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Leader-Follower Formation of Second-Order Agents via Delayed Relative Displacement Feedback

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Abstract— **This paper investigates a leader-follower formation of a group of second-order agents, considering an undirected and connected topology. It is assumed that no velocity information is available, and that a uniform delay affects the processing of the displacement information. To address these issues, a delayed relative displacement feedback is introduced. A necessary and sufficient condition for stability is derived in terms of a delay threshold, given that the delay-free controller is stable. Moreover, a stability region in the complex plane for the eigenvalues of the Laplacian interaction matrix is introduced. The results are illustrated through numerical examples.**

Index Terms— **distributed control, stability of linear systems, delay systems, networked control systems.**

I. INTRODUCTION

Consensus theory has gained significant attention in recent years as a distributed control method for multi-agent systems. One of the most studied applications of consensus is the leader-follower formation of autonomous vehicles. Some prominent results can be found in [1], where consensus is achieved employing a relative displacement and velocity feedback, coupled through a Laplacian interaction matrix.

However, the availability of velocity measurements may be unfeasible for some mission scenarios, as in [2]. Some examples, in which consensus protocols can rely solely on the displacement information, can be found in literature. The work of [3] augments the state of the controller with two internal variables to reach consensus through a passivity approach. In [4], an observer-type consensus controller is introduced for general linear systems, and it is extended to a leader-follower topology in [5]. The work of [6] deals with the unavailability of relative velocity measurements through an augmented state variable, despite assuming that each agent is aware of its own velocity information. Finally, in [7], an internal controller state is used to achieve consensus in both position and velocity, considering the unavailability of any velocity measurement. This result stems from a stability condition [8] of a dynamical system described by coupled differential equations of first- and second-order, and it is extended to a leader-follower framework in [9].

Another issue, frequently encountered in consensus, is the presence of delays in the interaction network. Time delays are classified into *communication delays*, affecting the information coming from neighboring agents as in [10], *input delays*, acting on the agents' self-states as in [11], and *information processing delays*, impacting both as in [12]. A trade-off between network connectivity and tolerance against delays is presented in [12] for first-order delayed linear systems. Moreover, in [10], it is shown that communication delays alone can not yield instability. However, the behaviour of autonomous vehicles is better modelled through a second-order dynamics. In this case, communication delays can lead the system to instability, and the delay-free controller has to satisfy certain conditions to ensure the possibility of reaching consensus in the delayed case, as proved in [13].

Frequency domain approaches are commonly employed to deal with delayed second-order systems, in which the factorization of the characteristic equation allows to decouple the system modes. These procedures simplify considerably the analytical formulation, and are able to provide the exact delay threshold for stability change. The work of [14] applies this method to a group of double-integrator agents performing a rendez-vous, and it is extended to a leader-follower topology in [15] and to generate a formation in [16]. A similar procedure is employed in [17] to deal with a general second-order dynamic system, and is further developed by introducing a stability region for the proportional-derivative control gains in [18], and for diverse time delays in [19]. Also the work of [20] employs a similar approach, and introduces an integral term in the consensus protocol. However, these methods consider the availability of velocity measurements. Also, the stability analysis has to be repeated for each of the system modes, which are as many as the number of agents.

In this paper, we consider the case in which only relative displacement sensors are installed on-board, so that no velocity information is available. Moreover, we assume that the controller is able to handle the measurements after a certain processing delay. A relative displacement feedback as in [7], [21] is employed to obtain a stable controller in the delay-free case. Then, it is proved that this controller is able to reach consensus in both position and velocity if and only if the delay is lower than a certain threshold, which is related to the largest eigenvalue of the Laplacian matrix. Different from [22], we consider the undelayed displacement information unavailable. Different from [23], we consider the agents' own velocities unavailable. Also, as a key point, we define a stability region in the complex plane for the eigenvalues of the Laplacian matrix, that allows to promptly assess the stability of the system even with a large number of agents. Indeed, the link between the delay threshold and the largest eigenvalue of the Laplacian matrix is explicitly

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stated. A similar stability region was discussed in [4] and [5] for an undelayed controller.

The paper is organized as follows. In Section II, the relative displacement feedback is introduced, along with the information processing delay. Section III presents the necessary and sufficient conditions for the stability of the system, while Section IV provides some numerical examples. In Section V, several concluding remarks are given. An Appendix contains some important theoretical results.

