

Bearing-based distributed pose estimation for multi-agent networks

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Abstract—In this paper, we address the distributed pose estimation problem for multi-agent systems under a directed graph topology, where two agents have access to their respective poses, and the other agents have unknown static positions and time-varying orientations. The proposed estimation scheme consists of two cascaded distributed observers, an almost globally asymptotically stable (AGAS) attitude observer and an input-to-state stable (ISS) position observer, leading to an overall AGAS distributed localization scheme. Numerical simulation results are presented to illustrate the performance of our proposed distributed pose estimation scheme.

I. INTRODUCTION

The distributed pose estimation problem for multi-agent networks consists of estimating the agents' poses (positions and orientations) in a distributed manner using some available absolute and relative measurements. Due to the importance of this problem in many applications related to multi-agent autonomous networks, significant research has been devoted to designing robust and reliable distributed pose localization schemes. According to the network's sensing capabilities, these schemes can be categorized as position-based, distance-based, and bearing-based schemes. Recently, the latter category is gaining in popularity because of the revolutionary development in bearing sensors. As a result, several bearing-based distributed position estimation solutions have been proposed in the literature [1], [2], [3], [4]. However, the aforementioned references assume that the bearings are expressed in a global reference frame (*i.e.*, knowledge of agents' orientations with respect to a global reference frame), which, unfortunately, is not the case in most of the practical applications since the bearing measurements are usually obtained locally from a sensor (*e.g.*, a camera) mounted on the agent. This motivated many authors to design a distributed attitude observer that can be fed into a position estimation scheme together with local bearing measurements to obtain an overall cascaded bearing-based distributed pose estimation scheme [5], [6]. The idea consists in using the estimated attitudes to transform the local relative bearing measurements into the global reference frame and then use the transformed bearings in the position estimation law. In [7], [8], [9], [6], the authors proposed several orientation estimation algorithms based on the consensus approach and the Gram-Schmidt procedure.

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Later, these algorithms were extended to deal with a time-varying orientation in an arbitrary dimensional space [10], [11]. Note that the Gram-Schmidt procedure sometimes fails when the estimated matrix is singular. In addition, most of the distributed attitude estimation schemes require relative attitude measurements, which are difficult to obtain since no low-cost setup can provide such measurements. To the best of the authors' knowledge, there are very few results in the literature that address the distributed pose estimation problem for multi-agent networks relying on local relative bearing measurements without using the relative attitude information. References [12], [13] proposed distributed pose estimation schemes based on local relative bearing measurements and some absolute measurements (angular and linear velocities). Unfortunately, reference [12] provides only convergence results, with some flaws in the proofs that have been pointed out in [14]¹, and [13] provides only local exponential stability guarantees. Note that, in contrast to [13], the present work and the work in [12] deal with constant inertial relative bearings. In this paper, we propose a new distributed pose estimation scheme, with almost global asymptotic stability guarantees, for multi-agent systems with two agents having access to their respective poses. The proposed estimations scheme relies on individual angular velocity measurements and local information exchange between neighboring agents (relative bearing measurements and estimated poses), according to a directed graph topology. To the best of our knowledge, no such strong stability result has been reported in the available literature for the problem under consideration.

II. PRELIMINARIES

A. Notations

The sets of real numbers and the n -dimensional Euclidean space are denoted by \mathbb{R} and \mathbb{R}^n , respectively. The set of unit vectors in \mathbb{R}^n is defined as $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid x^T x = 1\}$. Given two matrices $A, B \in \mathbb{R}^{m \times n}$, their Euclidean inner product is defined as $\langle\langle A, B \rangle\rangle = \text{tr}(A^T B)$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = \sqrt{x^T x}$, and the Frobenius norm of a matrix $A \in \mathbb{R}^{n \times n}$ is given by $\|A\|_F = \sqrt{\langle\langle A, A \rangle\rangle}$. The matrix $I_n \in \mathbb{R}^{n \times n}$ denotes the identity matrix.

The attitude of a rigid body is represented by a rotation matrix R which belongs to the special orthogonal group $SO(3) := \{R \in \mathbb{R}^{3 \times 3} \mid \det(R) = 1, R^T R = I_3\}$. The tangent space of the compact manifold $SO(3)$ is given by

¹Authors' reply can be found in [15].

