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Iterative Learning Control of Discrete Systems with Actuator Backlash using a Weighted Sum of Previous Trial Control Signals

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Abstract— **This paper considers iterative learning control design for discrete dynamics in the presence of backlash in the actuators. A new control design for this problem is developed based on the stability theory for nonlinear repetitive processes. An example demonstrates the effectiveness of the new design where the system model is constructed from data collected from frequency response tests on a physical system.**

I. Introduction

Iterative learning control (ILC) emerged from the problem of how to better the operation of robots that complete the same finite-duration task over and over again [1]. The key feature is that the dynamics repetitively operate over a finite duration, where one example is a pick-and-place robot undertaking the following tasks in sequence, i) collect the payload from a specified location, ii) transfer the payload over a fixed time interval, iii) place the payload on a moving conveyor under synchronization, iv) return to the starting location, and v) repeat i)-iv) as many times as required or until a halt for maintenance or other reasons are required.

Repetitions are termed trials (iterations or passes have also been used), and the finite duration of a trial is known as the trial length. Suppose that a reference trajectory is specified representing the desired behavior of the output on any trial. Then the error on each trial is defined as the difference between this trajectory and the output of this trial. Also the control problem is the construction of a sequence of trial inputs that force the sequence of trial errors to converge, under an appropriate norm, with the trial number either to zero (the ideal case) or within a specified tolerance.

A prevalent form of ILC law constructs the input for the subsequent trial as the sum of the previous trial input plus a correction. Once a trial is complete, all information generated during its execution is available, at storage cost, to update the control input applied on the subsequent trial. Consider discrete dynamics at sample p on trial k: Then the construction of the following trial input at this sample can, as one example, use information from sample $p+\lambda$ on the previous trial. Using such information is the distinguishing feature of ILC over alternative forms of control action.

The survey papers [2], [3] are possible sources for early ILC research. Since then, ILC has remained an active area of research both in developing new theoretical results and design methods and experimental validation and implementation. More recent developments include applications to additive manufacturing, e.g. high-precision multilayer laser deposition systems [4], robotic-assisted stroke rehabilitation, where the initial results are in [5] with more recent work in, e.g., [6]. Supporting clinical trial results have also been reported. Also, an application to heart ventricular support devices, e.g., [7], has been reported.

This paper considers the problem of actuator backlash arising in an implementation, which introduces nonlinearity into a linear design. The appearance of nonlinearities in the actuators can have, at the very least, a detrimental effect on the control signal applied to the system. Typical effects of nonlinear behavior in the actuators include reducing the achievable accuracy, slowing down the ILC law convergence from trial to trial or result in complete failure. Consequently, control design in the presence of implementation nonlinearities is a critical issue in some applications.

Previous research on ILC with backlash includes [8], where the application is a Timoshenko beam system described by a second order distributed parameter model, and the backlash term is divided into a linear term and an unknown bounded term, which is estimated. Also, in [9], a model of a two-link rigid-flexible manipulator with backlash is considered, where the analysis is in an identical manner to [8], and the effects of an external disturbance are also considered. Both of these designs apply only to the specific systems they consider. In [10], an adaptive ILC scheme for a particular class of nonlinear systems with unknown time-varying delays and control direction preceded by an unknown nonlinear backlash-like hysteresis is considered.

This paper develops a design for ILC in the presence of backlash in the actuator, where the control law includes the weighted sum of the control signals on a finite number of previous trials. The control law is one form of higher-order ILC. The approach is based on representing the dynamics as a repetitive process, a distinct class of 2D systems, and using the vector Lyapunov functions approach to the stability of nonlinear repetitive processes, see, e.g. [11]. A simulation-based case study using a model of a physical process constructed from measured frequency response data highlights the benefits of the new design.

