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Input-to-State Stable hybrid momentum observer for mechanical systems

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Abstract— **This paper examines the dynamic properties of a hybrid momentum observer for mechanical systems, extending the previously-reported results. The observer estimates the momentum vector from measurements of the configuration vector and is shown to be input-to-state stable with respect to external perturbations. In the absence of external perturbation the observer is shown to be globally exponentially stable, converging at a user-controlled rate. The observer is constructed from a port-Hamiltonian representation of mechanical systems and exhibits a passivity property with respect to an input-output port that can be utilised for subsequent control design. The theoretical results are demonstrated via numerical simulation on a 2-link vertical manipulator.**

I. Introduction

Energy-based methods for control have proved effective for developing solutions to a wide variety of multi-domain physical systems [1]. The approach considers the underlying physical structure of the system under study to derive control laws that have a large or global region of attraction. Mechanical systems exhibit rich structure that has been successfully exploited for a wide variety of control tasks such as stabilisation, tracking and path following [2], [3], [4].

These approaches, however, typically assume complete knowledge of the system's state vector for implementation. For mechanical systems, this corresponds to complete knowledge of both the configuration and momentum of the system under control. In practice it is often reasonable to assume the existence of high-fidelity position measurements, but direct measurement of the velocity or momentum is typically more difficult to obtain. To obtain this information observers are implemented that combine the configuration measurements with the system model to estimate the momentum of the system under control.

Several authors have considered observers to estimate the momentum or velocity of nonlinear mechanical systems using configuration measurements. Several smooth solutions to the observer problem have been proposed using the Immersion & Invariance (I&I) technique [5]. The technique was first applied to nonholonomic systems in [6] and then extended to a class of mechanical systems that are 'partially linearisable via change of coordinates' (PLvCC) in [7], [8]. This class was extended to general mechanical systems in the works [9], [10], [11] by considering a coordinate transformation that results in the kinetic energy being described independent of the configuration. The alternate approach of using hybrid observers to detect velocities from configuration measurements were considered in [12], [13]. In both cases, the hybrid dynamics were introduced to resolve the topology of the considered rotation spaces.

In this work we consider the hybrid momentum observer that was previously reported in [14]. The analysis is extended to include an unknown perturbation acting on the system under observation. It is shown that the observation error is input-to-state stable (ISS) with respect to the unknown perturbation. It is additionally shown that, in the absence of an external perturbation, the observation error is globally exponentially stable. In contrast with observers constructed with I&I, the proposed observer has significantly lower state dimension and is computationally simpler.

Notation. Function arguments are declared upon definition and are omitted for subsequent use. $0_{n\times m}$ denotes a $n\times m$ matrix where each entry is equal to zero and I_n is a $n\times n$ identity matrix. For a map $H:\mathbb{R}^n\to\mathbb{R}$ we denote the transposed gradient as $\nabla H := \left(\frac{\partial H}{\partial x}\right)^\top$. For a real matrix $A\in\mathbb{R}^{n\times n}$, we denote the symmetric component as $\mathrm{symm}(A)=\frac{1}{2}(A+A^\top)$. For a discrete event occurring at time T and a time-varying parameter $\phi(t)$, $\phi^- = \lim_{t\to T^-}\phi(t)$ whereas $\phi^+ = \lim_{t\to T^+}\phi(t)$. $\mathbb{R}_+$ indicates positive real numbers whereas $\mathbb{Z}_+$ indicates positive integers.

II. Background and problem formulation

In this section the considered system model is introduced and some relevant properties of hybrid systems are revised.

A. System model

In this note we consider the class of mechanical systems described in the port-Hamiltonian framework

$$\begin{aligned}\begin{bmatrix}\dot q\\ \dot p_0\end{bmatrix} &= \begin{bmatrix}0_{n\times n} & I_n\\ -I_n & -D_0(q)\end{bmatrix}\begin{bmatrix}\nabla_q H_0\\ \nabla_{p_0} H_0\end{bmatrix} + \begin{bmatrix}0_{n\times n}\\ G_0(q)\end{bmatrix}u - \begin{bmatrix}0_{n\times 1}\\ \delta_{p_0}(t)\end{bmatrix}\\ y &= G_0^\top(q)\nabla_{p_0}H_0\\ H_0(q,p_0) &= \frac{1}{2}p_0^\top M^{-1}(q)p_0 + V(q),\end{aligned} \tag{1}$$