$T_R SO(3) := \{R\Omega \mid \Omega \in \mathfrak{so}(3)\}$, where $\mathfrak{so}(3) := \{\Omega \in \mathbb{R}^{3 \times 3} \mid \Omega^T = -\Omega\}$ is the Lie algebra of the matrix Lie group $SO(3)$. The map $[\cdot]^\times : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined such that $[x]^\times y = x \times y$, for any $x, y \in \mathbb{R}^3$, where \times denotes the vector cross product on \mathbb{R}^3 . The inverse map of $[\cdot]^\times$ is $\text{vex} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ such that $\text{vex}([\omega]^\times) = \omega$, and $[\text{vex}(\Omega)]^\times = \Omega$ for all $\omega \in \mathbb{R}^3$ and $\Omega \in \mathfrak{so}(3)$. Let $\mathbb{P}_a : \mathbb{R}^{3 \times 3} \rightarrow \mathfrak{so}(3)$ be the projection map on the Lie algebra $\mathfrak{so}(3)$ such that $\mathbb{P}_a(A) := (A - A^T)/2$. Given a 3-by-3 matrix $C := [c_{ij}]_{i,j=1,2,3}$, one has $\psi(C) := \text{vex} \circ \mathbb{P}_a(C) = \text{vex}(\mathbb{P}_a(C)) = \frac{1}{2}[c_{32} - c_{23}, c_{13} - c_{31}, c_{21} - c_{12}]^T$. For any $R \in SO(3)$, the normalized Euclidean distance on $SO(3)$, with respect to the identity I_3 , is defined as $|R|_I^2 := \frac{1}{4}\text{tr}(I_3 - R) \in [0, 1]$. The angle-axis parameterization of $SO(3)$, is given by $\mathcal{R}_\alpha(\theta, v) := I_3 + \sin \theta [v]^\times + (1 - \cos \theta)([v]^\times)^2$, where $v \in \mathbb{S}^2$ and $\theta \in \mathbb{R}$ are the rotation axis and angle, respectively. The orthogonal projection map $P : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ is defined as $P_x := I_3 - \frac{xx^T}{\|x\|^2}$, $\forall x \in \mathbb{R}^3 \setminus \{0_3\}$. One can verify that $P_x^T = P_x$, $P_x^2 = P_x$, and P_x is positive semi-definite. Given a subset $\mathcal{S} \subset \mathbb{N}$, where \mathbb{N} is a set of natural numbers, the cardinality of \mathcal{S} is denoted by $|\mathcal{S}|$.

B. Graph Theory

Consider a network of n agents. The interaction topology between the agents is described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represent the vertex (or agent) set and the edge set, respectively. In directed graphs, $(i, j) \in \mathcal{E}$ does not necessarily imply $(j, i) \in \mathcal{E}$. The set of neighbors of agent i (from which this agent receives information) is defined as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

III. PROBLEM STATEMENT

Consider a network of n agents, where the motion of each agent $i \in \mathcal{V}$ is governed by the following rotational kinematic equation:

$$\dot{R}_i = R_i[\omega_i]^\times, \quad (1)$$

where $R_i \in SO(3)$ represents the orientation of the body-attached frame of agent i with respect to the inertial frame, and $\omega_i \in \mathbb{R}^3$ is the angular velocity of agent i measured in the body-attached frame of the same agent. Let $p_i \in \mathbb{R}^3$ denote the position of agent i with respect to the inertial frame. In this work, we assume that the positions of the agents are fixed and do not change with time, *i.e.*, $\dot{p}_i = 0$, for all $i \in \mathcal{V}$. The measurement of the local relative bearing between agent i and agent j is given by

$$b_{ij}^i := R_i^T b_{ij}, \quad (2)$$

where $b_{ij} := \frac{p_j - p_i}{\|p_j - p_i\|}$ and b_{ij}^i are the relative bearing measurements between agent i and agent j expressed in the inertial frame and the body-attached frame of agent i , respectively. The following assumptions are needed in our design:

Assumption 1: The angular velocity of each agent is available for measurement and bounded.

Assumption 2: By assigning a number to each agent, we assume that agents 1 and 2 are the leaders and the other agents are the followers. We also assume that

- The leaders know their pose, and have no neighbors *i.e.*, $\mathcal{N}_k = \{\emptyset\} \forall k = 1, 2$.
- Each agent $i \in \mathcal{V}_f$, where $\mathcal{V}_f := \mathcal{V} \setminus \{1, 2\}$ denotes the set of followers, measures b_{ij}^i and receives $(\hat{R}_j, \hat{p}_j, b_{ji}^j)$ from its neighbors $j \in \mathcal{N}_i$.
- No two agents are collocated, and the set of neighbors of each agent $i \in \mathcal{V}_f$ satisfies $\mathcal{N}_i \subseteq \{1, 2, 3, \dots, i-1\}$ with $|\mathcal{N}_i| \geq 2$, and each agent measures at least two non-collinear bearing vectors.

Assumption 2 defines a cascaded structure of the inter-agent interaction graph topology \mathcal{G} , which is instrumental in establishing our stability results via mathematical induction. Now, we will state the problem that is considered in this work.

Problem: Consider a network of n rotating agents with fixed positions. Suppose Assumptions 1-2 are satisfied. Design a bearing-based distributed pose (position and orientation) estimation scheme endowed with almost global asymptotic stability guarantees.

The above mentioned problem could be practically related to a motion capture system consisting of a network of cameras with fixed positions and time-varying orientations.

IV. MAIN RESULTS

A. Bearing-based Distributed Attitude Estimation on $SO(3)$

For $i \in \mathcal{V}_f$, we propose the following attitude observer on $SO(3)$:

$$\dot{\hat{R}}_i = \hat{R}_i \left[\omega_i - k_R \hat{R}_i^T \sum_{j \in \mathcal{N}_i} k_{ij} (\hat{R}_j b_{ij}^j \times \hat{R}_i b_{ij}^i) \right]^\times, \quad (3)$$

where $k_R, k_{ij} > 0$, $\hat{R}_i \in SO(3)$ is the estimate of R_i , and $\hat{R}_l = R_l$, $l \in \{1, 2\}$. Under Assumption 2, one has $\|p_i - p_j\| \neq 0$ and consequently the bearing measurement b_{ij}^i , for every $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$, is well defined. Defining the attitude estimation error $\tilde{R}_i := R_i \hat{R}_i^T$, the last term of (3) can be rewritten as follows:

$$\sum_{j \in \mathcal{N}_i} k_{ij} (\hat{R}_j b_{ij}^j \times \hat{R}_i b_{ij}^i) = \sum_{j \in \mathcal{N}_i} k_{ij} \left((\tilde{R}_j^T - I_3) b_{ij} \times \tilde{R}_i^T b_{ij} \right) - 2\psi(M_i \tilde{R}_i), \quad (4)$$

where $M_i := \sum_{j \in \mathcal{N}_i} k_{ij} b_{ij} b_{ij}^T$. The last equation was obtained using the fact $x \times y = 2\psi(yx^T)$, $\forall x, y \in \mathbb{R}^3$. It is always possible to choose $k_{ij} > 0$ such that the matrix M_i is positive semi-definite with three distinct eigenvalues. Since $[x + y]^\times = [x]^\times + [y]^\times$ and $[Rx]^\times = R[x]^\times R^T$, $\forall x, y \in \mathbb{R}^3$ and $\tilde{R} \in SO(3)$, and in view of (4), it follows that the time derivative of the attitude estimation error, for every $i \in \mathcal{V}_f$, is given by

$$\dot{\tilde{R}}_i = -2k_R \tilde{R}_i [\psi(M_i \tilde{R}_i)]^\times + k_R \tilde{R}_i \left[\sum_{j \in \mathcal{N}_i} k_{ij} g_{ij}(\tilde{R}_j) \right]^\times, \quad (5)$$

where $g_{ij}(\tilde{R}_j) := (\tilde{R}_j^T - I_3) b_{ij} \times \tilde{R}_i^T b_{ij}$. Note that the term $g_{ij}(\tilde{R}_j)$ is bounded and vanishes when $\tilde{R}_j = I_3$. Furthermore, system (5) can be viewed as a cascaded system, where the attitude estimation errors of the neighbors are considered as inputs to the following unforced system:

$$\dot{\tilde{R}}_i = -2k_R \tilde{R}_i [\psi(M_i \tilde{R}_i)]^\times. \quad (6)$$

The following lemma provides the stability properties of the equilibria of system (6).

Lemma 1: Let $k_{ij} > 0$ such that M_i is positive semi-definite with three distinct eigenvalues. Then, the following statements hold for all $i \in \mathcal{V}_f$:

- i) All solutions of (6) converge to the following set of isolated equilibria: $\Upsilon := \{I_3\} \cup \{\tilde{R}_i = \mathcal{R}_\alpha(\pi, v_i) | v_i \in \mathcal{E}(M_i)\}$, where $\mathcal{E}(M_i) \subset \mathbb{S}^2$ is the set of unit eigenvectors of matrix M_i .
- ii) The desired equilibrium $\tilde{R}_i = I_3$ is locally exponentially stable.
- iii) The linearized system of (6), at each undesired equilibrium $\Upsilon/\{I_3\}$, has at least one positive eigenvalue.
- iv) The undesired equilibria $\Upsilon/\{I_3\}$ are unstable and the desired equilibrium $\tilde{R}_i = I_3$ is AGAS.

Proof: Consider the following Lyapunov function candidate:

$$\mathcal{L}_i = \frac{1}{4} \text{tr} \left(M_i (I_3 - \tilde{R}_i) \right), \quad (7)$$

whose time-derivative, in view of (6), is given by

$$\dot{\mathcal{L}}_i = \frac{k_R}{2} \text{tr} \left(M_i \tilde{R}_i \left[\psi(M_i \tilde{R}_i) \right]^\times \right). \quad (8)$$

Using the facts that $\text{tr}(B[x]^\times) = \text{tr}(\mathbb{P}_\alpha(B)[x]^\times)$ and $\text{tr}([x]^\times [y]^\times) = -2x^T y$, for every $x, y \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3 \times 3}$, one has

$$\dot{\mathcal{L}}_i = -k_R \|\psi(M_i \tilde{R}_i)\|^2 \leq 0. \quad (9)$$

Since system (6) is autonomous, by virtue of LaSalle's invariance theorem, the attitude error \tilde{R}_i should converge to the largest invariant set contained in the set characterized by $\dot{\mathcal{L}}_i = 0$, i.e., $\psi(M_i \tilde{R}_i) = 0$. As per [16, Lemma 2], $\psi(M_i \tilde{R}_i) = 0$ implies that $\tilde{R}_i \in \Upsilon$.

Since the matrix M_i is positive semi-definite with three distinct eigenvalues, it follows that the equilibrium points, in the set Υ , are isolated. Moreover, following the same arguments as in [17, Theorem 1], one can show that the desired equilibrium $\tilde{R}_i = I_3$ is locally exponentially stable and the dynamics of the first order approximation of \tilde{R}_i around each undesired equilibrium has at least one positive eigenvalue. Accordingly, the desired equilibrium $\tilde{R}_i = I_3$ of (6) is AGAS. This completes the proof. ■

In the next lemma, we will study the ISS property of the forced attitude error dynamics (5), with respect to its inputs (i.e., \tilde{R}_j with $j \in \mathcal{N}_i$), using the notion of almost global ISS introduced in [18].