Throughout this paper, the notation for variables is of the

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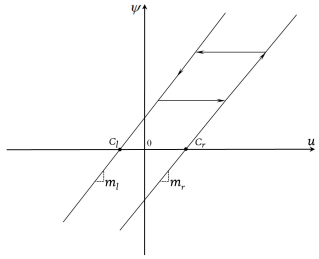

Fig. 1. Backlash model (shown for input u and output ψ.

form $h_k(p)$, $0 \leq p \leq N-1$, $k \geq 0$, where h denotes the scalar or vector-valued variable under consideration, N denotes the number of samples along a trial (N times the sampling period gives the trial length) and the integer k represents the trial number. Moreover, $\succ 0$ and $\prec 0$ denote a symmetric positive definite and a symmetric negative definite matrix. Also, $\succeq 0$ and $\preceq 0$ represent a symmetric positive semi-definite and a symmetric positive semi-definite matrix.

II. Problem Description

Consider a single-input single-output discrete-time system operating in a repetitive mode, where on trial k, the dynamics are described by the state-space model

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + B\Psi_k(p), \\ y_k(p) &= Cx_k(p),\ 0 \leq p \leq N-1,\ k \geq 1, \end{aligned} \quad (1)$$

where $x_k(p) \in \mathbb{R}^{n_x}$ is the state vector, $\Psi_k(p)$ is the control input, $y_k(p)$ is the trial profile (or output), and

$$\Psi_k(p) = \sum_{i=0}^{d} \tau_i \psi_{k-i}(p), \quad (2)$$

where (on any trial say h) $\psi_h(p)$ denotes the backlash function. This last function is illustrated in Fig. 1, and described [12] by

$$\begin{aligned} \psi_k(p) &= \text{back}(u_k(p)) \\ &= \begin{cases} m_l(u_k(p) - c_l), & \text{if } u_k(p) \leq \underline{u}_k(p), \\ m_r(u_k(p) - c_r), & \text{if } u_k(p) \geq \overline{u}_k(p), \\ \psi_k(p-1), & \text{if } \underline{u}_k(p) < u_k(p) < \overline{u}_k(p), \end{cases} \end{aligned} \quad (3)$$

where $\text{back}(u_k(p))$ is the backlash input (if there is no backlash then $u_k(p)$ is the control input to the system on trial k), m_l, m_r, c_r are positive constants, c_l is a negative constant, and

$$\underline{u}_k(p) = \frac{1}{m_l}\psi_k(p-1) + c_l, \quad (4)$$

$$\overline{u}_k(p) = \frac{1}{m_r}\psi_k(p-1)) + c_r. \quad (5)$$

The scalars τ_i in (2) determine the relative weightings of the previous trial contributions, and the integer $d > 1$ denotes the number of previous trial inputs used. After the designs are developed, these parameters for an application are considered in Section 5. This paper considers the case when $m_r = m_l = m$, and $c_l = -c_r$, and the other cases follow by appropriate/routine amendments to the analysis. Also, no loss of generality results from assuming that the boundary conditions are $x_k(0) = 0$ and $y_0(p) = f(p)$, where $f(p)$ is known scalar functions of p for $0 \leq p \leq N-1$. and $\psi_k(p) = 0$ if $k \in [-d, 0]$. It is also assumed that the pair $\{A, B\}$ is controllable and $CB \neq 0$. Let $y_{\text{ref}}(p) \in \mathbb{R}$, $0 \leq p \leq N-1$ denote the supplied reference signal and then

$$e_k(p) = y_{\text{ref}}(p) - y_k(p) \quad (6)$$

is the error on trial k. The control design problem is to construct a control input sequence $\{u_k\}$, such that

$$|e_k(p)| \leq \kappa\varrho^k + \mu,\ \kappa > 0,\ \mu \geq 0,\ 0 < \varrho < 1, \quad (7)$$

$$\lim_{k\to\infty} |u_k(p)| = |u_\infty(p)| < \infty,\ 0 \leq p \leq N-1, \quad (8)$$

where the bounded variable $u_\infty(p)$ is termed the learned control. If there is no backlash present, the design developed in this paper reduces to the case for linear dynamics, and $\lim_{k\to\infty} |e_k(p)| = 0$ results.

As discussed in the previous section, commonly used ILC law constructs the input for the subsequent trial as the sum of the previous trial input plus a correction term that uses the last trial data. This paper considers higher-order ILC, where

$$\psi_{k+1}(p) = \text{back}(\psi_k(p) + \delta u_{k+1}(p)), \quad (9)$$

and $\delta u_{k+1}(p)$ is the control update designed using information from the previous trial, and $\psi_k(p)$ is given by (3). The values of $\psi_k(p)$ are stored and form the control input (2) with different weights τ_i for the subsequent trial, i.e., a higher order ILC law. Next, the formulation of the dynamics as a nonlinear repetitive process is detailed.