where $q\in\mathbb{R}^n$ is the configuration, $p_0\in\mathbb{R}^n$ is the momentum, $V(q)\in\mathbb{R}^+$ is the potential energy, $M(q)=M^\top(q)\in\mathbb{R}^{n\times n}$ is the uniformly positive definite mass matrix satisfying

$$\underline{m}I_n \le M(q) = M^\top(q) \le \overline{m}I_n, \tag{2}$$

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for some $\overline{m} > \underline{m} > 0$ and all $q \in \mathbb{R}^n$, $H_0(q, p_0) \in \mathbb{R}^+$ is the Hamiltonian, $G_0(q) \in \mathbb{R}^{n\times n}$ is the full rank input mapping matrix, $D_0(q) = D_0^\top(q) \in \mathbb{R}^{n\times n}$ is the open-loop damping matrix which is positive semi-definite and $u, y \in \mathbb{R}^n$ are the input and natural passive output, respectively. We additionally assume that the mass matrix $M(q)$ is differentiable for all q and has continuous derivatives. The term $\delta_{p_0}(t) \in \mathbb{R}^n$ is an unknown time-varying force disturbance with upper bound given by

$$\gamma = \sup_t \|\delta_{p_0}(t)\|, \tag{3}$$

for some $\gamma \geq 0$. This force disturbance can represent the effects of external forces, input signal quantisation or modeling errors. Note that the bound γ does not need to be known for the observer implementation.

As design of the input term in (1) is not considered in this work, we cannot guarantee the existence of the forward solution for all time. For example, the input or disturbance could be such that the system exhibits a finite escape time. With this in mind we make the following assumption on the system solution.

Assumption 1: There exists a solution $(q(t), p_0(t))$ to the system (1) which is defined for all time on the domain $\mathcal{D}_L = [0, T_L)$, where $T_L \leq \infty$.

B. Hybrid systems

A hybrid system with state $x \in \mathbb{R}^n$ and input (disturbance) term $w \in \mathbb{R}^m$ is described by

$$\begin{aligned} \dot{x} &= f(x, w) \quad \text{for } (x, w) \in C \\ x^+ &= g(x, w) \quad \text{for } (x, w) \in D, \end{aligned} \tag{4}$$

where $C, D \subset \mathbb{R}^n \times \mathbb{R}^m$ that describe the domains of continuous and discrete dynamics, respectively. Solutions of hybrid systems, $x(t, j)$, are defined on hybrid time domains $E \subset \mathbb{R}_+ \times \mathbb{Z}_+$ where $E = \cup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some, possibly infinite, sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$ [15, Definition 2.3].

Solutions to hybrid systems combine the behaviours of continuous and discrete-time systems, with the possibility that the solutions are purely continuous or discrete. In this work we consider the design of a hybrid momentum observer for the system (1) which has a solution on the domain $\mathcal{D}_L$. As such, we are interested in ensuring that the proposed hybrid observer produces a solution on the same time domain. Given a hybrid time domain E we define

$$\sup_t E = \sup \{t \in \mathbb{R}_+ : \exists j \in \mathbb{N} \text{ such that } (t, j) \in E\}. \tag{5}$$

Other properties of solutions such as Zeno and eventually discrete behaviours can be found in [15, Chapter 2].

C. Passivity

Passivity for hybrid systems follows analogously from the standard continuous time definition. Considering the hybrid system (4), we introduce a storage function $S : \mathbb{R}^n \to \mathbb{R}$ and output $y(x) \in \mathbb{R}^m$.

Definition 1: [16, Definition 9.4] The hybrid system (4) with storage function S and output y is flow passive if

$$\begin{aligned} \dot{S}(x) &\leq -\rho(x) + y^\top w \;\; \text{for } (x, w) \in C \\ S^+(x) &= S^-(x) \qquad\quad\;\; \text{for } (x, w) \in D, \end{aligned} \tag{6}$$

where $\rho(x)$ is a positive semi-definite function of x. It is flow strictly passive if $\rho(x)$ is positive definite.

D. Problem statement and contributions

In this work we consider the momentum observer reported in [14] which estimates the momentum vector of mechanical systems described in the form (1) using measurements of the configuration vector $q(t)$. Here we significantly extend the previously-reported analysis to establish the following:

- An unknown disturbance term $\delta_{p_0}(t)$ is added to the system description and it is shown that the observer estimation error is ISS with respect to the disturbance.
- A technical assumption related to the underlying system dynamics has been removed, verifying that the origin of the observer error dynamics is globally exponentially stable for all mechanical systems of the form (1).

