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A Plug & Play Command Generator for Swarm Formation on Multiple Arbitrary Shaped Orbits: a Diffeomorphism-based approach

Antonio Bono[1], *Member, IEEE*, Luigi D'Alfonso[1], *Member, IEEE*, Giuseppe Fedele[1], *Member, IEEE* and Veysel Gazi[2], *Senior Member, IEEE*

***Abstract*— In this paper, we develop a strategy for arranging a team of agents in orbits of arbitrary shape that pass through a common point that may move. In particular, the shapes are defined by star-shaped sets. The solution is based on its transformation into the traditional circular formation problem and subsequent reconversion. Given the large number of solutions to such a problem, the developed algorithm is well suited as a plug & play command generator to extend these solutions to the promising field of aerial robotic swarms. Two examples of such an application show its ease of implementation.**

I. Introduction

The problem of distributed formation control in multi-agent systems has long attracted the interest of the scientific community and companies because of its numerous application areas: patrolling, search and rescue, agriculture just to name a few. This great interest, witnessed also by the numerous dedicated survey papers (e.g. [1][2]), provides us with different solving strategies that vary depending on some aspects such as: the mathematical model used for their description (kinematic, double integrator, nonlinear, etc.); the type of information available to the agents (position, displacement, bearing, etc.); and the communication flow described by graphs of various nature (static, dynamic or random). A fundamental and characteristic element of any distributed control strategy is *scalability*. Most of the proposed solutions, in fact, are designed for small teams of agents, i.e., a few dozen. The limitation of scalability may arise from the amount of communication data [3], computational resources [4], or the difficulty of specifying the positions of each individual agent in the desired formation [5]. In the robotic context, in particular, works that focus on the scalability aspect are part of a specific field that is known as *swarm robotics* [6]. A recent example is [7] where the authors present a method to control 2D static and time-varying formations among collective self-repelling ferromagnetic microrobots by spatially and temporally programming an external magnetic potential energy distribution at the air-water interface or on solid surfaces. The remarkable results obtained, however, are limited to the very specific case of planar-moving microrobots. The most of works in swarm robotics, in fact, are designed for the 2D space as reported in [1], [8]. *Aerial* robotic swarms, however, are among the most promising applications of formation control [9] along with, more recently, space robotics [10]. In this direction, our work provides a tool to exploit many existing strategies for two-dimensional space in three-dimensional space as well (*property i*). The problem considered, in particular, is the classical one of circular formation tracking (CFT). Making use of the disk transformation in a topologically equivalent space, in fact, it is possible to exploit the command signals of the original strategy, to obtain a formation in a curve defined by a star-shaped set which allow a much wider freedom of choice of shapes than polytopes or convex sets (*property ii*). Given the interest in aerial applications, we specifically employ not a single but a family of transformations (diffeomorphisms), capable of arranging agents in arbitrarily shaped orbits around a known reference that may be fixed or moving (*property iii*). Moreover, if, as is often the case, the original strategy involves constant rotational motion, that motion is preserved by the agents in their assigned orbits (*property iv*). Immediate applications of this ability are asteroid observation [11] and satellite formation [12]. The proposed strategy that adds properties i, ii and iii to the classical CFT problem is based on a transformation between equivalent topological spaces. This transformation was first used by the authors in [21] to solve the distributed region following problem. The algorithm proposed in this paper, however, extends the previous results in two different aspects. The first is the *new capabilities* offered to the team, in particular properties i and iii cannot be addressed using [21]. The second is the *ease of use*. Due to the complete independence between the original strategy for CFT and the proposed algorithm, it is possible to use the former as a black box. Therefore, using an analogy with the IT devices, we can consider the new algorithm as a *Plug & Play* component for the original strategy that generates the proper command signals. An important aspect, finally, is the choice of modeling the agents as single integrators, which is a kinematic model or basically points without mass. Considering only the kinematics, allows us to control any type of aerial robot if we consider the generated trajectory as the reference for a low-level on-board controller [13], [14].

II. Preliminaries

Notation: $p = \left[p^{(1)}, p^{(2)}, p^{(3)}\right]^\top \in \mathbb{R}^3$ is a point in $\mathbb{R}^3$; $\overline{p} = \left[p^{(1)}, p^{(2)}\right]^\top \in \mathbb{R}^2$ is the projection of the point $p \in \mathbb{R}^3$ on the x-y plane of its reference frame; $\angle(p)$ is the angle

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[1]A. Bono, L. D'Alfonso and G. Fedele are with the Department of Informatics, Modeling, Electronics and Systems Engineering, University of Calabria, Rende 87036, Italy. {antonio.bono, luigi.dalfonso, giuseppe.fedele}@unical.it