III. Representation as a Nonlinear Repetitive Process

Introduce the variables $\check{x}_{k,1}(p) = \psi_k(p)$, $\check{x}_{k,2}(p) = \psi_{k-1}(p), \ldots, \check{x}_{k,d}(p) = \psi_{k-d+1}(p)$, $\check{x}_{k,d+1}(p) = \psi_{k-d}(p)$ and the vector $\check{x}_k = \begin{bmatrix} \check{x}_{k,1} & \ldots & \check{x}_{k,d+1} \end{bmatrix}^\top$. Then by construction

$$\check{x}_k(p) = A_d\check{x}_{k-1}(p) + B_d\psi_k(p),\ \ k > 0, \quad (10)$$

where

$$A_d = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, B_d = [1\ 0\ 0 \ldots 0]^\top.$$

Using (10), the first equation in (1) can be written as

$$x_k(p+1) = Ax_k(p) + BC_d\check{x}_k(p), \quad (11)$$

where $C_d = \begin{bmatrix} \tau_0 & \tau_1 & \ldots & \tau_d \end{bmatrix}$.

Introduce, for the design purpose only, the vectors

$$\begin{aligned} \eta_k(p) &= x_k(p) - x_{k-1}(p), \\ \check{\eta}_k(p) &= \check{x}_k(p) - \check{x}_{k-1}(p). \end{aligned} \quad (12)$$

Then using (10) and (12) it follows that

$$\check{\eta}_k(p) = A_d\check{\eta}_{k-1}(p) + B_d\Delta\psi_k(p), \tag{13}$$

where $\Delta\psi_k(p) = \psi_k(p) - \psi_{k-1}(p)$. Also

$$\begin{aligned}\eta_k(p+1) &= A\eta_k(p) + BC_dA_d\check{\eta}_{k-1}(p)\\ &+ BC_dB_d\Delta\psi_k(p).\end{aligned} \tag{14}$$

Using (6), $e_k(p) = y_{ref}(p) - Cx_k(p)$, and then (14) gives the following system of equations in terms of the incremental variables

$$\begin{aligned}\eta_k(p+1) &= A\eta_k(p) + BC_dA_d\check{\eta}_{k-1}(p)\\ &\quad + BC_dB_d\Delta\psi_k(p),\\ \check{\eta}_k(p) &= A_d\check{\eta}_{k-1}(p) + B_d\Delta\psi_k(p),\\ \bar{e}_k(p) &= -CA\eta_k(p) - CBC_dA_d\check{\eta}_{k-1}(p)\\ &\quad + \bar{e}_{k-1}(p) - CBC_dB_d\Delta\psi_k(p),\end{aligned} \tag{15}$$

where $\bar{e}_k(p) = e_k(p+1)$. Also, consider (9) with

$$\delta u_k(p) = K_1\eta_k(p) + K_2\bar{e}_{k-1}(p), \tag{16}$$

where K_1 and K_2 are matrices of compatible dimensions to be designed. Then, using (15) and (16), the model of the controlled dynamics can be written as

$$\begin{aligned}\eta_k(p+1) &= A_c\eta_k(p) + BC_dA_d\check{\eta}_{k-1}(p)\\ &\quad + BC_dB_dK_2\bar{e}_{k-1}(p) + BC_dB_d\varphi_k(p),\\ \check{\eta}_k(p) &= B_dK_1\eta_k(p) + A_d\check{\eta}_{k-1}(p)\\ &\quad + B_dK_2\bar{e}_{k-1}(p) + B_d\varphi_k(p),\\ \bar{e}_k(p) &= -CA_c\eta_k(p) - CBC_dA_d\check{\eta}_{k-1}(p)\\ &\quad + (1 - CBC_dB_dK_{2i})\bar{e}_{k-1}(p)\\ &\quad - CBC_dB_d\varphi_k(p),\end{aligned} \tag{17}$$