Lemma 2: Let M_i be positive semi-definite with three distinct eigenvalues. Then, system (5) is almost globally ISS, for each $i \in \mathcal{V}_f$, with respect to I_3 and inputs \tilde{R}_j .

Proof: Since system (5), subject to the bounded input $\sum_{j \in \mathcal{N}_i} k_{ij} g_{ij}(\tilde{R}_j)$, evolves on the compact manifold $SO(3)$, condition A0, given in [18], is fulfilled. Moreover, according to Lemma 1, conditions A1 and A2, given in [18], are also fulfilled. Now, consider the following real-valued function:

$$|\tilde{R}_i|_I^2 = \frac{1}{4} \text{tr}(I_3 - \tilde{R}_i), \quad (10)$$

whose time-derivative, in view of (5), is given by

$$\frac{d}{dt} |\tilde{R}_i|_I^2 = -k_R \psi(\tilde{R}_i)^T (\psi(M_i \tilde{R}_i) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} k_{ij} g_{ij}(\tilde{R}_j)).$$

The last equation was obtained using the following identities: $\text{tr}(B[x]^\times) = \text{tr}(\mathbb{P}_\alpha(B)[x]^\times)$ and $\text{tr}([x]^\times [y]^\times) = -2x^T y$, for every $x, y \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3 \times 3}$. Moreover, since $\psi(\tilde{R}_i)^T \psi(M_i \tilde{R}_i) = \psi(\tilde{R}_i)^T Q_i \psi(\tilde{R}_i)$, with $Q_i := \sum_{j \in \mathcal{N}_i} k_{ij} ([b_{ij}]^\times)^T [b_{ij}]^\times$, one has

$$\frac{d}{dt} |\tilde{R}_i|_I^2 = -k_R \psi(\tilde{R}_i)^T (Q_i \psi(\tilde{R}_i) - \frac{1}{2} \sum_{j \in \mathcal{N}_i} k_{ij} g_{ij}(\tilde{R}_j)).$$

Using the fact that $\|\psi(\tilde{R}_i)\|^2 = 4(1 - |\tilde{R}_i|_I^2) |\tilde{R}_i|_I^2 \leq 1$ and at least two bearing vectors are noncollinear, one obtains

$$\begin{aligned} \frac{d}{dt} |\tilde{R}_i|_I^2 &\leq -4k_R \lambda^{Q_i} (1 - |\tilde{R}_i|_I^2) |\tilde{R}_i|_I^2 + \frac{k_R}{2} \left\| \sum_{j \in \mathcal{N}_i} k_{ij} g_{ij}(\tilde{R}_j) \right\| \\ &\leq -4k_R \lambda^{Q_i} |\tilde{R}_i|_I^2 + 4k_R \lambda^{Q_i} + \frac{k_R}{2} \sum_{j \in \mathcal{N}_i} k_{ij} \|g_{ij}(\tilde{R}_j)\|, \end{aligned}$$

where λ^{Q_i} is the smallest eigenvalue of Q_i , which is positive definite under the assumption that at least two bearing vectors are noncollinear. Furthermore, using the fact that $2\sqrt{2}|R|_I = \|I - R\|_F$, one verifies that $\|g_{ij}(\tilde{R}_j)\| \leq 2\sqrt{2}|\tilde{R}_j|_I$. Hence, it follows from the above inequality that

$$\frac{d}{dt} |\tilde{R}_i|_I^2 \leq -4k_R \lambda^{Q_i} |\tilde{R}_i|_I^2 + 4k_R \lambda^{Q_i} + \sqrt{2} k_R \sum_{j \in \mathcal{N}_i} k_{ij} |\tilde{R}_j|_I. \quad (11)$$

It is clear that, in view of (11), system (5) satisfies the ultimate boundedness property defined in [18, Proposition 3]. Consequently, as per [18, Proposition 2], one can conclude that system (5) is almost globally ISS with respect to I_3 and inputs \tilde{R}_j . ■

In the remainder of this section, we will prove almost global asymptotic stability of the equilibrium ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3$) of the n -agent network governed by the cascaded dynamics (5). Thanks to the cascaded structure of the interaction graph topology \mathcal{G} , as per Assumption 2, a mathematical induction argument, together with Lemma 1 and Lemma 2, can be used to prove the result.

Stability analysis of a network with one follower

For a network with one follower, according to (5), one has the following attitude dynamics:

$$\dot{\tilde{R}}_3 = -2k_R \tilde{R}_3 [\psi(M_3 \tilde{R}_3)]^\times. \quad (12)$$

where M_3 is positive semi-definite with three distinct eigenvalues. Note that $\sum_{j \in \mathcal{N}_3} k_{3j} g_{3j}(\tilde{R}_j) = 0$ since $\mathcal{N}_3 = \{1, 2\}$ and $\tilde{R}_l = R_l$ (i.e., $\tilde{R}_l = I_3$), $l \in \{1, 2\}$, as per Assumption 2. It follows from Lemma 1 that the equilibrium point $\tilde{R}_3 = I_3$ of system (12) is AGAS.