[2]V. Gazi is with the Department of Control and Automation Engineering, Faculty of Electrical and Electronics Engineering, Yıldız Technical University, Esenler 34220, Turkey. vgazi@yildiz.edu.tr

between $p \in \mathbb{R}^n, n \in \{2, 3\}$ and the unit vector of the x-axis, i.e., $\angle(p) = \arccos(p^{(1)}/\|p\|)$; given any set $\mathcal{S}$ we denote its boundary with $\tilde{\mathcal{S}}$; $\mathcal{B}(q, \rho) = \{p \in \mathbb{R}^2 | \|p - q\| \leq \rho\}$ is the disk of center $q \in \mathbb{R}^2$ and radius $\rho \in \mathbb{R}_{>0}$; given the interval $\mathcal{A} = [-\pi, \pi)$ and a set $\mathcal{S}$, the function $b_{\mathcal{S}} : \mathcal{A} \to \mathbb{R}^2, \tilde{\mathcal{S}} = \{b_{\mathcal{S}}(\theta), \theta \in \mathcal{A}\} = b_{\mathcal{S}}(\mathcal{A})$ defines its boundary. For the sake of notation, the parametrization of $\tilde{\mathcal{S}}$ will be denoted as $\tilde{\mathcal{S}}(\theta)$.

Star-shaped sets. This work uses the concept of *star-shaped sets*. To facilitate the comprehension of the following results, we summarize here some important properties of such sets. Firstly, let us give a formal definition of a star-shaped set in the 2D Euclidean space according to [15]. A set $\mathcal{S}$ in the Euclidean space $\mathbb{R}^2$, with non empty interior, is *star-shaped* at $v \in \mathcal{S}$ if, for all $p \in \mathcal{S}$, the line segment joining v and p is contained in $\mathcal{S}$. A star-shaped set $\mathcal{S}$ compliant with the previous definition is a *strictly* star-shaped set if for each $p \in \tilde{\mathcal{S}}$, the line segment joining p and v intersects $\tilde{\mathcal{S}}$ only at p. Usually, v is called the *vantage point*, and the set of all vantage points is the *kernel* of the star shaped set, namely $\ker(\mathcal{S})$. A distinctive property of the star-shaped sets is the possibility to be shrunk into themselves. Formally a compact set $\mathcal{Q} \in \mathbb{R}^2$ is defined shrinkable iff $\forall \alpha \in [0, 1)\ \exists\, p \in \mathbb{R}^2$ s.t. $\alpha \mathcal{Q} + p \in \mathcal{Q}$. Finally, it is known that star shaped sets are homeomorphic to a disk [15], which is a useful property.

The following choice about star-shaped sets is made:

Assumption A. *We consider only strictly star-shaped sets whose border is a simple, closed curve (also known as Jordan curve [16]) that admits a differentiable radial parametrization with respect to the chosen vantage point v.*

It is then assumed that there exists a radial parametrization of the boundary curve [17]. Since the sets are strictly star-shaped, there is one and only one point of the boundary connected to the vantage point for any angle, and therefore, the parametrization is a bijective function.

III. Problem definition

The main purpose of the paper is organizing a swarm of agents into groups that occupy different spatial regions, all passing through a point known to all agents. This moving point represents the target that the swarm aims to envelop and thus, in a sense, capture. Each group is assigned a region configured as two-dimensional ring of arbitrarily small thickness whose outer and inner edges are formed by a star curve. The inner edge, in particular, is a shrinked version of the outer curve. These rings lie in different planes all passing through the reference and moving alongside it, similarly to what happens with orbits of celestial bodies or satellites. As model for the agents we use a higher-level or kinematic one that ignores the dynamics of the individual agent. This choice, which at first seems simplistic, has proven effective in realistic contexts through its use as a reference trajectory for real robots (some aerial robots, all differentially flat systems [18]) provided that at least one point of the body of the robot can asymptotically track any (smooth) trajectory [19]. A good overview of the wide potential of the kinematic models is provided in [20] and the references therein. To formally define this problem some definitions are required.

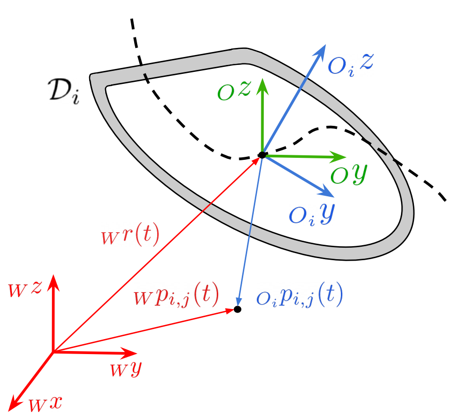

Fig. 1: An example of a star-shaped ring $\mathcal{D}_i$ and the related reference frames. The dashed line, in particular, represents the path of the target $r(t)$ that is the origin of $\{O\}$ and $\{O_i\}$.