where $A_c = A + BC_dB_dK_1$, $\varphi_k(p) = \Delta\psi_k(p) - \delta u_k(p)$. Also, it follows from (3) and Fig. 1 that $\Delta\psi_k(p) = \psi_k(p) - \psi(u_{k-1}(p)$ satisfies the constraints

$$m\delta u_k(p) - m\Delta c \le \Delta\psi_k(p) \le m\delta u_k(p) + m\Delta c,$$

where $\Delta c = c_r - c_l$. Moreover, $\varphi_k(p)$ satisfies the constraints

$$m_1\delta u_k(p) - m\Delta c \le \varphi_k(p) \le m_1\delta u_k(p) + m\Delta c,$$

or

$$m^2(\Delta c)^2 - [\varphi_k(p) - m_1\delta u_k(p)]^2 \ge 0, \tag{18}$$

where $m_1 = m - 1$. If (as assumed in the previous section) $m_r = m_l = m$, back$(mu) = m$back(u) and hence, without the loss of generality, $m = 1$ is considered, and the quadratic constraint (18) can then be written as

$$(\Delta c)^2 - \varphi_k^2(p) \ge 0. \tag{19}$$

In the presence of backlash, the ILC dynamics are nonlinear, and there has been recent work on developing a stability theory for nonlinear repetitive processes. One approach is to use vector Lyapunov functions [11]. This paper uses this theory for ILC design, starting from the convergence conditions in the next section. The model (17) is a discrete, repetitive process, a particular class of 2D systems.

IV. Analysis and Design

This paper uses a Lyapunov function approach to control design. In the standard systems case, the Lyapunov method, a scalar function, is used, and its gradient (full increment in the case of a discrete system) along the trajectories of the system is the basis for analysis and design. In the case of repetitive processes, two scalar functions are needed, one for the state dynamics along a trial and the other for the trial-to-trial updating. Moreover, computing the gradient of a Lyapunov function (or its discrete counterpart), in this case, is only possible if the solution of the defining equations is known, which is not feasible. Instead, the two functions define a column vector, and analysis then uses the divergence. This method is known as vector Lyapunov function analysis (other formulations are possible, but the critical issue is using the discrete counterpart of the divergence operator, termed the divergence operator, in the rest of this paper for ease of presentation).

Consider the model (17), then the vector $\eta_k(p)$ is updated along the trials and the vectors $\check{\eta}_k(p)$ and $\bar{e}_k(p)$ are updated from trial-to-trial. Introduce the $\epsilon_k(p) = \begin{bmatrix} \check{\eta}_{k-1}^\top(p) & \bar{e}_{k-1}^\top(p) \end{bmatrix}^\top$ and the vector Lyapunov function for dynamics described by (17) is taken as

$$V(\eta_k(p), \epsilon_k(p)) = \begin{bmatrix} V_1(\eta_k(p)) \\ V_2(\epsilon_k(p)) \end{bmatrix}, \tag{20}$$

where $V_1(\eta_k(p)) > 0$, $\eta_k(p)) \neq 0$, $V_2(\epsilon_k(p)) > 0$, $\epsilon_k(p) \neq 0$, $V_1(0) = 0$, $V_2(0) = 0$ and define the divergence operator along the trajectories of (17) as

$$\begin{aligned}\mathcal{D}V(\eta_k(p), \epsilon_k(p)) &= V_1(\eta_k(p+1)) - V_1(\eta_k(p))\\ &+ V_2(\epsilon_{k+1}(p)) - V_2(\epsilon_k(p)).\end{aligned} \tag{21}$$

and the following result can be established, where $||\cdot||$ denotes the chosen norm for vectors.

Theorem 1: Suppose that there exists a vector Lyapunov function (20) and positive scalars c_1, c_2, c_3 and γ for dynamics described by (17) such that

$$c_1||\eta_k(p)||^2 \le V_1(\eta_k(p)) \le c_2||\eta_k(p)||^2, \tag{22}$$

$$c_1||\epsilon_k(p)||^2 \le V_2(\epsilon_k(p)) \le c_2||\epsilon_k(p)||^2, \tag{23}$$

$$\mathcal{D}V(\eta_{k+1}(p), e_k(p)) \le \gamma - c_3(||\eta_{k+1}(p)||^2 + ||\epsilon_k(p)||^2). \tag{24}$$

Then the error convergence conditions of (7) hold under the ILC law (9) where $\delta u_{k+1}(p)$ is given by (16).

