IEEE Control Systems Letters paper presented at
2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Adaptive Observer-Based Output Regulation with Non-smooth Non-periodic Exogenous Signals

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Abstract— **In this paper, we address the error-feedback output regulation problem for linear systems with non-smooth non-periodic exogenous signals. We design an adaptive observer under a relaxed persistence of excitation (PE) condition and we solve the error-feedback problem in the non-smooth case. In addition, we show that this relaxed PE condition is equivalent to a complete observability condition that can be checked *a priori* by means of an exogenous excitation (EE) condition. We finally show that, if the exogenous signals are generated by a traditional linear time-invariant implicit model, the EE condition is equivalent to a non-resonance-like condition.**

I. Introduction

Given a system of interest, output regulation is a fundamental problem consisting in designing a control law that guarantees closed-loop stability and achieves disturbance rejection and/or reference tracking. In this problem, the disturbances and reference signals, named "exogenous signals", are modeled by a known autonomous system called "exosystem". The problem was initially introduced in the linear time-invariant (LTI) context [1], [2], and then extended to nonlinear systems [3]–[6], hybrid systems [7], [8], stochastic systems [9], and so on.

In this paper, we focus on the error-feedback output regulation problem for a special class of input signals. In the traditional setting, the exogenous signals are smooth and can be modeled by autonomous differential equations, which herein we call *implicit generator*. On the contrary, when exogenous signals are non-smooth, those implicit forms do not apply. In this case the exosystem can be expressed in explicit form as [10], [11, Section 5.1]

$$\omega(t) = \Lambda(t,t_0)\,\omega(t_0), \qquad \omega(t_0) = \omega_0, \tag{1}$$

where $\omega(t) \in \mathbb{R}^{\nu}$, and $\Lambda(t,t_0) \in \mathbb{R}^{\nu\times\nu}$, which is not assumed continuous, is such that $\Lambda(t_0,t_0) = I$ and $\Lambda(t_2,t_0) = \Lambda(t_2,t_1)\Lambda(t_1,t_0)$ for any t_1, t_2 greater than t_0. Non-smooth (possibly non-periodic) signals, such as sawtooth or pulse width modulation (PWM) signals, are commonly encountered in real-life applications, *e.g.* robotic manipulation [12]–[14]. The full-information problem[1] has been solved in [15]. Since $\Lambda(t,t_0)$ is not assumed to be continuous, standard approaches based on the internal model principle are difficult to extend to this setting because of the difficulty in stabilizing the closed-loop system. Note also that since Λ is not assumed to be periodic, this problem cannot be solved with possibly equivalent approaches currently available for hybrid systems, see *e.g.* [8].

Herein, we note that as the information of the exosystem (*i.e.* $\Lambda(t,t_0)$) is *a priori* known, the error-feedback problem can be approached by online estimating the system states and an unknown constant parameter ω_0. Exploiting this observation, this problem is then solvable by recursive algorithms known as adaptive observers, which have been widely studied for continuous-time linear systems [16]–[19]. However, in most adaptive methods, an assumption called persistence of excitation (PE) is necessary for the correct parameter estimation. As this condition generally requires *a posteriori* knowledge of experiments, see [20], various studies have focused on the relaxation of such condition [21]–[23].

Contributions. The paper makes the following key contributions: (a) We propose a solution to the error-feedback regulation problem with non-smooth non-periodic exogenous signals by an adaptive observer under an ultimate PE (UPE) condition (Section II). (b) We show that this UPE condition is equivalent to a complete observability condition that can be checked *a priori* by a condition called exogenous excitation (EE). (c) We provide an interesting interpretation of such EE condition showing that, when the exosystem is in the traditional LTI implicit form, the EE condition is equivalent to a non-resonance-like condition (Section III).

Notation. We use standard notation. $\mathbb{C}_{<0}$ denotes the set of complex numbers with a strictly negative real part and $\mathbb{C}_{\geq 0}$ denotes $\mathbb{C}\backslash\mathbb{C}_{<0}$. The symbol I_n denotes an $n\times n$ identity matrix, $0_{m\times n}$ denotes an $m\times n$ zero matrix, $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The superscript $\top$ denotes the transposition operator.

II. Error Feedback Non-smooth Non-periodic Output Regulation Problem

In this section, we consider the output regulation problem of linear systems with non-periodic and non-smooth exogenous signals, and we show that the error-feedback problem can be solved by using an adaptive observer under a PE condition.

Consider a class of single-input single-output LTI systems in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E\omega(t), \\ e(t) &= Cx(t) + Du(t) + F\omega(t), \end{aligned} \tag{2}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $E \in \mathbb{R}^{n\times\nu}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$, $F \in \mathbb{R}^{1\times\nu}$, $x(t) \in \mathbb{R}^n$ the state, $u(t) \in \mathbb{R}$ the control input, $e(t) \in \mathbb{R}$ the regulation error available to the observer and regulator, and $\omega(t) \in \mathbb{R}^{\nu}$ the exogenous signal that represents disturbances

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[1]This means that the states of the linear system and of the exosystem are available for feedback.

and/or references generated by an exosystem. The (possibly) non-smooth non-periodic exogenous signals are generated by the system in explicit form (1). We introduce a general assumption to make the exogenous signal ω well-behaved in the context of output regulation.