A. Multiple Rings Tracking Problem

Let's consider a swarm of N agents moving in the 3D space organized in M groups such that $\sum_{i=1}^{M} N_i = N$ where N_i is the size of the ith group. We assume that some agent pairs know (or exchange information about) their relative position through a bidirectional information/interaction network which can be modelled by an undirected simple graph $\mathcal{G}(\mathcal{V}, \mathcal{E}), \quad |\mathcal{V}| = N$. Here the vertexes set $\mathcal{V}$ models the agents and we assume the network to be *connected*.

We label the fixed world reference frame $\{W\}$ and the moving reference frame corresponding to the point to be tracked $\{O\}$, both with right-handed coordinates. We assume both with the same orientation and with a displacement described by $r(t) \in \mathbb{R}^3$ with the function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ continuously differentiable. All the planes where the star-shaped rings lie pass through such a point and their orientation is described with M reference frames centered at $r(t)$ and with the z-axis perpendicular to the plane. We label such frames as $\{O_i\}, i = 1, \ldots, M$ and their orientation is expressed by the rotation matrices $R_i \in SO(3), i = 1, \ldots, M$. Figure 1 provides an example of such planes and the discussed reference frames. Agents are modelled as

$$_{W}\dot{p}_{i,j}(t) = {_{W}u_{i,j}}(t) \quad i = 1, \ldots, M,\ j = 1, \ldots, N_i \qquad (1)$$

where $_{W}p_{i,j}(t), {_{W}u_{i,j}}(t) \in \mathbb{R}^3$ account for agent's position and local control law, respectively. In particular, the first and second right subscripts denote the agent's group and its number within the group, respectively; the left subscript, instead, indicates the reference frame with respect to which a quantity is expressed (in this case the world reference frame). Using the function $\psi : \mathbb{R}_{\geq 0} \to [0, 2\pi)$ we define

$$\psi_{i,j}(t) \triangleq \angle({_{O_i}\overline{p}_{i,j}}(t)) \qquad (2)$$

the angle formed by the projection of the agent $\{i, j\}$ on the plane $_{O_i}x$-$_{O_i}y$. In order to define the star-shaped ring where the agents of the ith group must converge, we introduce

$$ {}_{O_i}\hat{\mathcal{S}}_i \triangleq \{s \in \mathbb{R}^3 \mid \left[s^{(1)},\, s^{(2)}\right]^\top \in \mathcal{S}_i,\; s^{(3)} = 0\} \quad (3) $$

i.e., the star shaped set defined by $\mathcal{S}_i$ lying on the x-y plane of the ith reference frame $\{O_i\}$. Then, the star-shaped ring will be described by the set difference

$$ {}_{O_i}\mathcal{D}_i \triangleq {}_{O_i}\hat{\mathcal{S}}_i \setminus (1-\sigma)\,{}_{O_i}\hat{\mathcal{S}}_i, \quad \sigma \in (0,1] \quad (4) $$

where σ is the *shrinkability* factor of the star-shaped set. Looking at Figure 1, it is easy to appreciate the two transformations needed to express such a set in the world fixed frame: the first one is the rotation with respect to $\{O\}$ that, for the sake of notation, we improperly denote as ${}_O\mathcal{D}_i \triangleq R_i\,{}_{O_i}\mathcal{D}_i$; the second one is the translation w.r.t to $\{W\}$ that leads to ${}_W\mathcal{D}_i(t) \triangleq {}_O\mathcal{D}_i \oplus r(t)$, where the plus symbol '$\oplus$' denotes the Minkowski sum operator. It is worth noting that, in order to express the ring region in the world reference frame, it is necessary to introduce the time dependence since the reference $r(t)$ moves. Finally, we can summarize these two transformations as

$$ {}_W\mathcal{D}_i(t) \triangleq {}_O\mathcal{D}_i \oplus r(t) = \left(R_i\,{}_{O_i}\mathcal{D}_i\right) \oplus r(t). \quad (5) $$

It is now reasonable to make the following assumption:

Assumption B. *The target position described by $r(t)$, with the function $r : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ continuously differentiable, is a vantage point of any $\mathcal{S}_i$, i.e., $r(t) \in \ker(\mathcal{S}_i)$, $i = 1, \ldots, M$.*

The main goal of the proposed strategy is just to design a control law for each agent, ${}_W u_{i,j}(t)$, that drives it inside ${}_W\mathcal{D}_i(t)$ and, once inside, it never leaves it again. With this notation we can now formally define the addressed problem:

Problem 1 (Multiple Orbits Tracking Problem (MOTP))**.** *Consider a swarm of N agents organized in M groups according to $\sum_{i=1}^{M} N_i = N$, modelled as single integrators whose positions are controlled via* (1)*. Recalling assumptions A and B and definitions* (2)-(5)*, for any arbitrary small $\sigma \in (0,1]$, design a control law that $\forall i = 1, \ldots, M, \forall j = 1, \ldots, N_i$ meets the following requirement*

$$ {}_W p_{i,j}(\underline{t}) \in {}_W\mathcal{D}_i(\underline{t}) \to {}_W p_{i,j}(t) \in {}_W\mathcal{D}_i(t),\; \forall t \geq \underline{t} \quad (6) $$

and respects alternatively one of the two following conditions

$$ \begin{aligned} &\forall\, m \in \mathbb{Z}_{\geq 0},\; !\exists\, T \in \mathbb{R}_{>0},\; \exists\, t^* \in \mathbb{R}_{>0}\ s.t. \\ &\{\psi_{i,j}(t+T) = \psi_{i,j}(t), \forall t \geq t^*\} \wedge \\ &\left\{\psi_{i,j}(\cdot) \text{ is piecewise-monotonic in } [t^*+mT,\, t^*+(m+1)T]\right\} \end{aligned} \quad (7a) $$

$$ \lim_{t\to\infty} {}_{O_i}p_{i,j}(t) \in {}_{O_i}\mathcal{D}_i \quad (7b) $$

Condition (6), in particular, implies that once any agent enters the related ring, it cannot escape from it anymore. Condition (7a), instead, expresses how the projection of individual agent on the ${}_{O_i}x$-${}_{O_i}y$ plane exhibits a rotational movement, since the angle is described by a periodic function that is also piecewise-monotonic. Please note that all the agents rotate cohesively since the evolutions of their angles have the same period T. Finally (7b) describes the case when the position of the single agent relative to the reference becomes constant thus reaching an overall *consensus of velocity among the agents and the target*. It is worth noting that, in both the variants of the problem described by (7a) and (7b), our interest is towards rings with a very small thickness ($\sigma \ll 1$) because we aims at arranging the agents in specific shapes around the target. Since because of the scalability issue, in this strategy we do not set a specific position for each agent with respect to the reference, the more space there is among the boundaries of the rings, the less probably agents will show the desired shape. In order to solve the MOTP, the following conditions are prescribed: i) each agents has information about the ring to reach via the triple $\{\mathcal{S}_i,\, R_i,\, \sigma\}$; ii) each agent knows the velocity of target to be tracked ${}_W r(t)$, ${}_W\dot{r}(t)$ and its own position ${}_W p_{i,j}(t)$; iii) agents are able to exchange their own position data with their neighbors according to the network described by $\mathcal{G}$.

B. Equivalent problem

Driving a group of agents in a ring of any arbitrary star-shaped curve is not a trivial task. Following the same lines of [21], we will redefine our problem in a *topologically equivalent space* where designing the control law is much simpler and then convert it back to the original space. **Diffeomorphism.** The main idea is to define a bijection $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ that maps any point on the boundary of the star-shaped set into a point on a circle

$$ p \in \tilde{\mathcal{S}} \to \phi(p) \in \tilde{\mathcal{B}}(v, \rho) \quad (8) $$

where $v \in \mathcal{S}$ is any selected vantage point of the star-shaped set $\mathcal{S}$ and $\tilde{\mathcal{B}}(v, \rho)$ is the circle related to the disk $\mathcal{B}(v, \rho)$. For our strategy it is fundamental that such a map has an inverse function ϕ^{-1} that is continuously differentiable, i.e., ϕ is a diffeomorphism [17]. We will refer to the topologically equivalent space described by $\phi(\cdot)$ as *circular space*. This map must satisfy one additional condition

$$ \phi\left(\mu\,\tilde{\mathcal{S}}(\theta)\right) \in \tilde{\mathcal{B}}(v, \mu\rho), \quad \mu \in \mathbb{R},\; p, v \in \mathbb{R}^2, \quad (9) $$

stating that a point in a scaled version of the star-shaped curve is translated in a circle scaled of the same factor.

Algorithm 1 Conversion from MOTP to COTP and viceversa

Require: ${}_W\dot{r}(t)$, ${}_O p_{i,j}(t)$, $\mathcal{S}_i$, R_i, σ, ρ.
Ensure: ${}_W u_{i,j}(t)$ such to solve MOTP.
1: ${}_{O_i}p_{i,j}(t) = R_i^\top\,{}_O p_{i,j}(t)$.
2: $q_k(t) = \phi_i\left({}_{O_i}\overline{p}_{i,j}(t)\right)$.
3: Compute $w_k(t)$ such to solve the COTP.
4: ${}_{O_i}\overline{u}_{i,j}(t) = J_{\phi_i}\left({}_{O_i}\overline{p}_{i,j}(t)\right)^{-1} w_k(t)$.
5: ${}_{O_i}u_{i,j}(t) = \left[{}_{O_i}\overline{u}_{i,j}(t)^\top, -\tau\,{}_{O_i}p_{i,j}^{(3)}(t)\right]^\top, \quad \tau \in \mathbb{R}_{>0}$.
6: ${}_O u_{i,j}(t) = R_i\,{}_{O_i}u_{i,j}(t)$.
7: ${}_W u_{i,j}(t) = {}_O u_{i,j}(t) + {}_W\dot{r}_i(t)$