Before giving the proof of this result, the following remark is given.

Remark 1: The backlash function contains a dead-zone that prevents trial-to-trial error convergence to zero, and this is the reason for the need to include the positive scalar γ in this last result. This fact prevents application of the (nonlinear repetitive process) results in [11].

Proof: Calculating divergence along the trajectories (17) and following the same steps as in the proof of Theorem

1 in [13] gives

$$|e_k(p)|^2 \leq ||\epsilon_k(p)||^2 \leq \lambda^k \sum_{q=0}^{p} \lambda^{p-q} ||\epsilon_0(q)||^2 + \frac{\gamma}{c_1(1-\lambda)^2}, \quad (25)$$

where $\lambda = 1 - \frac{\bar{c}_3}{c_2}$ and $c_3 \leq \bar{c}_3 < c_2$, which implies that (7) holds for $\varrho = \lambda$, $\kappa = \frac{\alpha}{1-\lambda}$, $\alpha = \max_q ||\epsilon_0(q)||^2$, $\delta = \frac{\gamma}{c_1(1-\lambda)^2}$. ■

Two ILC designs are developed in this paper, termed one-step and two-step, respectively.

A. One Step Design

Consider the vector Lyapunov function (20) for (17) in the case when

$$\begin{aligned} V_1(\eta_k(p)) &= \eta_k^\top(p) P_1 \eta_k(p), \\ V_2(\epsilon_k(p)) &= \epsilon_k^\top(p) P_2 \epsilon_k(p), \end{aligned} \quad (26)$$

where $P_1 \succ 0$ and $P_2 \succ 0$ and set $P = \mathrm{diag}[P_1 \; P_2]$. Also, introduce $\xi_k(p) = \begin{bmatrix} \eta_k^\top(p) & \check{\eta}_{k-1}^\top(p) & \bar{e}_{k-1}^\top(p) \end{bmatrix}^\top$ (where in what follows the dependence of some variables on k and p is omitted). Computing (21) for (17), gives

$$\begin{aligned} \mathcal{D}V_i(\eta, \epsilon) &= [(\bar{A} + \bar{B}KH)\xi + \bar{B}\varphi]^\top P[(\bar{A} \\ &+ \bar{B}KH)\xi + \bar{B}\varphi] - \xi^\top P \xi, \end{aligned} \quad (27)$$

where $K = [K_1 \; K_2]$,

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & BC_dA_d & 0 \\ 0 & A_d & 0 \\ -CA & -CBC_dA_d & 1 \end{bmatrix}, \\ \bar{B} &= \begin{bmatrix} BC_dB_d \\ B_d \\ -CBC_dB_d \end{bmatrix} \; H = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (28)$$

Since $V_1(\eta) \succ 0$ and $V_2(\epsilon) \succ 0$, (22) and (23) of Theorem 1 hold.

A sufficient condition for (24) to hold under the constraints (19) is that

$$\begin{aligned} \mathcal{D}V(\eta, \epsilon) &+ \vartheta((\Delta c)^2 - \varphi^2) \\ &\leq \gamma - \xi^\top [Q + (KH)^\top RKH]\xi, \end{aligned} \quad (29)$$

holds for all φ and ξ, where $Q \succ 0$ and $R \succ 0$ are matrices of compatible dimensions and $\vartheta > 0$ (see also [14]). The inequality (29) holds if $\gamma = \vartheta(\Delta c)^2$ and

$$\begin{bmatrix} (\bar{A} + \bar{B}KH)^\top P(\bar{A} + \bar{B}KH) - P + M & (\bar{A} + \bar{B}KH)^\top P\bar{B} \\ \bar{B}^T P(\bar{A} + \bar{B}KH) & \bar{B}^\top P\bar{B} - \vartheta \end{bmatrix} \preceq 0, \quad (30)$$