Assumption 1: The matrix-valued function Λ is bounded, piecewise continuous, and non-singular for all times, with Λ^{-1} bounded.

This assumption is not restrictive as standard traditional frameworks satisfy it (*e.g.* $\Lambda(t,t_0) = e^{S(t-t_0)}$ in the LTI case), see [15] for more detail. The output regulation problem is then formulated.

Problem 1 (Output Regulation Problem): Consider system (2) interconnected with the exosystem (1) under Assumption 1. The output regulation problem consists in designing a regulator u such that the following two conditions are satisfied.

(S) The closed-loop system obtained by interconnecting system (2), the exosystem (1), and the regulator with $\omega(t) \equiv 0$ is asymptotically stable.

(R) The closed-loop system obtained by interconnecting system (2), the exosystem (1), and the regulator satisfies $\lim_{t\to\infty} e(t) = 0$, uniformly for any $x(t_0) \in \mathbb{R}^n$, $\omega(t_0) \in \mathbb{R}^\nu$.

To solve Problem 1 when the states x and exogenous signals ω are unknown, we aim to find an error-feedback controller of the form

$$\dot{\xi}(t) = \mathscr{G}_1(t)\xi(t) + \mathscr{G}_2(t)e(t), \tag{3a}$$
$$u(t) = K_1(t)\xi(t) + K_2(t)e(t), \tag{3b}$$

where $\xi(t) \in \mathbb{R}^q$, $\mathscr{G}_1(t) \in \mathbb{R}^{q\times q}$, $\mathscr{G}_2(t) \in \mathbb{R}^{q\times 1}$, $K_1(t) \in \mathbb{R}^{1\times q}$, $K_2(t) \in \mathbb{R}$ are bounded piecewise continuous matrices to be specified later.

Another standard assumption introduced in traditional error-feedback output regulation problems is stated next [24, Section 1.4].

Assumption 2: The pair $(A,\, B)$ is stabilizable, and the pair $(C,\, A)$ is detectable.

From now on, when $t_0 = 0$, we write $\Lambda(t) = \Lambda(t,0)$ with a slight abuse of notation.

To solve the error-feedback output regulation problem, we first review the design method of a state-feedback regulator in the full-information case, which needs the following solvability condition (SC).

(SC) If $D = 0$, then Λ is assumed to be piecewise differentiable, and system (2) is assumed to be minimum-phase with a unitary relative degree. If $D \neq 0$, system (2) is assumed to be minimum-phase.

Then the following theorem holds.

Theorem 1 ([15]): Consider Problem 1. Suppose the pair $(A,\, B)$ is stabilizable and Assumption 1 and (SC) hold. Then the following statements hold and are equivalent.

(i) There exist a matrix K and a bounded piecewise continuous matrix Γ such that the regulator

$$u^*(t) = Kx(t) + \Gamma(t)\omega(t) \tag{4}$$

solves the full-information output regulation problem.

(ii) There exist bounded piecewise continuous matrices Π and Δ that satisfy the regulator equations

$$\begin{aligned} \Pi(t) &= \left(\int_{-\infty}^{t} e^{A(t-\tau)}(E + B\Delta(\tau))\Lambda(\tau)d\tau\right)\Lambda(t)^{-1}, \\ 0_{1\times\nu} &= \lim_{t\to+\infty} C\Pi(t) + D\Delta(t) + F. \end{aligned} \tag{5}$$

In particular, if solutions Π and Δ to (5) are obtained and the matrix K is such that $\sigma(A+BK) \subset \mathbb{C}_{<0}$, then the controller u^* in (4) solves the full-information problem by selecting $\Gamma = \Delta - K\Pi$.

The importance of Theorem 1 is that it provides a full-information regulator design method for Problem 1 when (SC) holds. With this result at hand, the error-feedback problem can be solved by designing an observer for both the system state x and the exogenous signal ω. As non-smooth exogenous signals cannot be represented by any differential equation, designing a Luenberger observer similar to the conventional method, see *e.g.* [25, Theorem 1.14] is not an option. However, since the exosystem (herein Λ) is known, the only unknown information about the exogenous signals is the initial value ω_0. This implies that the same observation problem can be approached by using an adaptive observer for the online estimation of both state x and initial value ω_0 from the output e. This is the idea that we pursue in this paper.

Consider system (2) interconnected with the explicit generator (1). Then, an adaptive observer for the joint estimation of both state x and initial value ω_0 can be constructed by following the method for LTV systems initially proposed by [18] and improved by [26]. In addition to the detectability condition in Assumption 2, the application of an adaptive observer in this context also requires a PE condition, which is instrumental in establishing the convergence of the adaptive observer to the correct parameter values [18]. In our context, we formalize a relaxed PE condition as follows.