II. PRELIMINARIES

Consider N *second-order agents* called *followers*

$$m\ddot{x}_i(t) = u_i(t), \quad i = 1,\dots,N, \tag{1}$$

being $m > 0$ the mass, $x_i(t) \in \mathbb{R}$ the position, and $u_i(t) \in \mathbb{R}$ the control input of agent i, and a free-motion *leader*

$$m\ddot{x}_0(t) = 0, \tag{2}$$

as in [9]. Denote as $h_i - h_0$ the desired relative displacement between agent i and the leader.

The objective of this work is to establish a consensus control law achieving a *leader-follower formation*

$$\lim_{t\to\infty}(\dot{x}_i(t) - \dot{x}_0(t)) = 0, \quad \lim_{t\to\infty}(x_i(t) - x_0(t)) = h_i - h_0,$$

for $i = 1,\dots,N$, assuming that no velocity information is available, and that a uniform delay affects the displacement information processing.

To this end, consider the following *delayed relative displacement feedback*

$$u_i(t) = -z_i(t) - \gamma p \sum_{j\in\mathcal{N}_i}\big((x_i(t-\tau) - h_i) - (x_j(t-\tau) - h_j)\big) \tag{3}$$

$$\frac{\dot{z}_i(t)}{\gamma} = -z_i(t) - (\gamma p - q)\sum_{j\in\mathcal{N}_i}\big((x_i(t-\tau) - h_i) - (x_j(t-\tau) - h_j)\big), \tag{4}$$

for $i = 1,\dots,N$, where $z_i(t) \in \mathbb{R}$ is the internal state of the controller, γ, p and q are positive parameters, and $\mathcal{N}_i \subseteq \{0,1,2,\dots,N\}$ is the set of agents adjacent to agent i. The delay $\tau \geq 0$ is assumed to be uniform and to affect the position information but not the internal state $z_i(t)$, which is promptly available to the controller as in [23].

The reason behind the choice of this controller is that, when $\tau = 0$, the substitution of the Laplace transform of the state (4) into the control input (3) leads to

$$u_i(s) = -\frac{ps+q}{(1/\gamma)s+1}\sum_{j\in\mathcal{N}_i}\big((x_i(s) - h_i) - (x_j(s) - h_j)\big).$$

This input can be treated as a PD-type control law, if γ is large enough that the denominator dynamics of the previous equation can be neglected. Indeed, the work of [7], [21] showed that the delay-free controller ensures consensus in both position and velocity if

$$\gamma > \frac{q}{p}.$$

Throughout the paper, we assume that this condition holds.

Let us define the graph Laplacian $L \in \mathbb{R}^{N\times N}$ whose $\{i,j\}$-th element ℓ_{ij} is -1 if $j \in \mathcal{N}_i$, $\sum_{j=1,j\neq i}^{N}|\ell_{ij}|$ if $j = i$, or 0 otherwise. Similarly, we define the interaction vector between leader and followers as $l_0 \in \mathbb{R}^N$, whose i-th element ℓ_{i0} is 1 if $0 \in \mathcal{N}_i$, or 0 otherwise.

In this work, we assume that the interaction topology among followers is undirected and connected, so that

$$L = L^T \geq 0, \quad L\mathbf{1}_N = 0, \quad \mathrm{rank}(L) = N-1, \tag{5}$$

and that at least one follower has access to the leader information, so that $l_0 \neq 0$. Finally, define $\hat{L} = L + \mathrm{diag}(l_0)$.

III. MAIN RESULT

The main result of this paper is the following theorem providing a necessary and sufficient condition for stability of the second-order multi-agent system (1), following a leader (2), under the delayed relative displacement feedback (3) and (4).

Theorem 1: The system composed of (1)–(4) achieves a consensus

$$\begin{cases}\lim_{t\to\infty}(\dot{x}_i(t) - \dot{x}_0(t)) = 0\\ \lim_{t\to\infty}(x_i(t) - x_0(t)) = h_i - h_0 \qquad \forall i = 1,\dots,N\\ \lim_{t\to\infty} z_i(t) = 0\end{cases}$$

if and only if

$$\tau < \tau_N^*,$$

where τ_N^* is the largest delay such that the complex region enclosed by the first crossing of the boundary

$$\tilde{\lambda} = -ms^2\cdot\frac{(1/\gamma)s+1}{ps+q}\cdot e^{\tau s}, \qquad s = j\omega$$

with the real axis contains all the eigenvalues of $\hat{L}$.