where $M = Q + (KH)^\top RKH$. Rewriting this last inequality as

$$\begin{bmatrix} -P & 0 \\ 0 & -\vartheta \end{bmatrix} + \begin{bmatrix} (\bar{A} + \bar{B}KH)^\top & I & (KH)^\top \\ \bar{B}^\top & 0 & 0 \end{bmatrix} \times \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} (\bar{A} + \bar{B}KH) & \bar{B} \\ I & 0 \\ KH & 0 \end{bmatrix} \preceq 0, \quad (31)$$

and applying the Schur's complement formula gives

$$\begin{bmatrix} -X & 0 & (\bar{A} + \bar{B}YH)^\top & X & (YH)^\top \\ 0 & -\vartheta & \bar{B}^\top & 0 & 0 \\ (\bar{A} + \bar{B}YH) & \bar{B} & -X & 0 & 0 \\ X & 0 & 0 & -Q^{-1} & 0 \\ YH & 0 & 0 & 0 & -R^{-1} \end{bmatrix} \preceq 0, \quad (32)$$

where $X = P^{-1}$, $Y = KW$ and W is a solution of

$$HX = WH. \quad (33)$$

If the linear matrix inequality (LMI) (32) and the linear matrix equation (33) are solvable for the variables X, Y, ϑ, and W, then the ILC law (9) ensures that the convergence condition (7) holds where $\delta u_{k+1}(p)$ is given by (16) and

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = YW^{-1}. \quad (34)$$

To prove the boundedness condition (8) for this ILC law, first note that the convergence condition (7) and the definition of the error on trial k given in (6) imply that $|Cx_\infty(p)| = \lim_{k\to\infty} |Cx_k(p)|$ is bounded for all p. Also, it follows from (1) and using (2) that

$$Cx_k(p+1) = CAx_k(p) + CB\Psi_k(p)$$

and

$$\Psi_k(p) = (CB)^{-1}(Cx_k(p+1) - CAx_k(0)). \quad (35)$$

Hence

$$|\Psi_\infty(0)| \leq |(CB)^{-1}|(|Cx_\infty(1)| + |CAx_\infty(0)|) < \infty,$$

because $|Cx_\infty(p)| < \infty$ for all p, and

$$\begin{aligned} |\Psi_\infty(1)| &\leq |(CB)^{-1}|(|Cx_\infty(2)| + |CA^2x_\infty(0)| \\ &+ |CAB||\Psi_\infty(0)|) < \infty, \end{aligned} \quad (36)$$

since $|Cx_\infty(p)| < \infty$ for all p, and $|\Psi_\infty(0)| < \infty$.

Continuing this procedure gives for $1 \leq p \leq N$,

$$\begin{aligned} |\Psi_\infty(p-1)| &\leq |(CB)^{-1}|(|Cx_\infty(p)| + |CA^pB||x_\infty(0)| \\ &+ |\sum_{q=0}^{p-2} |CA^{p-1-q}B||\Psi_\infty(q)|) < \infty, \end{aligned} \quad (37)$$

since $|Cx_\infty(p)| < \infty$ for all p, and (from above) $|\Psi_\infty(q)| < \infty$, $q = 0, 1, \ldots, p-2$. Finally by (2) and by the definition of the inverse backlash function [12] the boundedness condition (8) holds.

B. Two Step Design

Depending on the specific choice of the entries in the vector Lyapunov function (20), various sufficient convergence conditions result from Theorem 1. Moreover, it is a nontrivial task to determine in advance the level of conservativeness of each of them. For example, suppose an ILC law based on a linear model approximation ensures convergence. Then it is possible that this law also ensures convergence for the nonlinear dynamics. This section develops an alternative design to that in the previous section.

Consider the discrete Riccati inequality

$$\bar{A}^\top \bar{P}\bar{A} - (1-\sigma)\bar{P} - \bar{A}^\top \bar{P}\bar{B}[\bar{B}^\top \bar{P}\bar{B} + R]^{-1}\bar{B}^\top \bar{P}\bar{A} + Q \preceq 0 \quad (38)$$

relative to the matrix $\bar{P} = \text{diag}\begin{bmatrix} P_1 & P_2 \end{bmatrix} \succ 0$, where $P_1 \in \mathbb{R}^{n_x \times n_x}$, $P_2 \in \mathbb{R}^{(d+2)\times(d+2)}$, $0 < \sigma < 1$. Applying Schur's complement formula gives that if the LMI

$$\begin{bmatrix} (1-\sigma)\bar{X} & X\bar{A}^\top & \bar{X} \\ \bar{A}\bar{X} & \bar{X} + \bar{B}R^{-1}\bar{B}^\top & 0 \\ \bar{X} & 0 & Q^{-1} \end{bmatrix} \succeq 0, \quad (39)$$

is feasible for $X = \text{diag}[X_1 \; X_2] \succ 0$, where X_1 and X_2 have the same dimensions as P_1 and P_2, respectively, then $P = X^{-1}$.