Stability analysis of a network with two followers

The attitude dynamics of a network with two followers are given by

$$\dot{\tilde{R}}_3 = -2k_R \tilde{R}_3 [\psi(M_3 \tilde{R}_3)]^\times \quad (13)$$

$$\dot{\tilde{R}}_4 = -2k_R \tilde{R}_4 [\psi(M_4 \tilde{R}_4)]^\times + k_R \tilde{R}_4 \left[\sum_{j \in \mathcal{N}_4} k_{4j} g_{4j}(\tilde{R}_j) \right]^\times, \quad (14)$$

where M_3 and M_4 are positive semi-definite with three distinct eigenvalues.

Proposition 1: Under Assumption 2, the equilibrium point ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3$) of system (13)-(14) is AGAS.

Proof: According to Assumption 2, one has either $3 \in \mathcal{N}_4$ or $3 \notin \mathcal{N}_4$. In the case where $3 \notin \mathcal{N}_4$, the \tilde{R}_3 -subsystem and the \tilde{R}_4 -subsystem are independent, and the almost global asymptotic stability of the equilibrium point ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3$) can be directly deduced from Lemma 1. On the other hand, if $3 \in \mathcal{N}_4$, the two subsystems (\tilde{R}_3 -subsystem and \tilde{R}_4 -subsystem) are cascaded, and as such, in view of Lemma 1 and Lemma 2, it follows that the \tilde{R}_4 -subsystem is almost globally ISS with respect to I_3 and input \tilde{R}_3 , and the \tilde{R}_3 -subsystem is AGAS at $\tilde{R}_3 = I_3$. Finally, in view of [19, Theorem 2], it follows that the cascaded system (13)-(14) is AGAS at ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3$).

Stability analysis of a network with $n - 2$ followers

Consider the following attitude dynamics of a network with $n - 2$ followers:

$$\dot{\tilde{R}}_3 = -2k_R \tilde{R}_3 [\psi(M_3 \tilde{R}_3)]^\times \quad (15)$$

$$\dot{\tilde{R}}_4 = -2k_R \tilde{R}_4 [\psi(M_4 \tilde{R}_4)]^\times + k_R \tilde{R}_4 \left[\sum_{j \in \mathcal{N}_4} k_{4j} g_{4j}(\tilde{R}_j) \right]^\times \quad (16)$$

⋮

$$\begin{aligned} \dot{\tilde{R}}_{n-1} &= -2k_R \tilde{R}_{n-1} [\psi(M_{n-1} \tilde{R}_{n-1})]^\times \\ &\quad + k_R \tilde{R}_{n-1} \left[\sum_{j \in \mathcal{N}_{n-1}} k_{(n-1)j} g_{(n-1)j}(\tilde{R}_j) \right]^\times \end{aligned} \quad (17)$$

$$\dot{\tilde{R}}_n = -2k_R \tilde{R}_n [\psi(M_n \tilde{R}_n)]^\times + k_R \tilde{R}_n \left[\sum_{j \in \mathcal{N}_n} k_{nj} g_{nj}(\tilde{R}_j) \right]^\times, \quad (18)$$

where $M_i, i \in \mathcal{V}_f$, is positive semi-definite with three distinct eigenvalues. Now, we can formally state the stability properties of the equilibrium point ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3$) of the n -agent cascaded system (15)-(18).

Theorem 1: Under Assumption 2, the equilibrium point ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3$) of the n -agent cascaded system (15)-(18) is AGAS.

Proof: We will prove the claimed result by induction. First, it follows from Proposition 1 that the cascaded system (15)-(16) is AGAS at ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3$). Second, we assume that the cascaded $(n - 1)$ -agent subsystem (15)-(17) is AGAS at ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_{n-1} = I_3$). Finally, using the proof by induction, with the fact that the \tilde{R}_n -subsystem is almost globally ISS with respect to I_3 and inputs from the cascaded $(n - 1)$ -agent subsystem (15)-(17), one can show that the n -agent cascaded system is AGAS at ($\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3$) according to [19]. This completes the proof. ■

B. Bearing-based Distributed Pose Estimation

Consider the distributed attitude observer (3) together with the following distributed position estimation law:

$$\begin{aligned} \dot{\hat{p}}_i &= -k_R \left[\sum_{j \in \mathcal{N}_i} k_{ij} \left(\hat{R}_j b_{ij}^j \times \hat{R}_i b_{ij}^i \right) \right]^\times \hat{p}_i \\ &\quad - k_p \sum_{j \in \mathcal{N}_i} \hat{R}_i P_{b_{ij}}^i \hat{R}_i^T (\hat{p}_i - \hat{p}_j), \end{aligned} \quad (19)$$

for each $i \in \mathcal{V}_f$, where $k_p, k_R, k_{ij} > 0$, $\hat{p}_i \in \mathbb{R}^3$ is the estimate of p_i , $\hat{R}_i \in SO(3)$ is the estimate of R_i obtained from the distributed attitude observer (3), and $(\hat{R}_l, \hat{p}_l) = (R_l, p_l), l \in \{1, 2\}$. Define the position estimation error as $\tilde{p}_i := p_i - \hat{R}_i \hat{p}_i$. Its time derivative, in view of (5) and (19), is given by

$$\dot{\tilde{p}}_i = -k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} \tilde{p}_i + k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} (p_j - \hat{R}_i \hat{p}_j), \quad (20)$$

