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Exponential Stability of Parametric Optimization-Based Controllers via Lur'e Contractivity

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***Abstract*— In this letter, we investigate sufficient conditions for the exponential stability of LTI systems driven by controllers derived from parametric optimization problems. Our primary focus is on parametric projection controllers, namely parametric programs whose objective function is the squared distance to a nominal controller. Leveraging the virtual system method of analysis and a novel contractivity result for Lur'e systems, we establish a sufficient LMI condition for the exponential stability of an LTI system with a parametric projection-based controller. Separately, we prove additional results for single-integrator systems. Finally, we apply our results to state-dependent saturated control systems and control barrier function-based control and provide numerical simulations.**

I. INTRODUCTION

Controllers solving optimization problems are ubiquitous in systems and control. One large class of optimization-based controllers are based upon (i) solving an optimal control problem offline, such as LQR, LQG, or Hamilton-Jacobi PDE and (ii) closing the loop with the resulting controller. Recent interest has focused on a different class of optimization-based controllers, that solve optimization problems at every time-step of the dynamic evolution of the plant. Namely, such controllers are solutions to *parametric optimization problems*, i.e., programs that are functions of the state of the system. Examples of these controllers include model-predictive control [1], online feedback optimization [2], and control barrier (or Lyapunov) function-based control [3]. While stability and robustness properties of the first class of optimization-based controllers are well understood, fewer studies have focused on stability and robustness properties of parametric optimization-based controllers.

Literature Review: Parametric optimization is a rich subdiscipline of optimization which studies solutions of optimization problems as a function of a parameter; see the textbook [4]. Parametric optimization is ubiquitous in systems and control, especially in model predictive control [1] and CBF-based control [3]. Closed-form solutions for certain classes of parametric programs were studied in [1, Chapter 5]. However, closed-form solutions are not always attainable. Regularity of solutions to parametric programs, namely establishing smoothness properties of their solutions, is a classical problem and has even pervaded systems and control [5], [6]. Compared to regularity results, there are fewer results on the stability of control systems with parametric optimization-based controllers.

One class of systems for which there have been results on stability and safety of systems driven by parametric optimization-based controllers are those coming from CBFs and control Lyapunov functions (CLFs) [3]. In these works, CLF and CBF constraints are jointly enforced in a state-dependent quadratic program (QP). To guarantee feasibility of the QP when the CLF and CBF inequalities cannot be jointly satisfied, the stability is commonly relaxed by introducing a slack variable. This relaxation results in a lack of stability guarantee even for arbitrarily large penalties on the slack variable [7]. Recent work, [8], studied a variant of the CLF-CBF QP controller and demonstrates how to estimate the basin of attraction of the origin.

Contributions: We consider LTI systems equipped with parametric projection-based controllers. As our main contribution, assuming linearity of the nominal controller and various well-posedness conditions, we obtain LMI-based sufficient conditions for exponential stability and the existence of a global Lyapunov function. Our proposed sufficient conditions generalize those presented in [9] which focus only on special classes of parametric QPs, whereas our controllers can incorporate more general convex constraints. Our results can also be seen as using similar ideas to those in [10] regarding sector bounds for projection operators.

Our analysis is based upon the virtual system methods in contraction theory and contractivity of Lur'e systems. For context, contraction theory is a computationally-friendly notion of robust nonlinear stability [11] and the virtual system method, first proposed in [12], is an analysis approach to establish exponential convergence for systems satisfying certain weak contractivity properties. As a tutorial contribution, we provide a novel review of the virtual system method in Section II-B. Specifically, we show that LTI systems with parametric projection-based controllers are in Lur'e form with state-dependent nonlinearity and that an appropriate virtual system can be designed in standard Lur'e form.

As our second main contribution, we establish in Theorem 1 a novel necessary and sufficient condition for absolute contractivity of Lur'e systems with cocoercive nonlinearities. In contrast, in [13] and [14, Proposition 4], monotone and Lipschitz nonlinearities are considered yet only sufficient conditions are provided. By focusing on cocoercive nonlinearities, we propose a relaxed LMI condition that is necessary and sufficient. See the related discussion in [15, Theorem 4.2] for other sufficient conditions.