Introduce

$$L = [\underbrace{L_1}_{n_x} \; \underbrace{L_2}_{d+1} \; \underbrace{L_3}_{1}] = -[\bar{B}^\top \bar{P}\bar{B} + R]^{-1}\bar{B}^\top \bar{P}\bar{A}, \quad (40)$$

$$F = [\underbrace{F_1}_{n_x} \; \underbrace{0}_{d+1} \; \underbrace{F_3}_{1}] = L\Theta, \quad (41)$$

where the under brace denotes the column dimension of each block entry, and

$$\Theta = \begin{bmatrix} \Theta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Theta_3 \end{bmatrix}, \quad (42)$$

is a matrix with blocks of compatible dimensions, such that the LMI

$$\begin{bmatrix} M - M\Theta - \Theta M - Q & \Theta\sqrt{M} \\ \sqrt{M}\Theta & -I \end{bmatrix} \preceq 0, \quad (43)$$

with $M = \bar{A}^\top \bar{P}\bar{B}[\bar{B}^\top \bar{P}\bar{B} + R]^{-1}\bar{B}^\top \bar{P}\bar{A}$ is feasible. The following result can now be established.

Theorem 2: Assume that for matrices $Q \succ 0$ and $R \succ 0$ and scalar $0 < \sigma < 1$ the LMI's (39), (43) and

$$\begin{bmatrix} (\bar{A}+\bar{B}KH)^\top S(\bar{A}+\bar{B}KH) - S & (\bar{A}+\bar{B}KH)^\top S\bar{B} \\ \bar{B}^\top S(\bar{A}+\bar{B}KH) & \bar{B}^\top S\bar{B} - \vartheta \end{bmatrix} \prec 0, \quad (44)$$

where

$$K = [F_1\Theta_1 \; F_3\Theta_3], \quad (45)$$

are feasible for X, $\Theta, \vartheta > 0$ and $S = \text{diag}\begin{bmatrix} S_1 & S_2 \end{bmatrix} \succ 0$ with block entrys of the same dimensions as P_1 and P_2, respectively. Then (9) where $\delta u_{k+1}(p)$ is given by (16) and $K = [K_1 \; K_2]$ by (45) ensures that both (7) and (8) hold under the higher order ILC law (2).

Proof: The inequality (44) implies that for all ξ and φ including those satisfying (19)

$$[(\bar{A}+\bar{B}KH)\xi + \bar{B}\varphi]^\top S[(\bar{A} + \bar{B}KH)\xi + \bar{B}\varphi] - \xi^\top S\xi - \vartheta\varphi^2 < 0. \quad (46)$$

Since the left-hand side of this last expression is a quadratic form relative to ξ and φ, and since $S \succ 0$, it follows by applying the same steps as the analysis of the one-step method that all the conditions of the Theorem 1 is satisfied, and the convergence condition (6) holds for $\gamma = \vartheta(\Delta c)^2$. Finally, the proof of the boundedness condition (7) follows in the same way as in the one step design given above. ■

Consider the system (1) in the absence of backlash. In such a case, the ILC law has the form

$$u_{k+1}(p) = u_k(p) + \delta u_{k+1}(p). \quad (47)$$

Suppose that $\delta u_{k+1}(p)$ is obtained as in Theorem 2. In that case, it follows as a corollary of Theorem 2 from [13] that for this linear case, conditions (7) and (8) hold with $\gamma = 0$. For this reason, this theorem's condition could be less conservative than that in the result of the previous section.

