantee the passivity of linear perturbation parameter-dependent MIMO SPSs based on the dynamics of their reduced-order and extended quotient boundary layer subsystems. It also provided quantitative frequency-domain characterizations of how the full SPS is affected by the structural modifications caused by the transition from $\varepsilon = 0$ to $\varepsilon > 0$. Our approach allows all matrices of (4) to depend on ε, while requiring fewer assumptions. The main future research direction is to extend this framework to analyze the passivity of nonlinear SPSs.

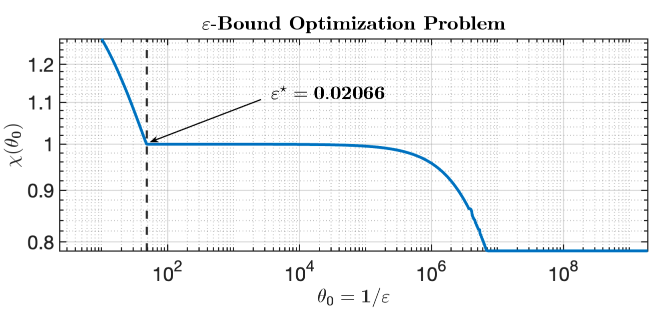

Fig. 2. The ε-bound problem reformulated as a unique root-finding of the monotone decreasing scalar function $\chi(\theta_0) - 1 = 0$ from (39).

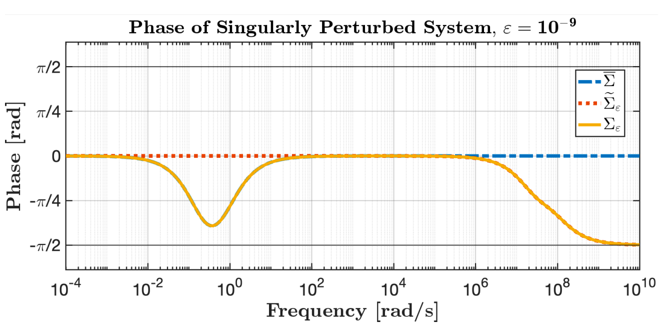

Fig. 3. Phase response of Σ_ε for $\varepsilon = 10^{-9}$.

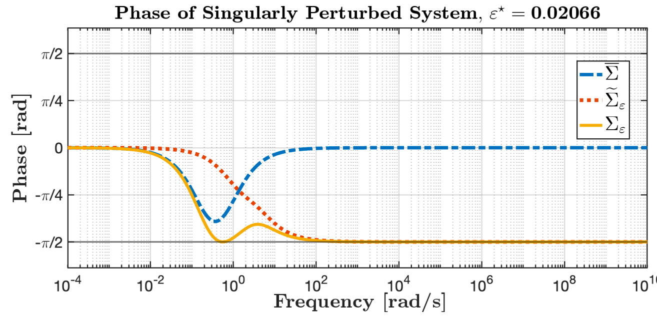

Fig. 4. Phase response of Σ_ε for $\varepsilon^\star = 0.02066$.

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