with $i \in \mathcal{V}_f$. We have used the fact that $P_{b_{ij}}(p_i - p_j) = 0$ and $P_{b_{ij}}^i = \hat{R}_i^T P_{b_{ij}} \hat{R}_i$ to obtain the last equality. It follows from (20) that

$$\dot{\tilde{p}}_i = -k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} \tilde{p}_i + k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i), \quad (21)$$

where $f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i) := \left((\tilde{R}_j - I_3) - (\tilde{R}_i - I_3) \right) \tilde{R}_j^T (p_j - \tilde{p}_j) + \tilde{p}_j$. It is clear that $f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i) = 0$ for $\tilde{p}_j = 0$ and $\tilde{R}_j = \tilde{R}_i = I_3$. Again, system (21) can be seen as a cascaded system, where the attitude and position estimation errors of the neighbors as well as the attitude estimation error of agent i are considered as inputs to the following unforced system:

$$\dot{\tilde{p}}_i = -k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} \tilde{p}_i. \quad (22)$$

Next, we study the stability of the equilibrium point $\tilde{p}_i = 0$ of system (22) and the ISS property of system (21).

Proposition 2: Consider system (22) under Assumption 2. The equilibrium point $\tilde{p}_i = 0, i \in \mathcal{V}_f$, is globally exponentially stable (GES).

Proof: Under the assumption that at least two bearing vectors are non-collinear (Assumption 2), for every $i \in \mathcal{V}_f$, the matrix $\sum_{j \in \mathcal{N}_i} P_{b_{ij}}$ is positive definite, and hence, the equilibrium $\tilde{p}_i = 0$ of the unforced position error dynamics (22) is GES. ■

Lemma 3: Suppose Assumption 2 is satisfied. Then, for every $i \in \mathcal{V}_f$, system (21) is ISS with respect to its inputs \tilde{p}_j, \tilde{R}_j and \tilde{R}_i .

Proof: Consider the following Lyapunov function candidate:

$$V_i = \frac{1}{2} \tilde{p}_i^T \tilde{p}_i, \quad (23)$$

whose time-derivative, in view of (21), is given by

$$\dot{V}_i = \tilde{p}_i^T \left(-k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} \tilde{p}_i + k_p \sum_{j \in \mathcal{N}_i} P_{b_{ij}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i) \right).$$

Since $\|Ax\| \leq \|A\|_F \|x\|$, for every $A \in \mathbb{R}^{3 \times 3}$ and $x \in \mathbb{R}^3$, and in view of the positive definiteness of the matrix $\sum_{j \in \mathcal{N}_i} P_{b_{ij}}$ (implied from Assumption 2), one has

$$\dot{V}_i \leq -k_p \lambda_i^P \|\tilde{p}_i\|^2 + k_p \bar{P} \sum_{j \in \mathcal{N}_i} \|\tilde{p}_j\| \|f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i)\|, \quad (24)$$

where λ_i^P denotes the smallest eigenvalue of the matrix $\sum_{j \in \mathcal{N}_i} P_{b_{ij}}$ and \bar{P} denotes the upper bound of the projection

matrix norm, *i.e.*, $\|P_{b_{ij}}\|_F \leq \bar{P}$. Applying Young's inequality on the last two terms of (24), leads to

$$\begin{aligned} \dot{V}_i &\leq -k_p \lambda_i^P \|\tilde{p}_i\|^2 + k_p \bar{P} \sum_{j \in \mathcal{N}_i} \left(\xi_i \|\tilde{p}_i\|^2 \right. \\ &\quad \left. + \frac{1}{4\xi_i} \|f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i)\|^2 \right) \\ &\leq -k_p (\lambda_i^P - \bar{P} \xi_i |\mathcal{N}_i|) \|\tilde{p}_i\|^2 + \frac{k_p \bar{P}}{4\xi_i} \sum_{j \in \mathcal{N}_i} \|f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_i)\|^2. \end{aligned}$$

Choosing $0 < \xi_i < \frac{\lambda_i^P}{\bar{P}|\mathcal{N}_i|}$, for every $i \in \mathcal{V}_f$, and using the fact that $8\|R\|_I^2 = \|I - R\|_F^2$ and $|R|_I \leq 1$, for every $R \in SO(3)$, one can show that

$$\dot{V}_i \leq -\alpha_1 (\|\tilde{p}_i\|) + \sum_{j \in \mathcal{N}_i} \left(\alpha_2 (\|\tilde{p}_j\|) + \alpha_3 (|\tilde{R}_j|_I) + \alpha_4 (|\tilde{R}_i|_I) \right),$$

where $\alpha_k(\cdot) \in \mathcal{K}_\infty$ with $k \in \{1, 2, 3, 4\}$. Hence, system (21) is ISS with respect to inputs \tilde{p}_j , \tilde{R}_j and \tilde{R}_i . ■

Thanks again to the cascaded nature of the interaction graph topology \mathcal{G} which allows the use of a similar induction proof as in the previous section, together with Proposition 2 and Lemma 3 as well as a result from Theorem 1, to establish the stability property of the equilibrium point $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3, \tilde{p}_3 = 0, \tilde{p}_4 = 0, \dots, \tilde{p}_n = 0)$ of the n -agent network governed by the cascaded dynamics (5) and (21).