As our third main contribution, we study the special LTI

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case of single integrators. We establish that all trajectories of the closed-loop system converge to the set of equilibria and that all trajectories converging to the origin do so exponentially fast with a known rate. While there are related results in the CBF/CLF literature [16], [17], this convergence result for the general class of parametric projection-based controllers is novel, to the best of our knowledge.

Finally, we study two applications, namely state-dependent saturated control systems and CBF-based control. For state-dependent saturated control systems, the maximal control effort depends on the state of the system and we demonstrate that our sufficient condition can be readily applied to yield a condition for global exponential stability. In CBF-based control, we consider a single integrator avoiding an obstacle and demonstrate that the results hold and provide evidence that the estimated exponential rate of convergence is tight. Specifically, we numerically observe that, in the case of single integrator dynamics, one does not need to enforce any CLF decrease condition to guarantee stability to the origin.[1]

II. Prerequisite Material

A. Contraction Theory

Given a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$, we let $\|\cdot\|_P$ be P-weighted ℓ_2 norm $\|x\|_P := \sqrt{x^\top P x}$, $x \in \mathbb{R}^n$ and write $\|\cdot\|_2$ if $P = I_n$.

Let the vector field $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in both of its arguments. Further let $P \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, and let there exist a constant $c > 0$ referred as *contraction rate*. We say that F is *strongly infinitesimally contracting with respect to* $\|\cdot\|_P$ *with rate* c if for all $x_1, x_2 \in \mathbb{R}^n$ and $t \geq 0$,

$$(F(t,x_1) - F(t,x_2))^\top P(x_1 - x_2) \leq -c\|x_1 - x_2\|_P^2. \quad (1)$$

If $x(\cdot)$ and $y(\cdot)$ are two trajectories satisfying $\dot{x}(t) = F(t, x(t)), \dot{y}(t) = F(t, y(t))$, then $\|x(t) - y(t)\|_P \leq \mathrm{e}^{-c(t-t_0)}\|x(t_0) - y(t_0)\|_P$ for all $t \geq t_0 \geq 0$. Although we present an integral definition of contractivity in (1), we remark that an equivalent differential characterization is available; see, e.g., [11] for a recent review of these tools.

B. Virtual System Method for Convergence Analysis

The *virtual system* analysis approach is a method to study the asymptotic behavior of a dynamical system that may not enjoy contracting properties. The virtual system approach was first proposed in [12], but we follow the systematic procedure advocated for in [11, Section 5.7]. For completeness sake, we describe this procedure below.

The virtual system analysis approach is as follows. We are given a dynamical system

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n \quad (2)$$

and we let $\phi_{x_0}(t)$ denote a solution from initial condition $x(0) = x_0$. The analysis proceeds in three steps:

[1]For more details on technical arguments, we refer to the extended version: https://arxiv.org/abs/2403.08159.

(i) *design* a time-varying dynamical system, called the *virtual system*, of the form

$$\dot{y} = f_{\text{virtual}}(y, \phi_{x_0}(t)), \quad y \in \mathbb{R}^d \quad (3)$$

satisfying a strong infinitesimal contractivity property with respect to an appropriate norm, e.g., the existence of a positive definite matrix $P \in \mathbb{R}^{d\times d}$ and a scalar $c > 0$ such that for all $y_1, y_2 \in \mathbb{R}^d, z \in \mathbb{R}^n$:

$$(f_{\text{virtual}}(y_1, z) - f_{\text{virtual}}(y_2, z))^\top P(y_1 - y_2) \leq -c\|y_1 - y_2\|_P^2;$$

(The vector field is called virtual since it is different from the nominal vector field, f, and does not correspond to any physically meaningful variation of f.)

(ii) *select* two specific solutions of the virtual system and state their incremental stability property:

$$\|y_1(t) - y_2(t)\|_P \leq \mathrm{e}^{-ct}\|y_1(0) - y_2(0)\|_P; \quad (4)$$

(iii) *infer* properties of the trajectory, $\phi_{x_0}(t)$, of the nominal system.