Transformation. A diffeomorphism compliant with conditions (8) and (9) is $\phi(p) = v + \frac{\rho}{\left\|\tilde{\mathcal{S}}(\angle(p-v))-v\right\|}(p-v)$, where $\angle(p-v)$ is the angle formed by the displacement between the point p to be transformed and the vantage point v. Condition (8) is respected and, thanks to the homogeneity property of the norm function, also condition (9). Let us now consider only the projections of the agents on the plane where the star-shaped ring lies, i.e., ${}_{O_i}\overline{p}_{i,j}$. For the control law design, we need to *transform all the agents to the* same *circular space*. Such a result is viable using a family of diffeomorphisms ϕ_i centered on the same vantage point v and with the same scaling factor ρ. Thanks to Assumption B we can choose $r(t)$ as common vantage point and, recalling that in $\{O_i\}$ the target $r(t)$ coincides with the origin $\left({}_{O_i}r(t) = 0\right)$, then

$$\phi_i\left({}_{O_i}\overline{p}_{i,j}\right) = \frac{\rho}{\left\|\tilde{\mathcal{S}}_i\left(\angle\,{}_{O_i}\overline{p}_{i,j}\right)\right\|}\,{}_{O_i}\overline{p}_{i,j},\ i = 1,\ldots,M. \tag{10}$$

For the sake of the readability, we denote 2D points converted to the circular space as $q_k(t) \triangleq \phi_i\left({}_{O_i}\overline{p}_{i,j}(t)\right)$, where the index $k = 1,\ldots,N$ is given by $k = j + \sum_{l=1}^{i-1} M_l$ (note how knowing all groups size N_i, (i,j) can be retrieved from k). Now, since all the agents are mapped to the same circular space, it is necessary to guarantee that two agents with different initial conditions in the original space are not mapped to the same point using (10). Therefore, let's develop the condition $q_k(t_0) \neq q_l(t_0)$ as

$$\rho\,\frac{\left\|{}_{O_i}\overline{p}_{i,j}(t_0)\right\|\,{}_{O_i}d_{i,j}(t_0)}{\left\|\tilde{\mathcal{S}}_i\left(\angle\,{}_{O_i}\overline{p}_{i,j}(t_0)\right)\right\|} \neq \rho\,\frac{\left\|{}_{O_m}\overline{p}_{m,n}(t_0)\right\|\,{}_{O_m}d_{m,n}(t_0)}{\left\|\tilde{\mathcal{S}}_m\left(\angle\,{}_{O_m}\overline{p}_{m,n}(t_0)\right)\right\|}$$

where and ${}_{O_i}d_{i,j}, {}_{O_m}d_{m,n} \in \mathbb{R}^2$ are direction vectors of ${}_{O_i}\overline{p}_{i,j}$ and ${}_{O_m}\overline{p}_{m,n}$, respectively. This condition is true if at the initial time t_0, for any agent pairs in the swarm labeled ${}_{O_i}p_{i,j}$ and ${}_{O_m}p_{m,n}$, it holds true that ${}_{O_i}d_{i,j}(t_0) \neq {}_{O_m}d_{m,n}(t_0)$ or that $\frac{\left\|\tilde{\mathcal{S}}_i\left(\angle\,{}_{O_i}\overline{p}_{i,j}(t_0)\right)\right\|}{\left\|\tilde{\mathcal{S}}_m\left(\angle\,{}_{O_m}\overline{p}_{m,n}(t_0)\right)\right\|} \neq \frac{{}_{O_i}\overline{p}_{i,j}(t_0)}{{}_{O_m}\overline{p}_{m,n}(t_0)}$. The agent motion in the same space will be $\dot{q}_k(t) = w_k(t),\ k=1,\ldots,N$, where $w_k(t)$ is the control law to be designed in the circular space. **Problem Definition.** Using the defined transformation and $\psi_k(\cdot)$ in the same way of (2), the MOTP translates into:

Problem 2 (Circular Orbit Tracking Problem (COTP))**.** *Consider a swarm of N agents modelled as points without mass whose position is controlled as $\dot{q}_k(t) = w_k(t), k = 1,\ldots,N$. Given $\mathcal{C} \triangleq \mathcal{B}(0,\rho) \setminus \mathcal{B}(0,(1-\sigma)\rho)$, for any arbitrary small $\sigma \in \mathbb{R}_{>0}$, design a control law that $\forall k = 1,\ldots,N$ meets the requirement $q_k(\underline{t}) \in \mathcal{C} \to q_k(t) \in \mathcal{C} \quad \forall t \geq \underline{t}$ and respects alternatively one of the two following conditions*

$$\begin{aligned}&\forall\, m \in \mathbb{Z}_{\geq 0},\ !\exists\, T \in \mathbb{R}_{>0},\ \exists\, t^* \in \mathbb{R}_{>0}\ s.t.\\ &\{\psi_k(t+T) = \psi_k(t), \forall t \geq t^*\} \wedge\\ &\{\psi_k(\cdot) \textit{ is piecewise-monotonic in } [t^* + mT,\ t^* + (m+1)T]\}\end{aligned} \tag{11a}$$

$$\lim_{t\to\infty} q_k(t) \in \mathcal{C}. \tag{11b}$$

Comparing this problem with the original one, it becomes clear how the role of the multiple star-shaped rings is taken over by the single ring $\mathcal{C}$ by virtue of the family of diffeomorphisms (10) that bridges them. The rotational movement inside the ring is described by (11a) in the same way used in the MOTP. It is worth nothing that all the agents have the same angular velocity (period of rotation) so that there is an orderly movement. Since this angular velocity is maintained by the diffeomorphism, the agents in the original space will have the same angular velocity but different linear velocities according to the different shapes of the orbits. The swarm fixed configuration, instead, is described by (11b) and corresponds to the velocity consensus between agents and target requested in (7b). Finally, the assumptions for this problem are in line with the ones for the MOTP: each agent is aware of the ring to reach as $\{\rho,\sigma\}$; each agent knows its own position $q_k(t)$; agents are able to exchange their own position data with their neighbors as prescribed by the connected graph $\mathcal{G}$; for any agent pair $(k,l), k,l = 1,\ldots,N,\ k \neq l$, at the initial time t_0, $q_k(t_0) \neq q_l(t_0)$. **Conversions.** Algorithm 1 can be defined to convert the problem of interest into the circular space and to reconvert the obtained solution. First of all, the agent's position with respect to the target is trasformed in the reference frame related to the ith ring, i.e., $\{O_i\}$ (line 1). As a second step, the diffeomorphism $\phi_i(\cdot)$ is applied to the projection of the agents on the x-y plane of $\{O_i\}$ (line 2). The control law that solves the COTP is the obtained (line 3) and it is converted (line 4, according to [22], through the jacobian matrix $J_{\phi_i}\left({}_{O_i}\overline{p}_{i,j}\right) \triangleq \frac{\partial}{\partial_{O_i}\overline{p}_{i,j}}\phi_i\left({}_{O_i}\overline{p}_{i,j}\right)\Big|_{{}_{O_i}\overline{p}_{i,j}}$, that is always invertible since a diffeomorphism for the radial mapping is assumed). Since we want the agents to lie on the x-y plane of $\{O_i\}$, the control law is completed by adding a decaying term for the third dimension (line 5). The result

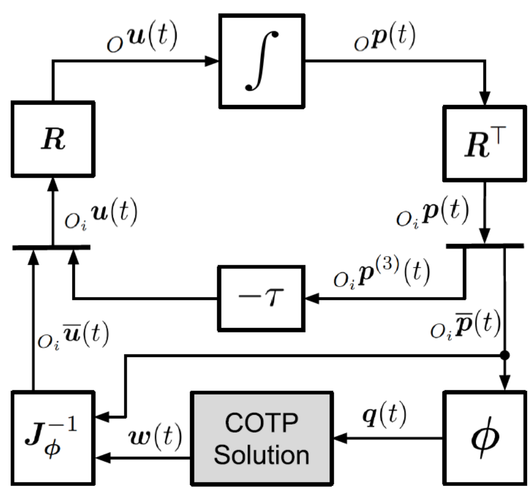

Fig. 2: A block diagram representing Algorithm 1. Bold letters indicate the vectors in which the relative values of the individual agents are contained (e.g. $\boldsymbol{q}(t) = [q_1(t)^\top, q_2(t)^\top, \ldots, q_N(t)^\top]^\top$).

is transformed in the frame $\{O\}$ attached to the target $r(t)$ (line 6) and, as a final step, the command in the world frame is obtained by adding the velocity of the target (line 7). These steps are shown in Figure 2 through a block diagram.

A specular interpretation. In what has been presented so far, we have described how to solve MOTP through a conversion to a topologically equivalent space by transforming the problem into a much simpler version (COTP). This sequence of conversion and reconversion, however, can also be read in a specular way: given a solution for the COTP problem, we can extend these results to generic forms (defined by star-shaped sets) in 3D space. This interpretation is quite interesting when we consider the many works that have solved the two versions of COTP in recent years (e.g. [23], [21]). For such solutions our algorithm is, in fact, *a Plug & Play command generator*. Considering the broad application perspective, the proposed strategy could be helpful to use the properties of an existing swarm formation protocol and extend it to a more general scenario.