III. Momentum observer

In this section we review the hybrid momentum observer for the system (1), which was previously reported in [14]. The analysis is significantly extended when compared to that work, considering the effects of unmodeled disturbances and removing the previously-used the technical assumption.

A. Momentum transformation

The momentum observer for (1) is defined in a non-canonical set of coordinates with the property that, under the transformation, the kinetic energy can be described independently of q. As $M(q)$ is uniformly positive definite, there exists a unique uniformly positive definite matrix square root $T(q) \in \mathbb{R}^{n\times n}$ satisfying

$$M^{-1}(q) = T^2(q) = T(q)T(q), \tag{7}$$

where

$$\overline{m}^{-\frac{1}{2}} I_n \leq T(q) = T^\top(q) \leq \underline{m}^{-\frac{1}{2}} I_n. \tag{8}$$

A momentum transformation is defined using the matrix $T(q)$ as

$$p := T(q)p_0, \tag{9}$$

which normalises the kinetic energy's dependence on the configuration q.

As the mapping $f : T \to \underbrace{TT}_{M^{-1}}$ is differentiable and invertible, we note that the inverse map $f^{-1} : TT \to T$ is differentiable for all positive definite T by the inverse function theorem [17, Theorem C.34]. Consequently, $T(q)$ and $T^{-1}(q)$ are differentiable due to the differentiability of $M(q)$. As $M(q)$ has continuous derivatives, $T(q)$ must have continuous derivatives also. Noting this, the dynamics (1) can be written in the coordinates (q,p) as

$$\begin{aligned}\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} &= \begin{bmatrix}0_{n\times n} & T(q)\\ -T(q) & S(q,p)-D(q)\end{bmatrix}\begin{bmatrix}\nabla_q V\\ p\end{bmatrix}+\begin{bmatrix}0_{n\times n}\\ G(q)\end{bmatrix}u\\ &\quad -\begin{bmatrix}0_{n\times n} & T(q)\end{bmatrix}^\top \delta_{p_0}\\ H(q,p) &= \frac{1}{2}p^\top p + V(q),\end{aligned} \tag{10}$$

where

$$\begin{aligned}D(q,p) &= T(q)D_0(q)T(q)\\ S(q,p) &= T(q)\left[\frac{\partial^\top}{\partial q}(T^{-1}(q)p)-\frac{\partial}{\partial q}(T^{-1}(q)p)\right]T(q)\\ G(q) &= T(q)G_0(q).\end{aligned} \tag{11}$$

The matrix $S(q,p)$ is linear in the second argument p. Using this property we implicitly define a matrix $\bar{S}(q,p)\in\mathbb{R}^{n\times n}$ such that for any two vectors $a,b\in\mathbb{R}^n$

$$\bar{S}(q,a)b = S(q,b)a. \tag{12}$$

While $S(q,p)$ is skew-symmetric, $\bar{S}(q,p)$ does not have the same property. The matrix $\bar{S}(q,p)$ is linear in its second argument, implying that

$$\bar{S}(q,a+b) = \bar{S}(q,a)+\bar{S}(q,b). \tag{13}$$

We additionally note that as the entries of the matrix $S(q,p)$ are continuous with respect to q,p the entries of $\bar{S}(q,p)$ are continuous also.

Remark 1: From Assumption 1 and the bounds on $T(q)$ in (8), the solution $(q(t),p(t))$ for (10) exists on the domain $\mathcal{D}_L$.

Remark 2: Implementation of the proposed observer requires the terms $T(q)$, $S(q,p)$ and $\bar{S}(q,p)$, which are difficult to compute in closed-form. The term $T(q)$ can be evaluated numerically point-wise as the matrix square root of $M^{-1}(q)$. The partial derivatives $\frac{\partial}{\partial q_i}\left(T^{-1}(q)\right)$, which are required to evaluate $S(q,p)$, can be evaluated point-wise as the solution to the Lyapunov equation

$$\frac{\partial}{\partial q_i}(M(q)) = \frac{\partial}{\partial q_i}\left(T^{-1}(q)\right)T^{-1}(q)+T^{-1}(q)\frac{\partial}{\partial q_i}\left(T^{-1}(q)\right). \tag{14}$$