Proof: Define matrices

$$A = \begin{bmatrix}0 & 0 & -\frac{1}{m}\\ 1 & 0 & 0\\ 0 & 0 & -\gamma\end{bmatrix}, \qquad A_d = \begin{bmatrix}0 & -\frac{\gamma p}{m} & 0\\ 0 & 0 & 0\\ 0 & -\gamma(\gamma p - q) & 0\end{bmatrix}$$

and state vector

$$X_i(t) = \begin{bmatrix}\dot{x}_i(t) - \dot{x}_0(t) & (x_i(t) - h_i) - (x_0(t) - h_0) & z_i(t)\end{bmatrix}^T$$
$$X(t) = \begin{bmatrix}X_1^T(t) & X_2^T(t) & \cdots & X_N^T(t)\end{bmatrix}^T.$$

The system composed of (1)–(4) can be rewritten as

$$\dot{X}(t) = (\mathbf{I}_N \otimes A)\cdot X(t) + (\hat{L}\otimes A_d)\cdot X(t-\tau),$$

where $\otimes$ denotes the Kronecker product. Since (5) and $l_0 \neq 0$ hold, $\hat{L}$ is symmetric and positive definite. Thus, it can be diagonalized through an orthonormal matrix $S \in \mathbb{R}^{N\times N}$, so that $S^T\hat{L}S = \Lambda$ is a diagonal matrix whose entries are the real positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ of $\hat{L}$.

Now, we can employ a variable transformation such as

$$\xi(t) = (S^T \otimes \mathbf{I}_3)\cdot X(t), \tag{6}$$

where

$$\xi(t) = \begin{bmatrix}\xi_1^T(t) & \xi_2^T(t) & \cdots & \xi_N^T(t)\end{bmatrix}^T \in \mathbb{R}^{3N},$$

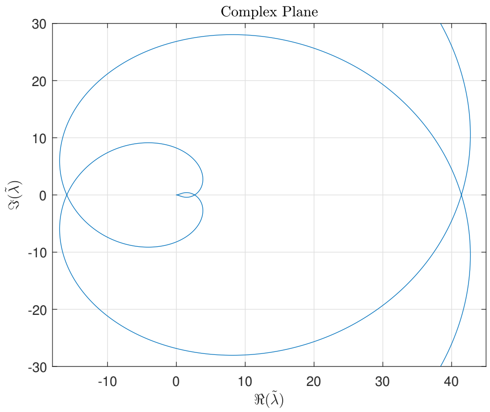

Fig. 1. Stability boundary $\tilde{\lambda}$.

so that system (6) can be decoupled as N independent subsystems like

$$\dot{\xi}_i(t) = A\cdot\xi_i(t) + \lambda_i A_d \cdot \xi_i(t-\tau). \tag{7}$$

Reminding that $\xi_i(t-\tau) \to \xi_i(s)e^{-\tau s}$ in the frequency domain, from the previous equation we get

$$s\xi_i(s) = A\cdot\xi_i(s) + \lambda_i\ A_d\cdot\xi_i(s)e^{-\tau s}.$$

Computing the determinant $\det(s\mathbf{I}_3 - A - \lambda_i A_d e^{-\tau s})$ finally leads to the Characteristic Equation (CE)

$$\lambda_i\,(ps+q)\,e^{-\tau s} + ms^2\left(\frac{s}{\gamma}+1\right) = 0. \tag{8}$$

Rewriting the previous equation for $s = j\omega$ as

$$\lambda_i = \frac{m\omega^2}{q}\cdot\frac{\frac{j\omega}{\gamma}+1}{\frac{j\omega p}{q}+1}\cdot\left(\cos(\tau\omega) + j\sin(\tau\omega)\right)$$

allows to decouple the dynamics, control, and delay parameters of the system from the eigenvalues of the interaction matrix. Besides, it provides a stability boundary in the complex plane for the eigenvalues of $\hat{L}$ that we denote as $\tilde{\lambda}$, depicted in Figure 1. Its real and imaginary parts are given by

$$\Re(\tilde{\lambda}) = \frac{m\omega^2}{q}\,\frac{\left(1+\frac{\omega^2 p}{\gamma q}\right)\cos(\tau\omega) + \left(\frac{p}{q}-\frac{1}{\gamma}\right)\omega\sin(\tau\omega)}{\left(1+\frac{\omega^2p^2}{q^2}\right)},$$

$$\Im(\tilde{\lambda}) = \frac{m\omega^2}{q}\,\frac{\left(1+\frac{\omega^2 p}{\gamma q}\right)\sin(\tau\omega) - \left(\frac{p}{q}-\frac{1}{\gamma}\right)\omega\cos(\tau\omega)}{\left(1+\frac{\omega^2p^2}{q^2}\right)}.$$