IV. EXAMPLES

The purpose of this section is to show the capability of the proposed algorithm by considering two different versions of COTP: in the first, the final position of the agents is fixed in compliance with (11b); in the second, instead, the agents exhibit rotatory motion thus satisfying (11a).

Swarm stationary configuration. This example aims at showing how the rings can be defined by totally different shapes and agents can maintain stationary positions. Concerning the COTP solution, the solution described in [23] has been used. There the authors use potential fields generated from a bivariate normal function $f(x, y) = e^{-\alpha\left((x-x_c)^2+\gamma(y-y_c)^2\right)}$ where (x_c, y_c) is the center of the ellipse and γ is the ratio of the minor axis (y-direction) to the major axis (x-direction). In this example $\gamma = 1$ to obtain a circle and the center is the origin, $(x_c, y_c) = (0, 0)$. Thanks to two sigmoid limiting functions, $S_{\text{in}}, S_{\text{out}}$, agents arrange themselves in a circular ring whose outer and inner borders are defined by the circles with radii $R^* + \Delta R_{\text{out}}$ and $R^* - \Delta R_{\text{in}}$, respectively. Once inside the ring, agents can equally disperse themselves thanks to another potential field SGN used together the limiting function $N_\perp$. Then the overall control law for the individual agent is

$$w_k = (S_{\text{in}}(q_k) - S_{\text{out}}(q_k))\,\nabla f\left(q_k^{(1)}, q_k^{(2)}\right) + SGN*$$
$$N_\perp(q_k)\nabla f\left(q_k^{(1)}, q_k^{(2)}\right) + \sum_{l=1, l\neq k}^{N} S_{\text{avoid}}(q_k, q_l)\begin{bmatrix} d_{x\_\text{avoid}} \\ d_{y\_\text{avoid}} \end{bmatrix}$$

where the last term is devoted to collision avoidance. This solution of COTP is used in the following example.

Example 1. Let's consider a team of $N = 16$ agents divided in two equal groups $N_1 = N_2 = 8$. The first group must arrange itself inside an ellipsoidal ring $\mathcal{D}_1$ based on ellipse $\mathcal{S}_1 = \left\{[x, y]^\top \in \mathbb{R}^2 \,|\, x^2/8^2 + y^2/5^2 \leq 1\right\}$. For the sake of simplicity, we set as vantage point the center of the ellipse, in such a way as to take advantage of the well-known radial parametrization of the ellipse $\tilde{\mathcal{S}}_1(\theta) = [a\cos(\theta), b\sin(\theta)]^\top$. The second group, instead, has to go inside a ring $\mathcal{D}_2$ defined by a set whose boundary is a rhodonea curve (see [24]) with five petals, i.e., $\tilde{\mathcal{S}}_2(\theta) = \left[(3+\sin(5\theta))\sin(\theta), -(3+\sin(5\theta))\cos(\theta)\right]^T$ where the vantage point is the center. The shrinkability factor of the curves is $\sigma = 0.1$. For the sake of readability we choose as target the origin (${}_W r(t) = [0,\,0,\,0]^\top$) and as planes for the rings the ${}_O x$-${}_O y$ plane, i.e., $R_1 = R_2 = I_3$. The agents start from the same plane, the communication graph is complete and $\rho = 25$. Using the COTP solution discussed above, in order to compute $w_k(\cdot)$ we used $\alpha = 0.01$, $\Delta R_{\text{avoid}} = 1$, $\kappa = 1$ and $\varepsilon = 1$. To make the circle with radius ρ the external border of the ring and $(1-\sigma)\rho$ the inner one, we chose $\Delta R_{\text{in}} = \Delta R_{\text{out}} = \Delta R$, $R^* = \rho - \Delta R$ and $\Delta R = \sigma R^*/(2-\sigma)$. Figure 3 shows the evolution of the agents from the initial position to the final one.