Definition 1 (Ultimate Persistence of Excitation (UPE)): Consider a matrix-valued function Υ such as

$$\dot{\Upsilon}(t) = (A+LC)\Upsilon(t) + (E+LF)\Lambda(t), \tag{6}$$

with $\Upsilon(t) \in \mathbb{R}^{n\times\nu}$, the pair $(C,\, A)$ detectable, and $L \in \mathbb{R}^{n\times 1}$ such that $\sigma(A+LC) \subset \mathbb{C}_{<0}$. The time-varying matrix Λ is *ultimately persistently exciting* if there exist positive constants $\hat{t}$, α_1, β_1, and T_1 such that

$$\alpha_1 I_\nu \leq \int_t^{t+T_1} (C\Upsilon_{ss}(\tau) + F\Lambda(\tau))^\top (C\Upsilon_{ss}(\tau) + F\Lambda(\tau)) d\tau \leq \beta_1 I_\nu, \tag{7}$$

for all $t \geq \hat{t}$, where Υ_{ss} is the steady-state[2] response of Υ.

Remark 1: The UPE condition provided above is a relaxed (steady-state) version of the classical PE condition[3] used in [26]. Compared with the classical PE condition, this UPE condition is independent of the initial condition Υ_0. A similar relaxed PE condition has been used in the context of adaptive

[2]We use the definition of steady state given in [27].

[3]The classical PE condition requires that there exist positive constants α_2, β_2, and T_2 such that $\alpha_2 I_\nu \leq \int_t^{t+T_2} (C\Upsilon(\tau) + F\Lambda(\tau))^\top (C\Upsilon(\tau) + F\Lambda(\tau))\, d\tau \leq \beta_2 I_\nu$, for all $t \geq 0$.

nonlinear output regulation, see [28, Lemma 3]. Additional details on this condition are provided in Section III.

We can now formulate the following assumption.

Assumption 3: The time-varying matrix Λ is ultimately persistently exciting.

Under Assumptions 1, 2, and 3, the adaptive observer in [26] for the interconnection of system (2) and generator (1) can be formulated as

$$\begin{aligned}\dot{\hat{x}}(t) &= (A+LC)\hat{x}(t)+(B+LD)u(t)\\ &\quad +(E+LF)\Lambda(t)\hat{\theta}(t)-Le(t)+\Upsilon(t)\dot{\hat{\theta}}(t),\\ \dot{\hat{\theta}}(t) &= M(t)(C\Upsilon(t)+F\Lambda(t))^{\top}(e(t)-C\hat{x}(t)-Du(t)-F\Lambda(t)\hat{\theta}(t)),\\ \dot{M}(t) &= \lambda M(t)-M(t)(C\Upsilon(t)+F\Lambda(t))^{\top}(C\Upsilon(t)+F\Lambda(t))M(t),\end{aligned} \tag{8}$$

where $\hat{x}(t)\in\mathbb{R}^n$ is the state estimate, $\hat{\theta}(t)\in\mathbb{R}^{\nu}$ the estimate of parameter ω_0, $\Upsilon(t)$ in (6) an auxiliary time-varying matrix, $M(t)\in\mathbb{R}^{\nu\times\nu}$ the parameter estimate gain matrix, and $\lambda>0$ a scalar forgetting factor. The observer is initialised with $\hat{x}(t_0)=\hat{x}_0$, $\hat{\theta}(t_0)=\hat{\theta}_0$, $\Upsilon(t_0)=\Upsilon_0$, $M(t_0)=M_0$, where $\hat{x}_0\in\mathbb{R}^n$, $\hat{\theta}_0\in\mathbb{R}^{\nu}$, $\Upsilon_0\in\mathbb{R}^{n\times\nu}$, $M_0\in\mathbb{R}^{\nu\times\nu}$ are (any) initial values of $\hat{x}(t)$, $\hat{\theta}(t)$, $\Upsilon(t)$, $M(t)$ respectively. Note that M_0 is required to be a symmetric positive definite matrix to enhance the convergence of the observer [29].

With this setting, we now show that the UPE condition is sufficient for the convergence of the adaptive observer (6)–(8). To this end, we need a preliminary lemma.

Lemma 1: Consider system (6). Suppose Assumption 1 holds. The steady-state response of Υ is given by $\Upsilon_{ss}(t)=\Pi_{\Upsilon}(t)\Lambda(t)$, where $\Pi_{\Upsilon}(t)\in\mathbb{R}^{n\times\nu}$ is the matrix-valued function

$$\Pi_{\Upsilon}(t)=\left(\int_{-\infty}^{t}e^{(A+LC)(t-\tau)}(E+LF)\Lambda(\tau)d\tau\right)\Lambda(t)^{-1}. \tag{9}$$

Proof: The proof is omitted because it is analogous to that of [15, Lemma 1] about the steady-state of the interconnection of (1), (2) and (4). ■

For brevity, define the estimation error $\varepsilon(t)=[\tilde{x}(t),\tilde{\theta}(t)]^{\top}=[x(t)-\hat{x}(t),\omega_0-\hat{\theta}(t)]^{\top}$. We are now ready to prove the convergence of the observer.