Due to the complex exponential, the stability boundary crosses the real axis an infinite number of times. Moreover, as the delay increases, these crossings move closer to the origin, i.e.

$$\left.\frac{\partial\Re(\tilde{\lambda})}{\partial\tau}\right|_{\Im(\tilde{\lambda})=0} < 0, \tag{9}$$

as proven in the Appendix. Whenever $\tilde{\lambda}$ crosses any λ_i of $\hat{L}$, the CE has two purely imaginary roots at $s = \pm j\omega_i^*$. This happens when $\Im(\tilde{\lambda}) = 0$, and $\Re(\tilde{\lambda}) = \lambda_i$.

The condition on the null imaginary part leads to

$$\tau = \frac{1}{\omega}\cot^{-1}\left(\frac{\omega^2 p + \gamma q}{(\gamma p - q)\omega}\right) + \frac{k\pi}{|\omega|}, \tag{10}$$

where $k \in \mathbb{N}$ takes into account the periodicity of the cotangent function. Substitute (10) into the real part of $\tilde{\lambda}$ to obtain

$$\left.\Re(\tilde{\lambda})\right|_{\Im(\tilde{\lambda})=0} = \frac{m\omega^2}{q}\cdot\frac{\sqrt{\frac{p^2}{\gamma^2q^2}\omega^4 + (\frac{p^2}{q^2}+\frac{1}{\gamma^2})\omega^2 + 1}}{1+\frac{\omega^2p^2}{q^2}}. \tag{11}$$

Since $\lambda_i \in \mathbb{R}^+$, k has to be even, for (11) to be positive. Calling τ_i^* the solution of (10) and (11) for $k = 0$ and $\left.\Re(\tilde{\lambda})\right|_{\Im(\tilde{\lambda})=0} = \lambda_i$, any delay

$$\tau_i^*,\ \tau_i^* + \frac{2\pi}{|\omega_i^*|},\ \tau_i^* + \frac{4\pi}{|\omega_i^*|},\ \dots$$

makes the stability boundary cross λ_i, while yielding two purely imaginary roots of the CE at $s = \pm j\omega_i^*$.

Now, the necessary condition of Theorem 1 can be proven by showing that the roots $s = \pm j\omega_i^*$ of the CE always move from the imaginary axis towards the unstable right-half complex plane as the delay increases. That is, the real part of the following sensitivity

$$\left.\frac{\partial s}{\partial\tau}\right|_{s=\pm j\omega_i^*,\tau=\tau_i^*+\frac{2k\pi}{|\omega_i^*|}} \tag{12}$$

is positive for any $k \in \mathbb{N}$ and for $i = 1,\dots,N$. Performing implicit differentiation with respect to τ in (8) leads to

$$\frac{\partial s}{\partial\tau} = \frac{\lambda_i(ps+q)e^{-\tau s}}{m\left(\frac{3s^2}{\gamma}+2s\right) + \lambda_i\big(p - \tau(ps+q)\big)e^{-\tau s}}.$$

Substituting $s = \pm j\omega_i^*$, $\tau = \tau_i^* + \frac{2k\pi}{|\omega_i^*|}$, and λ_i from (11), sensitivity (12) takes the form

$$\frac{a+jb}{c+jd} = \frac{ac+bd}{c^2+d^2} + j\frac{bd-ad}{c^2+d^2},$$

with $a,b,c,d \in \mathbb{R}$. Its real part is always positive because

$$ac+bd = \frac{\frac{3p^2}{\gamma^2q^2}\omega^4 + \left(\frac{2p^2}{q^2}+\frac{3}{\gamma^2}\right)\omega^2 + 2}{\frac{p^2}{\gamma^2q^2}\omega^4 + \left(\frac{p^2}{q^2}+\frac{1}{\gamma^2}\right)\omega^2 + 1}\cdot(q^2+p^2\omega^2) - p^2\omega^2$$

is positive regardless of $k \in \mathbb{N}$ and $i = 1,\dots,N$.