Stability analysis of a network with one follower

Consider the attitude error dynamics (12) cascaded with the following position error dynamics:

$$\dot{\tilde{p}}_3 = -k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} \tilde{p}_3 + k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_3), \quad (25)$$

where $f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_3) = -(\tilde{R}_3 - I_3)p_j$ since $(\hat{R}_l, \hat{p}_l) = (R_l, p_l)$, $l \in \{1, 2\}$, as per Assumption 2.

Proposition 3: Suppose Assumption 2 is satisfied. Then, the cascaded system (12) and (25) is AGAS at $(\tilde{R}_3 = I_3, \tilde{p}_3 = 0)$.

Proof: From Lemma 1 and Proposition 2, one can conclude that the \tilde{R}_3 -subsystem is AGAS at $\tilde{R}_3 = I_3$ and the \tilde{p}_3 -subsystem, with $\tilde{R}_3 = I_3$, is GES at $\tilde{p}_3 = 0$. Thus, as per Lemma 3, the cascaded system (12) and (25) is AGAS at $(\tilde{R}_3 = I_3, \tilde{p}_3 = 0)$. ■

Stability analysis of a network with two followers

Consider the attitude error dynamics (13)-(14) cascaded with the following position error dynamics:

$$\dot{\tilde{p}}_3 = -k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} \tilde{p}_3 + k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_3) \quad (26)$$

$$\dot{\tilde{p}}_4 = -k_p \sum_{j \in \mathcal{N}_4} P_{b_{4j}} \tilde{p}_4 + k_p \sum_{j \in \mathcal{N}_4} P_{b_{4j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_4). \quad (27)$$

Proposition 4: Suppose Assumption 2 is satisfied. Then, the cascaded system (13)-(14) and (26)-(27) is AGAS at $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \tilde{p}_3 = 0, \tilde{p}_4 = 0)$.

Proof: It follows from Propositions 1 and 3 that the equilibrium point $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3)$ of (13)-(14) is AGAS,

and the equilibrium point $(\tilde{R}_3 = I_3, \tilde{p}_3 = 0)$ of (13) and (26) is AGAS. Therefore, using the fact that the \tilde{p}_4 -subsystem is ISS with respect to \tilde{p}_3 , \tilde{R}_3 , and \tilde{R}_4 , one can establish the claim in Proposition 4. ■

Stability analysis of a network with $n - 2$ followers

For a network with $n - 2$ followers, we consider the attitude error dynamics (15)-(18) cascaded with the following position error dynamics:

$$\dot{\tilde{p}}_3 = -k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} \tilde{p}_3 + k_p \sum_{j \in \mathcal{N}_3} P_{b_{3j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_3) \quad (28)$$

$$\dot{\tilde{p}}_4 = -k_p \sum_{j \in \mathcal{N}_4} P_{b_{4j}} \tilde{p}_4 + k_p \sum_{j \in \mathcal{N}_4} P_{b_{4j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_4) \quad (29)$$

⋮

$$\begin{aligned} \dot{\tilde{p}}_{n-1} &= -k_p \sum_{j \in \mathcal{N}_{n-1}} P_{b_{(n-1)j}} \tilde{p}_{n-1} \\ &\quad + k_p \sum_{j \in \mathcal{N}_{n-1}} P_{b_{(n-1)j}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_{n-1}) \end{aligned} \quad (30)$$

$$\dot{\tilde{p}}_n = -k_p \sum_{j \in \mathcal{N}_n} P_{b_{nj}} \tilde{p}_n + k_p \sum_{j \in \mathcal{N}_n} P_{b_{nj}} f_j(\tilde{p}_j, \tilde{R}_j, \tilde{R}_n). \quad (31)$$

In the following theorem, we establish the stability properties of the overall rotational and translational cascaded estimation scheme.

Theorem 2: Considering the cascaded attitude and position estimation schemes given by (3) and (19), respectively, where Assumption 2 is satisfied. Suppose that the result in Theorem 1 holds. Then, the equilibrium point $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3, \tilde{p}_3 = 0, \tilde{p}_4 = 0, \dots, \tilde{p}_n = 0)$ of the overall cascaded system (15)-(18) and (28)-(31) is AGAS.

Proof: Similar to the proof of Theorem 1, we will prove the result of this theorem by induction. First, from Proposition 3, it is clear that the equilibrium point $(\tilde{R}_3 = I_3, \tilde{p}_3 = 0)$ of (15) and (28) is AGAS. Second, we assume that the cascaded $(n-1)$ -agent subsystem (15)-(17) and (28)-(30) is AGAS at $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_{n-1} = I_3, \tilde{p}_3 = 0, \tilde{p}_4 = 0, \dots, \tilde{p}_{n-1} = 0)$. Finally, using the induction arguments, in view of the result from Theorem 1 and the fact that the \tilde{p}_n -subsystem is ISS with respect to \tilde{R}_n as well as inputs from the cascaded $(n-1)$ -agent subsystem (15)-(17) and (28)-(30), one can show that the equilibrium point $(\tilde{R}_3 = I_3, \tilde{R}_4 = I_3, \dots, \tilde{R}_n = I_3, \tilde{p}_3 = 0, \tilde{p}_4 = 0, \dots, \tilde{p}_n = 0)$ of the overall cascaded system (15)-(18) and (28)-(31) is AGAS. This completes the proof. ■