Lemma 2: Consider system (2) interconnected with the exosystem (1) and the adaptive observer (8) with auxiliary system (6). Suppose Assumption 1 holds. If the UPE condition holds, the estimation error satisfies $\lim_{t\to\infty}\varepsilon(t)=0$.

Proof: Assume that the UPE condition is satisfied. Let $\eta=\tilde{x}-\Upsilon\tilde{\theta}$ and $\Psi=C\Upsilon+F\Lambda$. By Lemma 1, the steady-state response of Ψ is $\Psi_{ss}=C\Upsilon_{ss}+F\Lambda$. By similar derivations conducted in [26] in the context of the classical PE condition, the estimation errors $\tilde{x}$ and $\tilde{\theta}$ can be transformed into the system

$$\dot{\eta}(t)=(A+LC)\eta(t), \tag{10a}$$

$$\dot{\tilde{\theta}}(t)=-M(t)\Psi^{\top}(t)(C\eta(t)+\Psi(t)\tilde{\theta}(t)). \tag{10b}$$

Since $\lim_{t\to\infty}\eta(t)=0$, at steady-state (10b) becomes

$$\dot{\tilde{\theta}}(t)=-M(t)\Psi_{ss}^{\top}(t)\Psi_{ss}(t)\tilde{\theta}(t). \tag{11}$$

By the UPE condition, there exists a constant $\hat{t}$ such that (7) holds for all $t\geq\hat{t}$. Meanwhile, consider system (11) with M in (8) initialised by a symmetric positive definite M_0. Under Assumption 1 and the UPE condition, system (11) is exponentially stable for all $t\geq\hat{t}$ [29, Theorem 2]. Since the continuous function η decays to zero and Ψ is bounded and piecewise continuous, the continuous trajectory $M(t)$ always exists within the finite-time interval $t\in[0,\hat{t}]$ [30, Theorem 3.1]. Equivalently, $\tilde{\theta}(t)$ in (10b) also exists for all $0\leq t\leq\hat{t}$, leading to an arbitrary bounded value $\tilde{\theta}(\hat{t})$ at time $\hat{t}$. By the exponential stability of systems (10a) and (11), $\tilde{\theta}$ will finally converge to zero for all $t\geq\hat{t}$, resulting in $\tilde{x}=\eta+\Upsilon\tilde{\theta}\to 0$. ■

Then we are ready to provide the main result of this section, *i.e.* provide a solution to the error-feedback output regulation problem.

Theorem 2: Consider Problem 1. Suppose Assumptions 1, 2, 3, and (SC) hold. Then there exists a regulator (3) based on the adaptive observer (6)–(8) that solves the error-feedback output regulation problem.

Proof: Under Assumptions 1, 2, and 3, Lemma 2 implies that the adaptive observer (6)–(8) is able to estimate the system state x and the initial value ω_0, that is $\lim_{t\to\infty}(x(t)-\hat{x}(t))=0$ and $\lim_{t\to\infty}(\omega_0-\hat{\theta}(t))=0$. As (SC) holds, let $u=K\hat{x}+(\Delta-K\Pi)\Lambda\hat{\theta}$ with K such that $\sigma(A+BK)\subset\mathbb{C}_{<0}$ and Π and Δ bounded piecewise continuous mappings solving (5). Then $\lim_{t\to\infty}(u(t)-u^*(t))=0$. By Theorem 1 and boundedness of Λ, the bounded input u solves the output regulation problem asymptotically. Since both the adaptive observer and u are representable by (3) with proper choices of bounded piecewise continuous $\mathscr{G}_1$, $\mathscr{G}_2$, K_1, K_2, there exists an error-feedback regulator (3) solving Problem 1. ■

Remark 2: Designing an observer-based regulator is not the only way of solving the error-feedback problem. This kind of problems is usually addressed by directly designing an internal model, see [31], which also provides robustness. However, due to the time-varying nature of the problem, the stabilization of the resulting closed-loop system remains an open problem that hinders the use of the internal model design (see also [32] for a discussion of the so-called chicken-egg dilemma in the nonlinear non-minimum-phase case).

III. Insights into the UPE Condition

As the convergence of the adaptive observer requires the UPE assumption, in this section, we provide insights into this condition. We show that this UPE condition is equivalent to a complete observability condition. Moreover, we show that this observability condition can be checked *a priori* by a sufficient condition related to the solvability of a special differential-algebraic equation (DAE). We finally show that this sufficient condition is equivalent to a so-called exogenous non-resonance-like condition in the traditional LTI implicit generator case.

To begin, we show that the UPE condition is equivalent to a complete observability condition.