Thus, anytime $\tilde{\lambda}$ crosses any λ_i of $\hat{L}$, two unstable roots are added to the system. Since (9) holds, λ_N is the first eigenvalue of $\hat{L}$ crossed by the boundary, so that $\tau < \tau_N^*$ is a necessary condition for stability. Indeed, as long as it holds, $\tilde{\lambda}$ does not cross any λ_i, and no root of the CE can move towards the unstable right-half plane.

To prove the sufficient condition, we focus on the region shown in Figure 2, which is enclosed by the first crossing of $\tilde{\lambda}$ with the real axis, and contains a part of the positive real axis in a neighborhood of the origin. Indeed, the stability of the system can be assured if all the λ_i of $\hat{L}$ lie inside this region. Remind that, since (9) holds, this condition is equivalent to $\tau < \tau_N^*$.

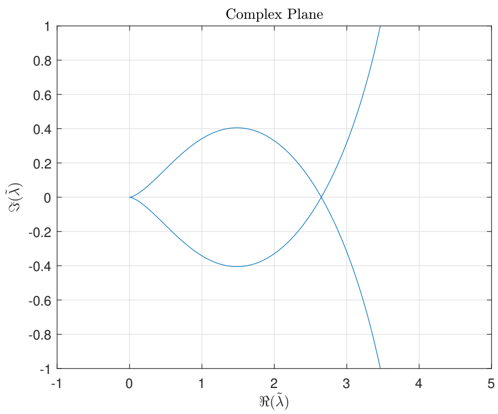

Fig. 2. Stability region.

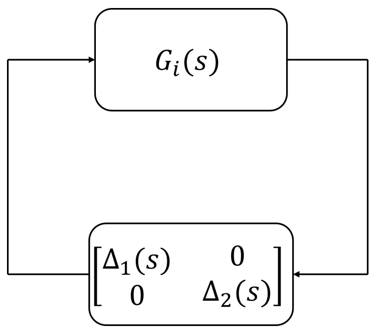

Fig. 3. Equivalent Closed Loop Subsystem.

Rewrite the subsystem (7) as proposed in [24]

$$\begin{aligned}\dot{\xi}_i(t) &= (A+\lambda_i A_d)\cdot\xi_i(t)+\tau_N^*\lambda_i A_d\cdot(\upsilon_{i,1}(t)+\upsilon_{i,2}(t))\\ y_{i,1}(t) &= A\cdot\xi_i(t)\\ y_{i,2}(t) &= \lambda_i A_d\cdot\xi_i(t)\\ \upsilon_{i,1}(t) &= \Delta_1\cdot y_{i,1}(t)\\ \upsilon_{i,2}(t) &= \Delta_2\cdot y_{i,2}(t),\end{aligned} \tag{13}$$

where $\Delta_1(s)=\frac{(e^{-\tau s}-1)}{\tau_N^* s}\mathbf{I}_3$ and $\Delta_2(s)=(e^{-\tau s}\cdot\Delta_1(s))\mathbf{I}_3$.

Through (13), it is possible to express the original system as N closed loop subsystems like the one in Figure 3, where

$$G_i(s)=C_i\big(s\mathbf{I}_3-(A+\lambda_i A_d)\big)^{-1}B_i \tag{14}$$

given

$$B_i=\tau_N^*\lambda_i[A_d \quad A_d], \qquad C_i=\begin{bmatrix}A\\ \lambda_i A_d\end{bmatrix}.$$

Notice that both $||\Delta_1(s)||_\infty\le 1$ and $||\Delta_2(s)||_\infty\le 1$ hold if $\tau/\tau_N^*\le 1$. Thus, if $||G_i(s)||_\infty<1$, we can ensure asymptotic stability of the closed loop subsystem in Figure 3 through the Small Gain Theorem, given that $A+\lambda_i A_d$ is Hurwitz.

The condition on the $\mathbf{H}_\infty$ norm of $G_i(s)$ can be provided by building the Hamiltonian H_i of subsystem (14)

$$H_i=\begin{bmatrix}A+\lambda_i A_d & -B_i\cdot B_i^T\\ C_i^T\cdot C_i & -(A+\lambda_i A_d)^T\end{bmatrix},$$

and showing that its eigenvalues do not belong to the imaginary axis, as proved in [25]. Using the Schur complement, the eigenvalues of H_i can be computed through

$$\det(s\mathbf{I}_3-(A+\lambda_i A_d))\cdot\det(s\mathbf{I}_3+(A+\lambda_i A_d)^T+\Psi_i)=0, \tag{15}$$