Using the partial derivatives we can define the matrix $A(q,p)$ as

$$A(q,p) = \begin{bmatrix}\frac{\partial}{\partial q_1}\left(T^{-1}(q)\right)p & \cdots & \frac{\partial}{\partial q_n}\left(T^{-1}(q)\right)p\end{bmatrix} \tag{15}$$

which can be used to construct $S(q,p)$ as

$$S(q,p) = T(q)\left[A^\top(q,p)-A(q,p)\right]T(q). \tag{16}$$

Due to the linearity of $S(q,p)$ it can be represented as

$$S(q,p) = \sum_{i=1}^{n} S(q,e_i)p_i. \tag{17}$$

Using this representation, the matrix $\bar{S}(q,p)$ can be computed as

$$\bar{S}(q,p) = \begin{bmatrix}S(q,e_1)p & S(q,e_2)p & \cdots & S(q,e_n)p\end{bmatrix}. \tag{18}$$

B. Momentum observer

In this section we consider the hybrid momentum observer previously reported in [14]. The momentum observer assumes configuration measurements are available for control purposes and generates an estimate for the momentum vector p, denoted by $\hat{p}(t)\in\mathbb{R}^n$. The observer additionally utilises a scalar piece-wise constant state $\phi(t)$ which acts to regulate the rate of convergence of the observer.

The observer dynamics are given by the equations

$$\begin{aligned}[\dot{x}_p,\dot{\phi}] &= [f_{x_p}(q,\hat{p},\phi),0], \qquad (x_p,\phi,q)\in\mathcal{C}\\ [x_p^+,\phi^+] &= [x_p-\kappa q,\phi+\kappa], \qquad (x_p,\phi,q)\in\mathcal{C}^c\\ \hat{p}(x_p,\phi,q) &= x_p+\phi q,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\mathcal{C} &:= \left\{(x_p,\phi,q)\mid \phi T(q)-\operatorname{symm}\left(\bar{S}(q,\hat{p})\right)\geq \kappa I_n\right\}\\ f_{x_p}(q,\hat{p},\phi,u,u_o) &:= \left[S(q,\hat{p})-D(q)-\phi T(q)\right]\hat{p}\\ &\quad - T(q)\nabla_q V(q)+G(q)\left[u+u_o\right],\end{aligned} \tag{20}$$

$\hat{p}\in\mathbb{R}^n$ is an estimate of the momentum vector p, $x_p\in\mathbb{R}^n$ is a piece-wise continuous observer state and $\phi\in\mathbb{R}_+$ is a piece-wise constant observer state and $\mathcal{C}^c$ is the set complement of $\mathcal{C}$. The tuning parameter $\kappa>0$ is chosen to set the ISS bounds and the rate of convergence. The input u is the same input used for the plant (10) whereas $u_o\in\mathbb{R}^n$ is an additional input that can be used for subsequent control design. The solution to the observer (19) is defined on a hybrid time domain denoted by $\mathcal{E}_o$. It will be shown in subsequent analysis that $\sup_t \mathcal{E}_o = T_L$.

The stability properties of the momentum observer (19) are now considered. It is shown that the error dynamics formed by taking the difference of the momentum estimate and the true momentum forms a set of passive hybrid dynamics where the input signals u_o,δ_{p_0} form passive inputs. It is then shown that the estimation error is ISS and, in the absence of any disturbance, converges to the origin at an exponential rate. The proof is inspired by [18, Theorem 2].

Proposition 1: Consider the hybrid momentum observer (19) for estimating the momentum of the mechanical system (10). The resulting estimation error system has the following properties:

1) The momentum estimation error $\tilde{p}:=\hat{p}-p$ has the dynamics

$$\begin{aligned}\dot{\tilde{p}} &= F_o(q,p,\tilde{p},\phi)\nabla_{\tilde{p}}H_o+G(q)u_o+T(q)\delta_{p_0},\\ &\quad (x_p,\phi,q)\in\mathcal{C}\\ \tilde{p}^+ &= \tilde{p}^-, (x_p,\phi,q)\in\mathcal{C}^c\\ F_o(q,p,\hat{p},\phi) &= S(q,p)+\bar{S}(q,\hat{p})-D(q)-\phi T(q)\\ y_o &= G^\top(q)\tilde{p}\\ H_o(\tilde{p}) &= \frac{1}{2}\|\tilde{p}\|^2\end{aligned} \tag{21}$$

and both $\tilde{p}(t),H_o(t)$ are continuous on $\mathcal{E}_o$.