For example, if $d = n$ and $f(x) = f_{\text{virtual}}(x, x)$, then one can see that $\phi_{x_0}(t)$ is a solution for both systems and is often selected as one of the two specific solutions in step (ii). Additionally, if $f_{\text{virtual}}(\mathbb{0}_n, z) = \mathbb{0}_n$ for all $z \in \mathbb{R}^n$, then $\mathbb{0}_n$ is an equilibrium point for the virtual system and can be selected as one of the specific solutions. We refer to [12] for example applications leveraging the virtual system method.

III. Absolute Contractivity of Lur'e Systems

Consider the Lur'e system

$$\dot{x} = Ax + B\varphi(t, Kx), \quad (5)$$

where $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous in its first argument and cocoercive second argument. Specifically, φ cocoercive in its second argument means there exists a constant $\rho > 0$ such that for all $y_1, y_2 \in \mathbb{R}^m, t \geq 0$,

$$(\varphi(t, y_1) - \varphi(t, y_2))^\top (y_1 - y_2) \geq \rho\|\varphi(t, y_1) - \varphi(t, y_2)\|_2^2. \quad (6)$$

Notably, cocoercivity, (6), implies that φ is monotone and Lipschitz continuous with constant ρ^{-1} in its second argument. Many standard nonlinearities satisfy cocoercivity including projections onto convex sets and nonlinearities of the form $\varphi(t, y) = (\varphi_1(t, y_1), \dots, \varphi_m(t, y_m))$ where each φ_i is slope-restricted between 0 and ρ^{-1} in its second argument.

Akin to the classical problem of absolute stability, *absolute contractivity* is the property that the system (5) is strongly infinitesimally contracting for any nonlinearity φ obeying the constraint (6).

Theorem 1 (Necessary and sufficient condition for absolute contractivity): Consider the Lur'e system (5) and let $P \in \mathbb{R}^{n\times n}$ be positive definite. The system (5) is strongly infinitesimally contracting with respect to $\|\cdot\|_P$ with rate $\eta > 0$ for any φ satisfying (6) if and only if there exists $\lambda \geq 0$ such that

$$\begin{bmatrix} A^\top P + PA + 2\eta P & PB + \lambda K^\top \\ B^\top P + \lambda K & -2\lambda\rho I_m \end{bmatrix} \preceq 0. \quad (7)$$

Proof: Employing the shorthand $\Delta x = x_1 - x_2, \Delta y = y_1 - y_2 = K\Delta x, \Delta u_t = \varphi(t, y_1) - \varphi(t, y_2)$, the contractivity condition (1) for the system (5) is equivalently rewritten as

$$\Delta x^\top (PA + A^\top P + 2\eta P)\Delta x + 2\Delta x^\top PB\Delta u_t \leq 0. \quad (8)$$

Moreover, the cocoercivity condition (6) is equivalent to

$$\Delta u_t^\top (\rho \Delta u_t - K\Delta x) \leq 0. \quad (9)$$

Asking when the inequality (9) implies (8) is equivalent to the inequality (7) in light of the necessity and sufficiency of the S-procedure [18]. ■

Note that the condition in [13, Theorem 2] corresponds to the inequality (7) with $\lambda = 1$. Moreover the matrix in (7) has $A^\top P + PA + 2\eta P$ in its $(1,1)$ block compared to $A^\top P + PA + \eta I_n$ in [13]. This modification ensures that the inequality (1) holds rather than a related inequality with $-\frac{\eta}{2}\|x_1 - x_2\|_2^2$ on the right-hand side. Thus, by restricting our focus to cocoercive nonlinearities, we are able to find the sharpest condition for absolute contractivity.