Lemma 3: Consider the matrix-valued funcion Υ in (6). Suppose Assumption 1 holds. The UPE condition is satisfied if and only if there exists a constant $\tilde{t}\geq 0$ such that the system

$$\begin{aligned}\omega(t)&=\Lambda(t)\omega_0,\\ \varphi(t)&=(C\Pi_{\Upsilon}(t)+F)\omega(t),\end{aligned} \tag{12}$$

with Π_Υ in (9), is completely observable[4] on $[t_a, +\infty)$ for all $t_a \geq \tilde{t}$.

Proof: By Lemma 1, $\Upsilon_{ss} = \Pi_\Upsilon \Lambda$ with Π_Υ in (9). As Λ is bounded, the equivalence between the UPE condition and complete observability of (12) follows by noticing that the inequality (7) with the substitution $\Upsilon_{ss} = \Pi_\Upsilon \Lambda$ is identical to the requirement that the observability Gramian of system (12) is bounded and positive definite. ■

Then we show that this complete observability condition can be checked *a priori* by means of a sufficient condition which is independent of the stabilization matrix L.

Definition 2 (Exogenous Excitation (EE) Condition): Consider the interconnection of (1) and (2). The exogenous excitation condition is said to hold if the DAE

$$\begin{aligned} \dot{z}(t) &= Az(t) + E\Lambda(t)\eta_\omega, \\ 0 &= Cz(t) + F\Lambda(t)\eta_\omega, \end{aligned} \tag{13}$$

does not admit a solution $z \in \mathbb{R}^n$ on the interval $[t_b, +\infty)$ for any $t_b \geq 0$ and $\eta_\omega \in \mathbb{R}^\nu \backslash \{0\}$.

Proposition 1: Consider system (6). Suppose Assumption 1 holds. Λ satisfies the UPE condition if the EE condition holds.

Proof: Assume the EE condition holds. Assume by contradiction that Λ is not UPE, which, by Lemma 3 means that there exist a constant $t_b \geq 0$ and a non-zero vector $\eta_\omega \in \mathbb{R}^\nu$ such that $(C\Pi_\Upsilon(t) + F)\Lambda(t)\eta_\omega = 0$ for all $t \geq t_b$. Since Lemma 1 implies that $\Upsilon_{ss} = \Pi_\Upsilon \Lambda$, right-multiplying η_ω to both sides of (6) gives

$$\dot{\Upsilon}_{ss}(t)\eta_\omega = A\Upsilon_{ss}(t)\eta_\omega + E\Lambda(t)\eta_\omega + L(C\Upsilon_{ss}(t) + F\Lambda(t))\eta_\omega. \tag{14}$$

Define $z_\omega(t) := \Upsilon_{ss}(t)\eta_\omega$ with $z_\omega(0) := \Upsilon_{ss}(0)\eta_\omega$. For all $t \geq t_b$, since $(C\Pi_\Upsilon(t) + F)\Lambda(t)\eta_\omega = 0$, (14) is equivalent to (13) with $z = z_\omega$. This contradicts the EE condition. Therefore, there does not exist $t_b \geq 0$ and a non-zero vector η_ω such that $(C\Pi_\Upsilon(t) + F)\Lambda(t)\eta_\omega = 0$ for all $t \geq t_b$, *i.e.* there exists some $\tilde{t} \geq 0$ such that the system is completely observable on the interval $[t_a, +\infty)$ for all $t_a \geq \tilde{t}$ [33, Section 3.3]. By Lemma 3, the UPE condition holds. ■

Lemma 3 and Proposition 1 suggest that the UPE condition can be checked *a priori* via the EE condition which only depends on system matrices and Λ. In the rest of this section, we show that this condition is equivalent to a so-called exogenous non-resonance-like condition when the exosystem is linear time-invariant.

We conclude this section by providing an interesting interpretation of the UPE condition and the EE condition in the traditional linear output regulation problem. To this end, consider the traditional case where (1) has a differential representation, namely

$$\dot{\omega}(t) = S\omega(t), \qquad \omega(0) = \omega_0, \tag{15}$$

where $S \in \mathbb{R}^{\nu\times\nu}$. In this case, $\Lambda(t) = e^{St}$ and Assumption 1 takes the following form.

[4]A system is completely observable at t_0 if the only state that is not observable at t_0 is the zero state. See [33, Section 3.3] for additional detail.

Assumption 4: All the eigenvalues of S are simple and lie on the imaginary axis.

Now, we provide insights on the UPE condition studying this case. Similarly to Lemma 3, the following lemma relates the UPE condition with an observability condition.