2) The observer error dynamics (21) are flow strictly passive with input-output pairs (u_o,y_o), (δ_{p_0},y_δ) and storage function $H_o(\tilde{p})$, satisfying

$$\begin{aligned}\dot{H}_o &\leq -2\kappa H_o+\underbrace{\tilde{p}^\top G(q)}_{:=y_o^\top}u_o+\underbrace{\tilde{p}^\top T(q)}_{:=y_\delta^\top}\delta_{p_0}, (x_p,\phi,q)\in\mathcal{C}\\ H_o^+ &= H_o^-, (x_p,\phi,q)\in\mathcal{C}^c\end{aligned} \tag{22}$$

3) If $u_o = 0_{n\times 1}$, the momentum estimate $\hat{p}(t)$ exists for all $t \in \mathcal{D}_L$ and the observer estimation error $\tilde{p}(t)$ is ISS with respect to an unknown input δ_{p_0}, satisfying the bound

$$\|\tilde{p}(t)\| \leq \sqrt{2H_o(0)}e^{-\frac{1}{2}\kappa t} + \underline{m}^{-\frac{1}{2}}\kappa^{-1}\gamma \tag{23}$$

for all $t \in \mathcal{D}_L$. If $\delta_{p_0} = 0_{n\times 1}$, $\gamma = 0$ and $\tilde{p} = 0_{n\times 1}$ is a globally exponentially stable equilibrium.

Proof: Claim 1 is similar to [14, Proposition 4] with the additional disturbance δ_{p_0}. Consider time derivative of $\tilde{p}$ and substitute the expressions for $\dot{p}$, $\dot{q}$ and $\dot{x}_p$, from (10) and (19). The resulting dynamics can be simplified as

$$\begin{aligned}\dot{\tilde{p}} =& \dot{\hat{p}} - \dot{p} \\ =& \dot{x}_p + \phi\dot{q} - \dot{p} \\ =& -\left[D(q) + \phi T(q) - \bar{S}(q,\hat{p}) - S(q,p)\right]\tilde{p} \\ &+ G(q)u_o + T(q)\delta_{p_0}(t)\end{aligned} \tag{24}$$

which agrees with (21).

To verify continuity of $\tilde{p}(t), H_o(t)$, notice that on the set $\mathcal{C}$ the dynamics are smooth so the claim holds on the same set. On the set $\mathcal{C}^c$ the momentum estimate $\hat{p}$ satisfies

$$\hat{p}^+ = x_p^+ + \phi^+ q = x_p^- - \kappa q + \left(\phi^- + \kappa\right)q = \hat{p}^-. \tag{25}$$

It then follows that

$$H_o^+ = \frac{1}{2}\left\|\hat{p}^+ - p\right\|^2 = \frac{1}{2}\left\|\hat{p}^- - p\right\|^2 = H_o^-, \tag{26}$$

verifying the claim.

Claim 2 can be verified by considering the flow dynamics (21) and noting that on the set $\mathcal{C}$ the function H_o satisfies

$$\begin{aligned}\dot{H}_o &= \frac{1}{2}\tilde{p}^\top\left[F_o(\cdot) + F_o^\top(\cdot)\right]\tilde{p} + \tilde{p}^\top G(q)u_o + \tilde{p}^\top T(q)\delta_{p_0} \\ &\leq -\kappa\|\tilde{p}\|^2 + y_o^\top u_o + y_\delta^\top\delta_{p_0}.\end{aligned} \tag{27}$$

Continuity of H_o through any jump event, verified in claim 1, completes the claim.

Now we turn our attention to Claim 3 where we must first verify that $\sup_t \mathcal{E}_o = T_L$ to ensure that the solution of the momentum observer exists for all $t \in \mathcal{D}_L$. This is done by ensuring that only finitely many jump events can happen on any closed subset of the time interval $\mathcal{D}_L$, ruling out any Zeno or eventually discrete behaviours on the interior of the time domain of interest. From the inequality (27) with $u_o = 0$ we have that on the set $\mathcal{C}$

$$\begin{aligned}\dot{H}_o &\leq -\kappa\|\tilde{p}\|^2 + \tilde{p}^\top T(q)\delta_{p_0} \\ &\leq -\kappa\|\tilde{p}\|^2 + \frac{c}{2}\tilde{p}^\top T(q)T(q)\tilde{p} + \frac{1}{2c}\delta_{p_0}^\top\delta_{p_0},\end{aligned} \tag{28}$$

where $c > 0$ is an arbitrary positive constant resulting from application of Young's inequality. Recalling the definition (7) and the bound (2) we take $c = \underline{m}\kappa$, resulting in

$$\dot{H}_o \leq -\kappa H_o + \frac{1}{2\underline{m}\kappa}\|\delta_{p_0}(t)\|^2. \tag{29}$$