IV. Parametric Projection-Based Controllers

We are interested in studying a continuous-time LTI system being driven by an parametric optimization-based controller. We say that the optimization problem is parametric since it is a function of the state. Specifically, we look at parametric projection-based controllers. More concretely, for $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, u^\star : \mathbb{R}^n \to \mathbb{R}^m, k : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, the LTI system and controller are:

$$\begin{aligned} \dot{x} &= Ax + Bu^\star(x), \\ u^\star(x) &:= \underset{u\in\mathbb{R}^m}{\arg\min} \quad \frac{1}{2}\|u - k(x)\|_2^2 \\ &\quad \text{s.t.} \qquad g(x,u) \leq \mathbb{0}_p. \end{aligned} \quad (10)$$

In the context of the parametric optimization problem in (10), k denotes a nominal feedback controller and g captures constraints on the controller as a function of the state. Such controllers commonly arise in CLF and CBF theory, where the parametric optimization problem in (10) is used to enforce that $u^\star$ either causes the closed-loop system to decrease a specified Lyapunov function or keep a certain set forward-invariant, respectively [3].

The question we aim to answer in this letter is the following: **What are conditions on the LTI system and the parametric optimization problem to ensure exponential stability of (10)?** Our main method for establishing sufficient conditions for exponential stability will be via the virtual system method in Section II-B.

A. Well-Posedness and Regularity of Solutions

In order to study the dynamical system (10), we need to ensure that it is well-posed. Several works in the literature have studied sufficient conditions for regularity of $u^\star$, e.g., continuity, Lipschitzness, or differentiability [5], [6], [19]. In this work, we utilize the following proposition from [6] which provides a sufficient condition for $u^\star$ to be continuous.

Proposition 2 ([6, Proposition 4]): Consider the map $u^\star : \mathbb{R}^n \to \mathbb{R}^m$ defined via the solution to the parametric optimization problem

$$\begin{aligned} u^\star(x) &:= \underset{u\in\mathbb{R}^m}{\arg\min} \quad f(x,u) \\ &\quad \text{s.t.} \qquad g(x,u) \leq \mathbb{0}_p. \end{aligned} \quad (11)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are each twice continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$. Further assume that for some $x_0 \in \mathbb{R}^n$, $f(x_0, \cdot)$ is strongly convex[2] and $g(x_0, \cdot)$ is convex and that there exists $\hat{u} \in \mathbb{R}^m$ such that $g(x_0, \hat{u}) \ll \mathbb{0}_p$[3]. Then there exists a neighborhood of x_0 such that $u^\star$ is continuous at every point in the neighborhood.

By existence theorems, we know that for the system $\dot{x} = Ax + Bu^\star(x)$, for each initial condition x_0 satisfying the assumptions of Proposition 2, there exists a positive constant $\tau_{\max}(x_0)$ and a continuously differentiable curve $\phi_{x_0} : [0, \tau_{\max}(x_0)) \to \mathbb{R}^n$ satisfying $\frac{d\phi_{x_0}}{dt}(t) = A\phi_{x_0}(t) + Bu^\star(\phi_{x_0}(t))$ for all $t \in [0, \tau_{\max}(x_0))$. We say that the solution ϕ_{x_0} is forward-complete if $\tau_{\max}(x_0) = +\infty$. While Proposition 2 ensures existence of solutions, we refer to [6] for discussions on conditions for uniqueness of solutions.

B. Stability Analysis for LTI Systems

Consider the dynamical system and its corresponding controller defined via a parametric optimization problem (10) and define the following sets

$$\Gamma(x) := \{u \in \mathbb{R}^m \mid g(x,u) \leq \mathbb{0}_p\} \quad \text{and} \quad (12)$$
$$\mathcal{K} := \{x \in \mathbb{R}^n \mid \exists \hat{u} \text{ s.t. } g(x,\hat{u}) \ll \mathbb{0}_p\}, \quad (13)$$

where $\Gamma(x)$ represents the feasible control actions at the state x and $\mathcal{K}$ denotes the points in state space where the feasible set, $\Gamma(x)$, has an interior.