Lemma 4: Consider the matrix-valued funcion Υ in (6) with $\Lambda(t) = e^{St}$. Suppose Assumption 4 holds. The UPE condition is satisfied if and only if the pair $(C\bar{\Pi} + F,\, S)$ is observable, where $\bar{\Pi} \in \mathbb{R}^{n\times\nu}$ is the unique solution to the Sylvester equation

$$\bar{\Pi}S = (A + LC)\bar{\Pi} + E + LF. \tag{16}$$

Proof: Under Assumption 4, as $\Lambda(t) = e^{St}$ and $\sigma(A + LC) \subset \mathbb{C}_{<0}$, $\sigma(A+LC) \cap \sigma(S) = \emptyset$ and [25, Lemma 1.4] gives that $\Upsilon_{ss}(t) = \bar{\Pi}\Lambda(t)$, where $\bar{\Pi}$ is the unique solution of (16). Since Λ is the transition matrix of system (15), by Lemma 3, the UPE condition and the observabilty of $(C\bar{\Pi} + F,\, S)$ are equivalent. ■

Then we are ready to show that the UPE condition in this LTI implicit generator case can be checked by a non-resonance-like condition.

Definition 3 (Exogenous Non-Resonance-Like Condition): Systems (2) and (15) are exogenously non-resonant if matrix

$$\begin{bmatrix} A - \lambda_s I_n & E \\ C & F \end{bmatrix} \tag{17}$$

has full row rank for any $\lambda_s \in \sigma(S)$.

Theorem 3: Consider system (6) with $\Lambda(t) = e^{St}$. Suppose Assumption 4 holds. Then the UPE condition holds if the exogenous non-resonance-like condition is satisfied.

Proof: Assume the exogenous non-resonance-like condition is satisfied. Let λ_s be an eigenvalue of S with its associated eigenvector $v_s \in \mathbb{R}^\nu$, *i.e.* $\lambda_s v_s = S v_s$. By right-multiplying v_s to both sides of (16), we get $(\lambda_s I_n - A)\bar{\Pi}v_s - Ev_s - L(C\bar{\Pi} + F)v_s = 0$. Then observability of the pair $(C\bar{\Pi} + F,\, S)$ can be proved analogously to the proof of its dual problem, see [34, Lemma 1] and [35, Theorem 1]. By Lemma 4, the UPE condition holds. ■

Remark 3: Condition (17) (similarly to the standard non-resonance condition) requires that the poles of the exosystem matrix S do not coincide with the transmission zeros of the system characterized by matrices (A, E, C, F). Note also that, the relation between non-resonance conditions and controllability/observability properties associated with the corresponding Sylvester equations is not a new story in the output regulation context. See [34, Lemma 1] and [35, Theorem 1] for more details.

The next theorem shows that the EE condition is a generalisation of the exogenous non-resonance-like condition to the non-smooth case.

Theorem 4: Consider system (6) with $\Lambda(t) = e^{St}$. Suppose Assumption 4 holds. The EE condition holds if and only if the exogenous non-resonance-like condition is satisfied.

Proof: The DAE in (13) is equivalent (by summing the second equation to the first pre-multiplied by L) to

$$\dot{z}(t) = (A + LC)z(t) + (E + LF)e^{St}\eta_\omega, \tag{18a}$$

$$0 = Cz(t) + Fe^{St}\eta_\omega. \tag{18b}$$

Necessity: By the proof of [25, Lemma 1.6], the DAE (18) has a solution z on the interval $[t_b, +\infty)$ for any $t_b \geq 0$ and $\eta_\omega \in \mathbb{R}^\nu \backslash \{0\}$ if and only if there exists a matrix $\Pi_z \in \mathbb{R}^{n\times\nu}$ solving $\Pi_z S = (A + LC)\Pi_z + E + LF$ and $0_{1\times\nu} = C\Pi_z + F$. These two equations can be equivalently written as

$$\begin{bmatrix} I_n & 0_{n\times\nu} \\ 0_{1\times n} & 0_{1\times\nu} \end{bmatrix}\begin{bmatrix} \Pi_z \\ I_\nu \end{bmatrix} S - \begin{bmatrix} A & E \\ C & F \end{bmatrix}\begin{bmatrix} \Pi_z \\ I_\nu \end{bmatrix} = \begin{bmatrix} 0_{n\times\nu} \\ 0_{1\times\nu} \end{bmatrix},$$

which is analogous to [25, Equation 1.25]. By a similar proof conducted in [25, Theorem 1.9], the solution Π_z does not exist if and only if the exogenous non-resonance-like condition holds. Therefore, if the EE condition is satisfied, the exogenous non-resonance-like condition must hold.

Sufficiency: Suppose the exogenous non-resonance-like condition holds. By Lemma 4 and Theorem 3, the pair $(C\bar{\Pi} + F,\ S)$, with $\bar{\Pi}$ in (16), is observable, *i.e.* there does not exist $\eta_\omega \in \mathbb{R}^\nu \backslash \{0\}$ such that $(C\bar{\Pi} + F)e^{St}\eta_\omega = 0$ for all $t \geq t_b$ with any $t_b \geq 0$. As the steady state of z in (18a) is characterized by $z_{ss}(t) = \bar{\Pi}e^{St}\eta_\omega$ [25, Lemma 1.4], the observability of the pair $(C\bar{\Pi} + F,\ S)$ implies that the DAE (18) (or (13)) does not admit a steady-state solution z_{ss} for any non-zero η_ω. Thus, the EE condition holds. ■

Remark 4: Up to this point, we have assumed Λ is bounded, which is also a standard assumption for matrices of LTV systems in conventional adaptive observer design problems. However, in our case, the existence of the forgetting factor in (8) still guarantees the exponential convergence of the observer even when Λ is just *exponentially bounded* (with bounded Λ^{-1}) [29, Theorem 2]. This will be illustrated by an example.