By the comparison Lemma [19, Lemma 3.4] and the solution to a LTI system [19, Chapter 4.9], $H_o(t)$ satisfies

$$\begin{aligned}H_o(t) &\leq H_o(t_j)e^{-\kappa(t-t_j)} + \int_{t_j}^t e^{-\kappa(t-\tau)}\frac{1}{2\underline{m}\kappa}\|\delta_{p_0}(\tau)\|^2\ d\tau \\ &\leq H_o(t_j)e^{-\kappa(t-t_j)} + \frac{1}{2\underline{m}\kappa}\gamma^2 e^{-\kappa t}\int_{t_j}^t e^{\kappa\tau}\ d\tau\end{aligned} \tag{30}$$

where j is the jump index, $t \in [t_j, t_{j+1}]$ is a subset of $\mathcal{E}_o$ with constant jump index and γ is the disturbance bound from (3). A global bound for the full time domain is now established via an induction argument. Suppose that for $t \in [t_{j-1}, t_j]$, $H_o(t)$ satisfies the bound

$$H_o(t) \leq H_o(0)e^{-\kappa t} + \frac{1}{2\underline{m}\kappa}\gamma^2 e^{-\kappa t}\int_0^t e^{\kappa\tau}\ d\tau. \tag{31}$$

It follows that at time $t = t_j$, H_o must satisfy the bound

$$H_o(t_j) \leq H_o(0)e^{-\kappa t_j} + \frac{1}{2\underline{m}\kappa}\gamma^2 e^{-\kappa t_j}\int_0^{t_j} e^{\kappa\tau}\ d\tau. \tag{32}$$

Substituting this value into (30) recovers the bound (31), but for the time interval $t \in [t_j, t_{j+1}]$. As this bound holds interval $t \in [0, t_1]$ directly from (30), it follows by induction that the bound (31) holds on the full hybrid time domain $\mathcal{E}_o$. By evaluating the integral in (31), it follows that any solution to the observer must satisfy

$$H_o(t) \leq H_o(0)e^{-\kappa t} + \frac{1}{2\underline{m}\kappa^2}\gamma^2 \tag{33}$$

on $\mathcal{E}_o$. Substitution of the expression for H_o in (21) and applying the Minkowski inequality recovers (23) on $\mathcal{E}_o$.

We now verify that $\sup_t \mathcal{E}_o = T_L$ by excluding the possibility of Zeno or eventually discrete behaviours of the observer on the interior of $\mathcal{D}_L$. Consider an finite time interval $T_{a,b} = [t_a, t_b]$ with $0 \leq t_a < t_b < T_L$ and note from Remark 1 that $(q(t), p(t))$ exists on this interval. From the definition of the flow domain in (20), the system is in the flow domain $\mathcal{C}$ provided that ϕ satisfies

$$\begin{aligned}\phi T(q) &\geq \kappa I_n + \mathrm{symm}\left(\bar{S}(q,\hat{p})\right) \\ &\geq \kappa I_n + \mathrm{symm}\left(\bar{S}(q,p)\right) + \mathrm{symm}\left(\bar{S}(q,\tilde{p})\right),\end{aligned} \tag{34}$$

where the linearity of $\bar{S}(q,p)$ in its second argument has been used to evaluate the second line. On the closed time interval $T_{a,b}$ the solutions $q(t), p(t)$ exist and are bounded. As the elements of $\bar{S}(q,p)$ are continuous functions of q, p and $T_{a,b}$ is closed, each element of $\mathrm{symm}\left(\bar{S}(q,\hat{p})\right)$ is bounded on $T_{a,b}$ by the extreme value theorem. Applying Gershgorin's circle theorem, each eigenvalue of $\mathrm{symm}\left(\bar{S}(q,p)\right)$ must be contained within a disk with radius defined by one of the matrix row sums [20, Theorem 6.1.1]. As each matrix entry is bounded the row sums, and hence the eigenvalues, are bounded also. Therefore there exists a value $\bar{\phi}_{a,b}$ such that the inequality (34) holds for on the interval $T_{a,b}$ for any $\phi \geq \bar{\phi}_{a,b}$. We conclude therefore that only finitely many jumps can occur in $T_{a,b}$, ruling out any Zeno or eventually discrete behaviour on the interval. As the interval $T_{a,b}$ is an arbitrary closed subset of $\mathcal{D}_L$, it follows that $\sup_t \mathcal{E}_o =$

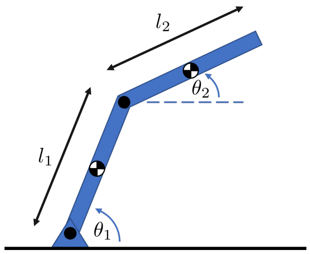

Fig. 1. Vertical 2 degree-of-freedom manipulator.