where $\Psi_i=C_i^T C_i\cdot((s\mathbf{I}_3-(A+\lambda_i A_d))^{-1})\cdot B_i B_i^T$ has the following structure

$$\Psi_i=\lambda_i^2\cdot\frac{2\gamma^2\tau_N^{*2}}{\det(s\mathbf{I}_3-(A+\lambda_i A_d))}\cdot\Phi,$$

with

$$\Phi=\begin{bmatrix}\frac{s(ps+q)}{m}\\ \frac{\gamma^2(ps+q)}{m}\left(\left(\frac{p}{m}\right)^2+(\gamma p-q)^2\right)\\ s^2(\gamma p-q)\left(\frac{1}{m^2}+\gamma^2\right)\end{bmatrix}\cdot\begin{bmatrix}\frac{p}{m} & 0 & (\gamma p-q)\end{bmatrix}$$

and

$$\det\big(s\mathbf{I}_3-(A+\lambda_i A_d)\big)=s^2(s+\gamma)+\frac{\lambda_i\gamma}{m}(ps+q).$$

Employing a first-order approximation for $\lambda_i\to 0^+$, one can see that matrix Ψ_i becomes negligible with respect to both matrices $s\mathbf{I}_3$ and $(A+\lambda_i A_d)^T$, so that (15) tends to

$$\det\big(s\mathbf{I}_3-(A+\lambda_i A_d)\big)\cdot\det\big(s\mathbf{I}_3+(A+\lambda_i A_d)^T\big)=0 \tag{16}$$

as $\lambda_i\to 0^+$. It is straightforward to notice that the roots of equation (16) are given by the eigenvalues of $A+\lambda_i A_d$ and by their symmetric with respect to the imaginary axis. The explicit form of (16) is

$$\left(s^2(s+\gamma)+\frac{\lambda_i\gamma}{m}(ps+q)\right)\cdot\left(s^2(s-\gamma)+\frac{\lambda_i\gamma}{m}(ps-q)\right)=0.$$

Since all the eigenvalues of $A+\lambda_i A_d$ have negative real part, we can conclude that the Hamiltonian H_i of the system does not have any purely imaginary eigenvalue for $\lambda_i\to 0^+$, which is equivalent to $||G_i(s)||_\infty<1$. Therefore, subsystem (13) is asymptotically stable when $\lambda_i\to 0^+$, i.e. when λ_i is close to the origin on the positive real axis.

This allows us to conclude that the whole region depicted in figure 2 is stable. If all the eigenvalues λ_i of $\hat{L}$ lie inside it, or, equivalently, if $\tau<\tau_N^*$, the asymptotic stability of the N subsystems (13) yields

$$\lim_{t\to\infty}\xi(t)=0.$$

This, together with (6), leads to

$$\begin{cases}\lim_{t\to\infty}(\dot{x}_i(t)-\dot{x}_0(t))=0\\ \lim_{t\to\infty}((x_i(t)-h_i)-(x_0(t)-h_0))=0 \qquad \text{for } i=1,\dots,N\\ \lim_{t\to\infty}z_i(t)=0\end{cases}$$

i.e. the sufficient condition of Theorem 1 holds true. ■

Hence, $\tau<\tau_N^*$ is a necessary and sufficient condition for stability. The region enclosed by the first crossing of $\tilde{\lambda}$ with the real axis and containing all the eigenvalues of $\hat{L}$ is the *only* stable region. This allows to easily assess the stability of the system even with a large N, by checking whether the eigenvalues of the Laplacian belong to the stability region.

IV. NUMERICAL RESULTS

In this section, we provide some numerical results to validate Theorem 1. Consider $N = 10$ agents with $p = 1$, $q = 2$, $m = 5$, $\gamma = 10$. Their initial positions $x_i(0)$ and velocities $\dot{x}_i(0)$ are randomly selected in the interval $[-10, 10]$m, $[-2, 2]$m/s, respectively. The leader starts at $x_0(0) = 0$, and travels with a constant velocity $\dot{x}_0 = 0.25\,\mathrm{m/s}$. The desired relative displacements of the formation are $h_i - h_j = 3(i - j)$, for $i, j = 0, \ldots, N$, and $h_0 = 0$.