IV. Illustrative Example

In this section, we use an RLC circuit to provide an example to illustrate how the adaptive observer solves the output regulation problem with non-smooth non-periodic exogenous signals. Consider the circuit depicted in Fig. 1, which consists of an independent voltage generator $u(t)$ connected to an RLC circuit described by a series connection of an inductor L_r, a resistor R_r, and a capacitor C_r. Note that the RLC circuit is also subject to the influence of an external circuit, which is equivalently represented by an independent voltage generator $d_{th}(t)$ in series with a resistor R_{th} using Thévenin's theorem. The goal of this example is to design $u(t)$ to regulate the input current $i_{in}(t)$ to track a given reference signal $r(t)$ while rejecting the external disturbance introduced by $d_{th}(t)$. By denoting $x_1(t)$ the current across the inductor $i_L(t)$, $x_2(t)$ the voltage across the capacitor $v_C(t)$, and the output $y(t) = i_{in}(t)$, such a circuit is described by

$$\begin{aligned} \dot{x}_1(t) &= \delta_2 x_1(t) - \tfrac{1}{L_r}x_2(t) + \delta_3 u(t) + \tfrac{R_{in}}{L_r}\delta_1 d_{th}(t), \\ \dot{x}_2(t) &= \tfrac{1}{C}x_1(t), \\ y(t) &= R_{th}\delta_1 x_1(t) + \delta_1 u(t) - \delta_1 d_{th}(t), \end{aligned} \tag{19}$$

where $\delta_1 = \frac{1}{R_{in}+R_{th}}$, $\delta_2 = -\frac{R_r}{L_r} - \frac{R_{in}R_{th}}{L_r}\delta_1$, $\delta_3 = \frac{1}{L_r} - \frac{R_{in}}{L_r}\delta_1$. With a given reference signal $r(t)$, the regulation error is defined by $e(t) = y(t) - r(t)$. Let both the reference and disturbance signals be described by $r(t) = \Lambda_r(t)r_0$ and $d_{th}(t) = \Lambda_d(t)d_0$. Then, we can define the exogenous signal $\omega(t) =$

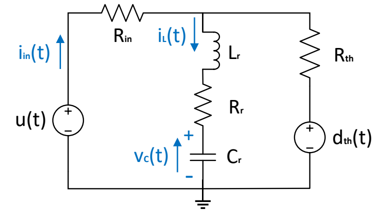

Fig. 1. Schematics of the RLC circuit.

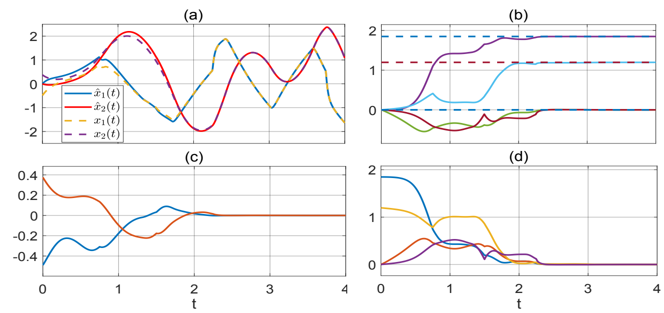

Fig. 2. **(a)** Time history of the state $x(t)$ (dashed lines) and of the estimate $\hat{x}(t)$ (solid lines). **(b)** Value of the initial condition ω_0 (dashed lines) and time history of the estimate $\hat{\theta}(t)$ (solid lines). **(c)** Time history of the difference $x(t) - \hat{x}(t)$. **(d)** Time history of the difference $\omega_0 - \hat{\theta}(t)$.

$[r(t),\ 0,\ d_{th}(t),\ 0]^\top = \Lambda(t)\omega_0$ with $\Lambda = \text{blkdiag}(\Lambda_1,\ \Lambda_2)$ where

$$\Lambda_i(t) = \begin{bmatrix} \Lambda_k(t) & -\tilde{\Lambda}_k(t) \\ \tilde{\Lambda}_k(t) & \Lambda_k(t) \end{bmatrix}, \tag{20}$$

with $i = 1, 2$. When $i = 1$, $\Lambda_k = \Lambda_r$ and $\tilde{\Lambda}_k = \tilde{\Lambda}_r$. When $i = 2$, $\Lambda_k = \Lambda_d$ and $\tilde{\Lambda}_k = \tilde{\Lambda}_d$ (Λ_r, Λ_d, $\tilde{\Lambda}_r$ and $\tilde{\Lambda}_d$ are defined later). Then the system is representable by (2) with