T_L the observer solution for $\hat{p}(t)$ exists for all $t \in \mathcal{D}_L$. Consequently, the inequality (23) holds for all $t \in \mathcal{D}_L$. If $\delta_{p_0} = 0_{n\times 1}$, we have that $\gamma = 0$ by (3). It follows from (23) the $\tilde{p} = 0_{n\times 1}$ is a globally exponentially stable equilibrium, verifying Claim 3. ∎

Remark 3: The observer input u_o is not directly used in this work but has applications for control purposes as studied in [14]. The passive input-output pair can be interconnected with a mechanical system, resulting in the momentum estimate being a passive output from the interconnected plant-observer system.

Remark 4: As was reported in [14], the hybrid momentum observer has dimension $n+1$, which is significantly lower than comparible I&I-based observers. For example, the solution reported in [10] has dimension $4n+1$. To understand why this is possible, note that the observer requires ϕ to be such that the inequality defining $\mathcal{C}$ in (20) is satisfied. This inequality can be tested using only position measurements, but the derivative of this expression requires knowledge of $\dot{q}$. Works that consider a smooth solution to the observer problem employ higher dimensions to compensate with the fact that $\dot{q}$ is unavailable whereas the hybrid observer avoids this difficulty by taking ϕ to be piece-wise constant.

Remark 5: Proposition 1 suggests that a large κ provides performance benefits due to an increased convergence rate and attenuation of force disturbances on the estimation error. Note from the observer dynamics in (20), however, that an increased κ will also increase sensitivity to imperfect measurements of the configuration vector q. The authors believe that there is a trade-off between sensitivity to external disturbances and measurement noise that will be investigated in future works.

IV. Vertical manipulator example

In this section, the proposed momentum observer is applied to a 2 degree-of-freedom manipulator (See Figure 1) to demonstrate both the exponential convergence and ISS properties. The system is driven by a known input torque at each joint, but an unknown time-varying disturbance is also assumed at each joint. This disturbance could represent unmodelled forces, input signal quantisation and parameter uncertainty. The Matlab code used to produce the presented results is available at `https://doi.org/10.24433/CO.0241413.v1`.

A. System model

The vertical manipulator can be described in the form (1) with configuration $q = (\theta_1, \theta_2)$, where θ_1, θ_2 describe the orientation of the first and second links with respect to the horizontal plane, respectively. The centre of mass of each link is assumed to be at the mid-point of the link. The mass matrix is described by

$$M_0(q) = \begin{bmatrix} J_1 + \frac{1}{4}m_1 l_1^2 + m_2 l_1^2 & \frac{1}{2}l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2}l_1 l_2 m_2 \cos(\theta_1 - \theta_2) & J_2 + \frac{1}{4}m_2 l_2^2 \end{bmatrix}, \tag{35}$$

where $l_1, l_2, m_1, m_2, J_1, J_2$ are the length, mass and moments of inertia of each link. The canonical momentum vector is then described by $p_0 = M(q)\dot{q}$. The potential energy is described by

$$V(q) = m_2 g \left(l_1 \sin\theta_1 + \frac{1}{2} l_2 \sin\theta_2 \right) + \frac{1}{2} m_1 g l_1 \sin\theta_1, \tag{36}$$

where g is the acceleration due to gravity. The damping matrix is given by

$$D_0(q) = \begin{bmatrix} d_{j_1}(q) + d_{j_2}(q) & -d_{j_2}(q) \\ -d_{j_2}(q) & d_{j_2}(q) \end{bmatrix}, \tag{37}$$

where $d_{j_1}(q)$ and $d_{j_2}(q)$ are the state-dependant coefficients of friction of the first and second joints, respectively. Finally, the input mapping matrix is described by

$$G_0 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{38}$$

Construction of the observer (19) requires the matrices $T(q), S(q,p), \bar{S}(q,p)$, defined in (7), (11), (12), respectively. For the presented example, these quantities were evaluated numerically point-wise using the methods described in Remark 2. This approach requires only a symbolic expression for the mass matrix for computation. The interested reader should refer to the linked simulation code for an example implementation.