V. SIMULATION

In this section, we present some numerical simulations to illustrate the performance of our proposed distributed pose estimation scheme. We consider an eight-agent system in a 3-dimensional space with the following positions: $p_1 = [0 \ 0 \ 0]^T$, $p_2 = [2 \ 0 \ 0]^T$, $p_3 = [2 \ 2 \ 0]^T$, $p_4 = [0 \ 2 \ 0]^T$, $p_5 = [0 \ 0 \ 2]^T$, $p_6 = [2 \ 0 \ 2]^T$, $p_7 = [2 \ 2 \ 2]^T$ and $p_8 = [0 \ 2 \ 2]^T$. The rotational subsystem is driven by the following angular velocities: $\omega_1 = [1 \ -2 \ 1]^T$, $\omega_2(t) = [-\cos 3t \ 1 \ \sin 2t]^T$, $\omega_3(t) = [-\cos t \ 1 \ \sin 2t]^T$, $\omega_4(t) = [-\cos 2t \ 1 \ \sin 5t]^T$, $\omega_5(t) = [-\cos 5t \ 1 \ \sin 9t]^T$, $\omega_6(t) = [-\cos 2t \ \sin 9t \ 1]^T$,

$\omega_7(t) = [-\cos 4t \ 1 \ 2]^T$ and $\omega_8(t) = [-2 \ 1 \ \sin 9t]^T$. The initial rotations of all agents are chosen to be the identity. We use a directed graph (see Assumption 2) to model the interaction graph of the eight-agent system as it is shown in Figure 1. Accordingly, the neighbors sets of the agents are given as $\mathcal{N}_1 = \mathcal{N}_2 = \{\emptyset\}$, $\mathcal{N}_3 = \{1, 2\}$, $\mathcal{N}_4 = \{2, 3\}$, $\mathcal{N}_5 = \{1, 4\}$, $\mathcal{N}_6 = \{2, 4, 5\}$, $\mathcal{N}_7 = \{3, 4, 6\}$ and $\mathcal{N}_8 = \{1, 7\}$. The initial conditions of our proposed estimation scheme are chosen as: $\hat{p}_3(0) = [-2 \ 0 \ -1]^T$, $\hat{p}_4(0) = [-1 \ 2 \ 2]^T$, $\hat{p}_5(0) = [-2 \ 2 \ 4]^T$, $\hat{p}_6(0) = [0 \ 0 \ 0]^T$, $\hat{p}_7(0) = [-4 \ 0 \ 1]^T$, $\hat{p}_8(0) = [-3 \ \frac{1}{2} \ 2]^T$, $\hat{R}_3(0) = \mathcal{R}_\alpha(0.1\pi, v)$, $\hat{R}_4(0) = \mathcal{R}_\alpha(0.2\pi, v)$, $\hat{R}_5(0) = \mathcal{R}_\alpha(0.3\pi, v)$, $\hat{R}_6(0) = \mathcal{R}_\alpha(0.9\pi, v)$, $\hat{R}_7(0) = \mathcal{R}_\alpha(0.4\pi, v)$ and $\hat{R}_8(0) = \mathcal{R}_\alpha(0.5\pi, v)$ with $v = [1 \ 0 \ 0]^T$. The gain parameters are taken as follows: $k_p = 1$, $k_R = 1$ and $k_{ij} = 1$ for every $(i, j) \in \mathcal{E}$. The time evolution of the average attitude and position estimation error norms are provided in Figure 2(a) and Figure 2(b), respectively.

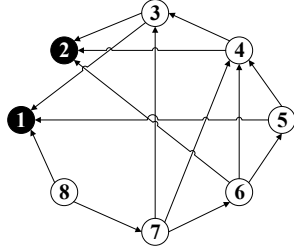


Fig. 1. The interaction graph (the black circles represent the leaders).

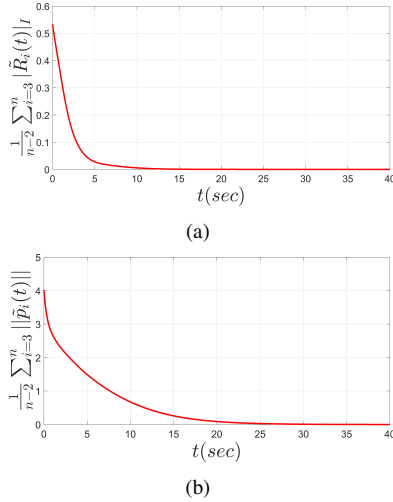


Fig. 2. Time evolution of the average estimation error norm for: (a) attitude (b) and position.

VI. CONCLUSIONS

In this work, we proposed an almost globally asymptotically stable bearing-based distributed pose estimation scheme for multi-agent networks, with a directed interaction graph topology, with two leaders that have access to their respective poses with respect to a global reference frame as per Assumption 2. The individual angular velocities and local inter-agent bearing measurements are assumed to be available.

The proposed scheme consists of a cascade of an almost globally asymptotically stable distributed attitude observer and an ISS distributed position observer. It is worth pointing out that the distributed attitude observer is an interesting contribution on its own right as it is stand-alone (*i.e.*, does not depend on the position estimation) and could be used in other applications that involve rotating multi-agent systems. An interesting extension of this work, would be the consideration of the case where the agents are allowed to have simultaneous translational and rotational motion.

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