It is assumed that 3 agents have access to the information from the leader, and that the interaction topology is represented by a path graph, as shown in figure 4. Thus, the maximum eigenvalue of $\hat{L}$ is $\lambda_N = 4.315$. With these conditions, solving (10) with $k = 0$ and (11) with $\Re(\tilde{\lambda})\Big|_{\Im(\tilde{\lambda})=0} = 4.315$ allows to compute $\tau_N^* = 0.333\,\mathrm{s}$, which is the maximum delay threshold the system can tolerate before losing stability.

Figures 5, 6 and 7 depict the leader and followers positions for $\tau = 0.320\,\mathrm{s}$, $\tau = 0.333\,\mathrm{s}$ and $\tau = 0.340\,\mathrm{s}$, respectively. The leader position is depicted in black. As it is clear from Figure 5, the system is stable for a delay value lower than τ_N^*. The followers travel at the leader velocity, and arrange themselves in the desired relative displacements. In Figure 6, for $\tau = \tau_N^*$, the followers positions display an oscillatory behaviour caused by the roots of the CE lying on the imaginary axis. Finally, in figure 7, the system is clearly unstable as $\tau > \tau_N^*$. Note that other frequency domain approaches, found in [14] – [20], would require the stability analysis to be repeated $N = 10$ times. In this sense, our method is more immediate when assessing the stability of large systems.

V. CONCLUSIONS

In this paper, a relative displacement feedback was introduced to achieve a leader-follower formation of a second-order multi-agent system. The work considered a uniform delay in the displacement information processing, and the unavailability of velocity measurements. A stability region in the complex plane for the eigenvalues of the Laplacian matrix allowed to define a necessary and sufficient condition for consensus in terms of a delay threshold. Numerical examples were provided to validate the results. Future works will extend the analysis to general linear systems expressed through a state-space representation. Furthermore, future studies will explore the achievement of finite-time consensus and the speed of consensus reaching in interconnected systems with delays, particularly in the context of real-life applications.

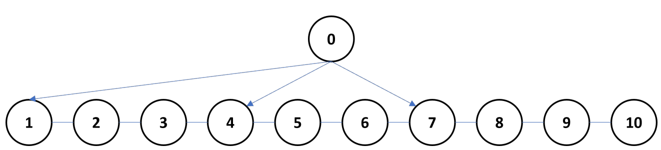

Fig. 4. Topology employed in simulations.

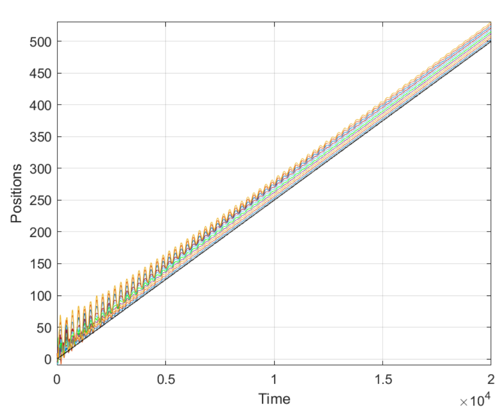

Fig. 5. Leader and followers positions with $\tau = 0.320s < \tau_N^*$.

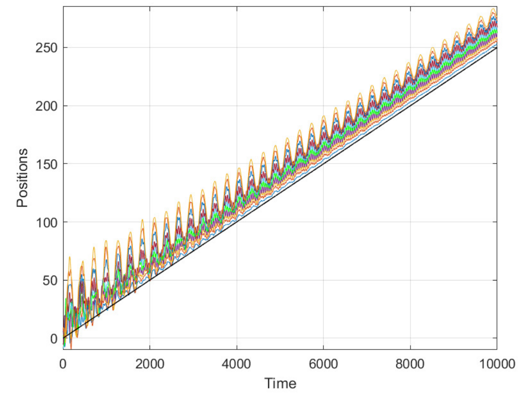

Fig. 6. Leader and followers positions with $\tau = 0.333s = \tau_N^*$.

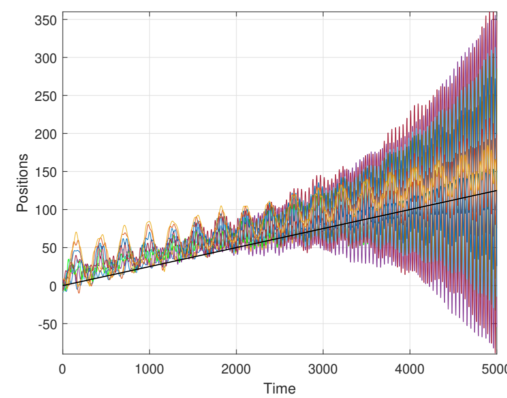

Fig. 7. Leader and followers positions with $\tau = 340s > \tau_N^*$.