The following model parameters are used to generate all subsequent simulation results:

$$\begin{aligned} m_1 &= 3 & m_2 &= 3 & d_{j_1}(q) &= 1 \\ l_1 &= 1 & l_2 &= 1 & d_{j_2}(q) &= 1 \\ J_1 &= \frac{3}{12} & J_2 &= \frac{3}{12} & g &= 9.8. \end{aligned}$$

For all simulations the system was initialised from the configuration $q(0) = [0, 0]^\top$ with the initial canonical momentum $p_0(0) = [-1, 2]^\top$. The manipulator and observer inputs were set to

$$u = \begin{bmatrix} 8\sin(t) \\ 4\cos(3t) \end{bmatrix}, \quad u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{39}$$

B. Exponential convergence

First the observer was tested in the case that there is no disturbance acting on the manipulator system ($\delta_{p_0}(t) = 0_{2\times 1}$). From Proposition 1 the observer should converge at an exponential rate, satisfying the bound (23) with $\gamma = 0$. As the value κ is a tuning parameter in the observer, it is expected that increasing values of κ should lead to a faster rate convergence.

Simulation results for several choices of κ are shown in Figure 2. The first plot shows the norm of the momentum estimation error on a log scale. As expected, increasing

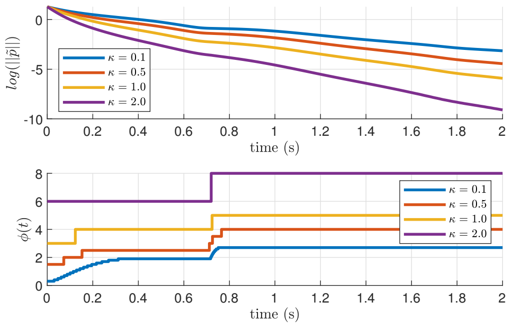

Fig. 2. Observer performance for several tuning gains in the case of no input disturbance. The first plot shows the square of the normed momentum estimation error on a log scale. The second plot shows the piece-wise constant observer state ϕ.

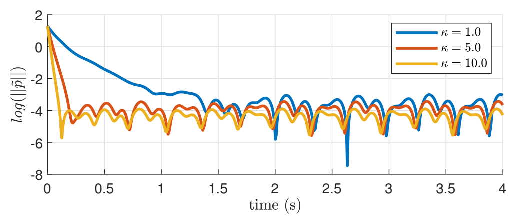

Fig. 3. Observer performance for several tuning gains in the case of an unknown input disturbance.

values of κ lead to an increased rate of convergence of the observe. The second plot shows the piece-wise constant observer state ϕ for each choice of κ. As the value of ϕ is updated to satisfy (20), it is not surprising that larger values of κ result in larger values of ϕ also.

C. *ISS results*

The observer was tested in the case that a input disturbance torque is acting on the robotic manipulator. The input disturbance was chosen to be

$$\delta_{p_0}(t) = \begin{bmatrix} \frac{1}{2}\sin(10t) \\ \frac{1}{2}\cos(20t) \end{bmatrix} \tag{40}$$

and was unknown to the observer. From Proposition 1 it is expected that the observer error is ISS with respect to the unknown disturbance, satisfying the bounds (23). In particular, increasing values of κ should increase the rate at which the observer converges to some neighbourhood of the true solution and decrease any perpetual error.

Simulation results for a variety of values for κ are shown in Figure 3. The figure shows the norm of the momentum estimation error, plotted on a log scale. As expected, increasing the tuning parameter κ results in a faster initial transient of the observer as it approaches a neighbourhood of the true solution. From the inequality (23) it is additionally expected that increasing the value κ should decrease the effect of the disturbance on the momentum estimate. It can be seen that after the initial transient, larger values for κ result in smaller peak errors as expected.

V. Conclusion

In this work, analysis for the momentum observer proposed in [14] was extended to verify that the observer error dynamics are both globally exponentially stable and ISS with respect to external perturbation. The results were demonstrated numerically on a 2 degree-of-freedom vertical manipulator, verifying the theoretical results. In future works the additional passive port (u_o, y_o) will be utilised for control purposes, interconnecting observer and controller subsystems. It is expected that Lyapunov functions for the joint observer/controller systems can be constructed without the use of separation principles.

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