V. Case Study

Consider the model of one of the axes of the multi-axis gantry robot described in [15]. Frequency response tests (also detailed in [15]) result in the following 3rd order continuous-time transfer function as an adequate dynamics model for control law design

$$G(s) = \frac{23.7356(s + 661.2)}{s(s^2 + 426.7s + 1.744 \cdot 10^5)}. \quad (48)$$

The reference trajectory is the same as in [15] with a trial length of 2 secs. For discrete design, the sampling period is 0.01 secs, and in the backlash nonlinearity (Fig. 1) $m = 1$ and $c_r = -c_l = c$.

As representative results two cases for $d = 1$ are given, and hence in (10)

$$A_d = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_d = \begin{bmatrix} \tau_0 & \tau_1 \end{bmatrix}. \quad (49)$$

Also in both cases $Q = \text{diag}[1 \; 1 \; 10^{-2} \; 10 \; 10 \; 1.0 \cdot 10^5]$, $R = 10^{-3}$, $\sigma = 0.0125$.

Case 1: $\tau_0 = 1, \tau_1 = 0.4$, and hence on trial $k+1$ information from both trials k and $k-1$ are used. In this case Theorem 2 gives

$$K_1 = \begin{bmatrix} -10.5 & -8.6 & -3508.3 \end{bmatrix}, \; K_2 = 124.2. \quad (50)$$

Case 2: $\tau_0 = 1$, $\tau_1 = 0$, and hence on trial $k+1$ only information from trial k is used. In this case Theorem 2 gives

$$K_1 = \begin{bmatrix} -15.0 & -12.5 & -5060.9 \end{bmatrix}, \; K_2 = 104.8. \quad (51)$$

Trial-to-trial error convergence of the designs is assessed using the root mean square error for each trial, i.e.,

$$\mathrm{RMS}(k) = \sqrt{\frac{1}{N}\sum_{p=0}^{N} ||e_k(p)||^2}. \tag{52}$$

Figure 2 shows that the Case 1 design accelerates the trial-to-trial error convergence. The results for these two cases show a slight dip. This feature is due to the dead-zone component of the nonlinearity, which is not present when the actuator does not have backlash.

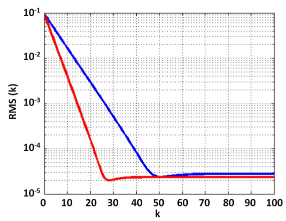

Fig. 2. The $\mathrm{RMS}(k)$ progression for $c = 0.003$: Case 1 (red line), Case 2 (blue line).

The parameter c in Fig. 1 determines the system's dead zone. It is interesting to examine the effects of varying this parameter, where here, interest is restricted to the Case 1 design. In this case, Fig. 3 shows the progression of the $\mathrm{RMS}(k)$ for two values of c, with the addition of measurement noise at the sensor accuracy level ($10^{-5}m$). A greater value of c results in the trial-to-trial error converging to a larger value (recall that if the nonlinearity is present, then convergence to zero error may not occur).

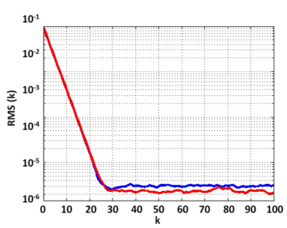

Fig. 3. The $\mathrm{RMS}(k)$ progression for the Case 1 design with $c = 0.001$ (red line) and $c = 0.003$ (blue line).

The analysis in this paper results in design algorithms, and two examples have demonstrated their performance. To apply them, however, requires the choice of d, the number of previous trials whose input contributes to the subsequent trial input, and the scalars τ_i that determine the relative weighting of corresponding trial data. Further research is required on these two aspects, but it may well be that these will have to be selected based on knowledge of the application under consideration. One possible approach to the second issue is to consider a forgetting factor approach based on the premise that the previous trial should contribute more than its predecessor and so on.

VI. Conclusions and Future Work

This paper has developed new results on the effects of actuator backlash on the performance of ILC designs for discrete linear systems. Numerical examples confirm the results obtained. Moreover, the use of a weighted sum of previous trial inputs in the computation of the control input for the subsequent trial has been considered. In addition to the discussion at the end of the last section, areas for future research include robustness, the effects of noise on measurements, and, in due course, experimental validation.

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