$$A = \begin{bmatrix} \delta_2 & -\frac{1}{L_r} \\ \frac{1}{C_r} & 0 \end{bmatrix},\quad B = \begin{bmatrix} \delta_3 \\ 0 \end{bmatrix},\quad C = \begin{bmatrix} R_{th}\delta_1 \\ 0 \end{bmatrix}^\top,$$

$$D = \delta_1,\quad E = \begin{bmatrix} 0_{2\times 2} & E_d & 0_{2\times 1} \end{bmatrix},\quad F = \begin{bmatrix} -1 & 0 & -\delta_1 & 0 \end{bmatrix},$$

where $E_d = [\frac{R_{in}}{L_r}\delta_1,\ 0]^\top$. In the following simulations, we set $R_{in} = 0.5\ \Omega$, $R_{th} = 3\ \Omega$, $R_r = 1\ \Omega$, $L_r = 0.1\ H$, and $C_r = 0.2\ F$. With those values, (SC) is satisfied. Moreover, the initial values of x and ω are randomly selected as $x_0 = [-0.4898,\ 0.3756]^\top$ and $\omega_0 = [r_0,\ 0,\ d_0,\ 0]^\top$, with $r_0 = 1.8481$ and $d_0 = 1.1964$. The observer is initialized with $\hat{x}_0 = 0_{2\times 1}$, $\hat{\theta}_0 = 0_{4\times 1}$, $\Upsilon_0 = 0_{2\times 4}$, $M_0 = I_4$ and $\lambda = 3$. The matrices K and L are selected such that the poles of $(A + BK)$ and $(A + LC)$ are located at $-5 \pm 3i$.

Now we consider the error-feedback output regulation problem, in which the external signal is assumed to be a magnified square wave represented by $\Lambda_d(t) = \log(t+2)\sqcap(\frac{2\pi}{3}t + \frac{\pi}{2})$ where $\sqcap(t) \triangleq \text{sign}(\sin(t))$. The expected reference signal is a triangular wave with $\Lambda_r(t) = \nabla(\frac{2\pi}{3}t^{\frac{3}{2}} + \frac{\pi}{2})$ where $\nabla(t) \triangleq \frac{4}{T}\int_0^t \sqcap(\tau)d\tau - 1$. $\Lambda = \text{blkdiag}(\Lambda_1,\ \Lambda_2)$ is set as (20) with $\tilde{\Lambda}_d(t) = \log(t+2)\sqcap(\frac{2\pi}{3}t)$ and $\tilde{\Lambda}_r(t) = \nabla(\frac{2\pi}{3}t^{\frac{3}{2}})$, and is unbounded in this case. For the selected $F\Lambda$(t) (containing time-varying triangular waves and time-varying square waves) and $E\Lambda(t)$ (containing only time-varying square waves), there is no solution to the DAE (13) on the interval $[t_b, +\infty)$ for

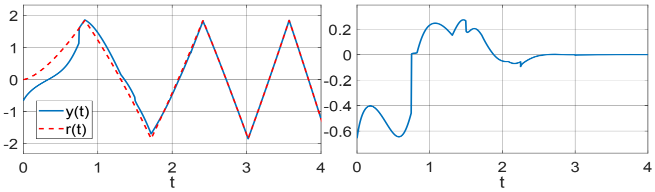

Fig. 3. Left graph: time history of the output $y(t)$ (solid blue) and of the reference signal $r(t)$ (red dashed). Right graph: time history of the regulation error $e(t)$.

any $t_b \geq 0$ and any non-zero η_ω, *i.e.* the EE condition holds. By Proposition 1, the UPE condition is satisfied. Then the proposed adaptive observer-based regulator is applied. Fig. 2 (a) and (b) compare the time histories of estimates $\hat{x}(t)$ and $\hat{\theta}(t)$ (solid lines) with the actual values $x(t)$ and ω_0 (dashed lines), respectively. The corresponding estimation errors that asymptotically converge to zero are depicted in Fig. 2 (c) and (d). This result shows that, under the UPE condition, the observer estimates $\hat{x}(t)$ and $\hat{\theta}(t)$ converge to $x(t)$ and ω_0, respectively, even if Λ is unbounded. Then, Fig. 3 (left) displays the time histories of the output $y(t)$ (solid line) versus the reference signal $r(t)$ (dashed line). Fig. 3 (right) shows the time history of regulation error $e(t)$. The figure illustrates that the regulation error decays asymptotically to zero.

V. Conclusion

In this paper, we have studied the use of adaptive observers in solving the error-feedback linear output regulation problem with non-smooth non-periodic exogenous signals. We have first solved this problem under a UPE condition. Then, we have shown that this UPE condition is equivalent to a complete observability condition, which can be checked *a priori* by means of an EE condition. We have finally demonstrated that, if the exosystem is in the traditional LTI implicit form, the EE condition is equivalent to a so-called exogenous non-resonance-like condition.

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