Swarm rotational movement. The ability of the algorithm to transform the rotatory motion within the starting circular ring $\mathcal{C}$ (see condition (11a)) into an ordered rotatory motion in the desired orbits (see condition (7a)) is now shown. As solution of COTP, the one we developed in [21] has been used. The agents kinematics evolve as:

$$\dot{q}_k = w_k = -\alpha q_k + (1-\sigma)^2 q_k \left(q_k^T q_k - (1-\sigma)^2\rho^2\right)^{-1}$$
$$+ \sum_{l=1, l\neq k}^{N} \varepsilon_I(\|q_k - q_l\|)(q_k - q_l) + h_k(t), \quad k = 1, \ldots, N.$$

The control law is formed by three main terms. The first one accounts for attraction, where $\alpha > \rho^{-2}\, W_0\left(e^{-1}\right)$, with $W_0(\cdot)$ the principal branch of the Lambert function [25], and $\sigma \in (0, 1]$ is the shrinkability factor used in (4). This term drives the agents to the disk $\mathcal{B}(0, (1-\sigma)\rho)$ simultaneously ensuring that they never enter it.As a second term, a summation that considers the interactions of the individual agent with all the other connected ones according to $\mathcal{G}$ is concerned, aiming at avoiding collisions among team

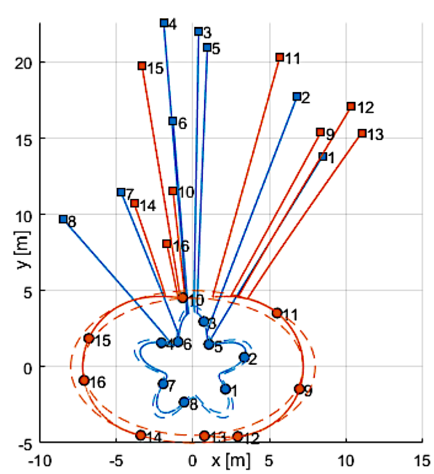

Fig. 3: Example 1: paths of the agents from the initial positions (squares) to the final ones (disks). The two groups are shown with different colors and the numbers are the k indexes of the agents.

members. As a last term, $h_k(t)$ provides a rotational behavior since it is designed to be always tangential to the boundary of $\mathcal{C}$, i.e., $h_k(t) \triangleq \omega A q_k(t), \quad \omega \in \mathbb{R}_{\geq 0}, A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ where ω is the desired angular speed for k-th agent. According to [21], $\alpha > \max\left\{W_0\left(e^{-1}\right)\rho^{-2}, \underline{\alpha}\right\}$ has to be used, with $\underline{\alpha} \triangleq (1-\sigma)^2 + N(N-1)\lambda\sigma(2-\sigma)\left(\rho^2\sigma(2-\sigma)\right)^{-1}$. This solution of COTP is used in the following example.

Example 2. In this example we consider a group of $N = 18$ agents divided in three groups whose sizes are $N_1 = 8$, $N_2 = 6$, $N_3 = 4$. We want them to rotate inside three rings $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ defined by the same ellipse of the previous example with a shrinkability factor $\sigma = 0.1$. These rings lay on three different planes defined by the rotation matrices. $R_1 = R_y(0) = I_3$, $R_2 = R_y(\pi/3)$, $R_3 = R_y(2/3\,\pi)$, where $R_y(\theta) \in SO(3)$ is the matrix describing a rotation of θ radians around the y-axis. This rings are centered on the common reference ${}_W r(t) = [0,\, 0,\, t/10]^\top$. Considering the agents starting from the ${}_W x$-${}_W y$ plane and a connected communication graph, we applied Algorithm 1 choosing $\rho = 30$. To compute $w_k(\cdot)$ we applied the proposed method with $\delta = 100, \eta = 0.9, \lambda = 1.8, \omega = 0.05, \tau = 1$. With these settings we computed $\alpha = 1.01\,\underline{\alpha}$. The evolution of the agents can be appreciated in the video: https://www.youtube.com/watch?v=MPL04Jc8Trg. A snapshot of the evolution is depicted in Figure 4.

V. Conclusions

In this paper, the authors provided an algorithm capable of arranging groups of agents in arbitrarily shaped orbits around a moving reference. Given the modeling of agents as generic points without mass and its connection to the traditional circular formation tracking problem, the presented algorithm succeeds in extending many existing solutions to 3D space by providing a Plug & Play scheme with freedom of choice on the desired shapes.

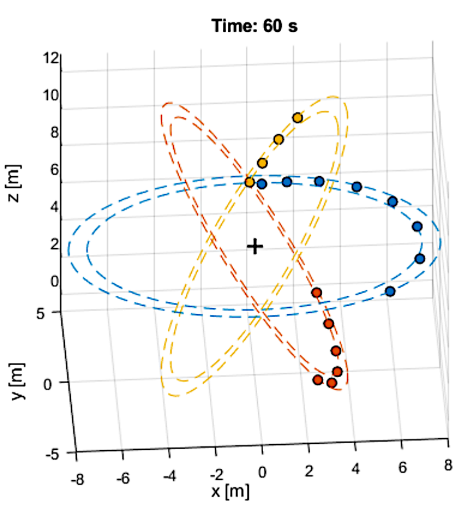

Fig. 4: Example 2: a snapshot of the agents evolution at $t = 60\,\text{s}$. The groups are shown in different colors together with the related ring. The black cross is $r(60) = [0,\, 0,\, 6]^\top$.

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