APPENDIX

Lemma 1: As τ increases, the stability boundary crosses the real axis closer to the origin, i.e.

$$\left.\frac{\partial \Re(\tilde{\lambda})}{\partial \tau}\right|_{\Im(\tilde{\lambda})=0} < 0. \tag{17}$$

Proof: To prove the lemma, it is convenient to express (17) as

$$\left.\frac{\partial \Re(\tilde{\lambda})}{\partial \tau}\right|_{\Im(\tilde{\lambda})=0} = \left.\frac{\partial \Re(\tilde{\lambda})}{\partial \omega}\right|_{\Im(\tilde{\lambda})=0} \cdot \left.\frac{\partial \omega}{\partial \tau}\right|_{\Im(\tilde{\lambda})=0}. \tag{18}$$

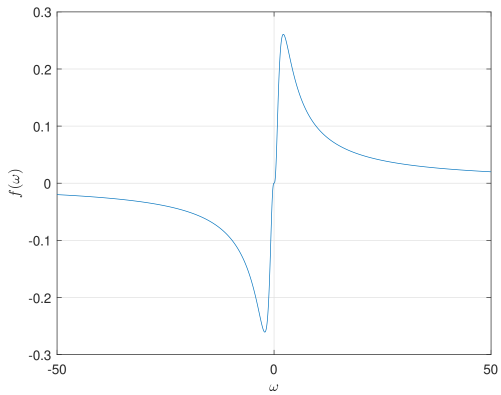

Fig. 8. Function $f(\omega)$ for $k=0$.

The first factor can be computed from (11), and is given by

$$\frac{\frac{p^4}{\gamma^2 q^4}\omega^7+\left(\frac{p^4}{2q^4}+\frac{5p^2}{2\gamma^2 q^2}\right)\omega^5+\frac{3}{2}\left(\frac{p^2}{q^2}+\frac{1}{\gamma^2}\right)\omega^3+\omega}{\frac{q}{2m}\left(1+\frac{\omega^2 p^2}{q^2}\right)^2\sqrt{\frac{p^2}{\gamma^2 q^2}\omega^4+\left(\frac{p^2}{q^2}+\frac{1}{\gamma^2}\right)\omega^2+1}},$$

which is positive for $\omega>0$ and negative for $\omega<0$.

The inverse of the second factor of (18) can be obtained from (10)

$$\left.\frac{\partial \tau}{\partial \omega}\right|_{\Im(\tilde{\lambda})=0}=-\frac{1}{\omega^2}f(\omega), \tag{19}$$

where

$$f(\omega)=\cot^{-1}\left(\frac{\omega^2 p+\gamma q}{(\gamma p-q)\omega}\right)+\frac{\left(\frac{\omega^2 p-\gamma q}{(\gamma p-q)\omega}\right)}{1+\left(\frac{\omega^2 p+\gamma q}{(\gamma p-q)\omega}\right)^2}+\frac{\omega}{|\omega|}k\pi$$

is depicted in figure 8. Function $f(\omega)$ is such that

$$\lim_{\omega\to 0^{\pm}} f(\omega)=\pm k\pi, \quad \lim_{\omega\to\pm\infty} f(\omega)=\pm k\pi,$$

while its derivative, being $\alpha=\frac{p}{q}-\frac{1}{\gamma}$ and $\beta=\frac{\gamma q}{p}$, is

$$\left.\frac{\partial f(\omega)}{\partial \omega}\right|_{\Im(\tilde{\lambda})=0}=2\alpha\beta\omega^2\cdot\frac{(-\omega^4+2\beta\omega^2+\alpha^2\beta^3+3\beta^2)}{(\omega^4+(2\beta+\alpha^2\beta^2)\omega^2+\beta^2)^2},$$

which is positive for small $|\omega|$ and negative for large $|\omega|$. Moreover, it has one real positive zero, one real negative zero and two zeros at the origin. Thus, we can conclude that $f(\omega)$ is positive for $\omega>0$ and negative for $\omega<0$ so that (19) is negative for $\omega>0$ and positive for $\omega<0$. Since (10) is an invertible mapping, the sign of (19) equals the one of the second factor of (18).

The signs of the two factors allow to state that (17) holds. ■

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