We make the following assumptions on our problem:

(A1) **(Regularity of g)** The map $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is twice continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ and $g(x, \cdot)$ is convex for all $x \in \mathbb{R}^n$,

(A2) **(Existence of equilibrium and feasibility of zero control)** $\mathbb{0}_n \in \mathcal{K}$ and $\mathbb{0}_m \in \Gamma(x)$ for all $x \in \mathcal{K}$,

(A3) **(Linearity of nominal controller)** the map $k : \mathbb{R}^n \to \mathbb{R}^m$ is linear, i.e., $k(x) = Kx$ for some $K \in \mathbb{R}^{m\times n}$,

(A4) **(Dynamical feasibility)** for every $x_0 \in \mathcal{K}$, $\phi_{x_0}(t) \in \mathcal{K}$ for all $t \in [0, \tau_{\max}(x_0))$.

We make comments about some of these assumptions. Assumption (A2) ensures that $\mathbb{0}_n$ is an equilibrium point and that $u = \mathbb{0}_m$ is a feasible control action. Assumption (A4) ensures that the controller $u^\star$ does not drive the system outside the set of points where the feasible set of (10) has an interior. Outside of this set, the controller may fail to be continuous and solutions of (10) may fail to exist. One simple way to verify Assumption (A4) is to ensure that $\mathcal{K} = \mathbb{R}^n$. Note further that $u^\star(x)$ can compactly be written $u^\star(x) = \text{Proj}_{\Gamma(x)}(Kx)$, where given a nonempty, closed, convex set $\Omega \subseteq \mathbb{R}^m$, $\text{Proj}_\Omega(z) := \arg\min_{v\in\Omega} \|z - v\|_2$.

[2]A map $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex if there exists $\nu > 0$ such that $\nabla^2 f(x) \succeq \nu I_n$ for all x, where $\nabla^2 f$ denotes the Hessian of f.

[3]For two vectors, $v, w \in \mathbb{R}^n$, $v \ll w$ if $v_i < w_i$ for all $i \in \{1, \dots, n\}$.

We are now ready to state our first main theorem establishing the exponential stability of the system (10).

Theorem 3 (Exponential stability for LTI systems with parametric projection-based controllers): Consider the dynamics (10) satisfying Assumptions (A1)-(A4). Further suppose that there exist $P = P^\top \succ 0$, $\eta > 0$, and $\lambda \geq 0$ satisfying the inequality

$$\begin{bmatrix} A^\top P + PA + 2\eta P & PB + \lambda K^\top \\ B^\top P + \lambda K & -2\lambda I_m \end{bmatrix} \preceq 0. \quad (14)$$

Then from any $x_0 \in \mathcal{K}$,

(i) solutions to (10), ϕ_{x_0}, are forward-complete,

(ii) the origin is globally exponentially stable with bound

$$\|\phi_{x_0}(t)\|_P \leq \mathrm{e}^{-\eta t}\|x_0\|_P, \quad (15)$$

(iii) the mapping $V : \mathcal{K} \to \mathbb{R}_{\geq 0}$ given by $V(x) = x^\top P x$ is a global Lyapunov function for the dynamics (10).

Proof: We apply the virtual system method. Let $x_0 \in \mathcal{K}$ be arbitrary and consider the virtual system

$$\dot{y} = Ay + B\,\mathrm{Proj}_{\Gamma(\phi_{x_0}(t))}(Ky). \quad (16)$$

Note that for all $t \in [0, \tau_{\max}(x_0))$, $\mathrm{Proj}_{\Gamma(\phi_{x_0}(t))}$ obeys the inequality (6) with $\rho = 1$ due to cocoercivity of projections, see, e.g., [20, Eq. (2)]. Therefore the virtual system is a Lur'e system of the form (5). Theorem 1 implies that this virtual system is contracting with respect to $\|\cdot\|_P$ with rate $\eta > 0$. In other words, any two trajectories $y_1(\cdot), y_2(\cdot)$ for the virtual system satisfy for all $t \in [0, \tau_{\max}(x_0))$,

$$\|y_1(t) - y_2(t)\|_P \leq \mathrm{e}^{-\eta t}\|y_1(0) - y_2(0)\|_P. \quad (17)$$

First note that ϕ_{x_0} is a valid trajectory for the virtual system so we set $y_1(t) = \phi_{x_0}(t)$. Additionally note that $y_2(t) = \mathbb{0}_n$ is a valid trajectory for the virtual system since $\mathbb{0}_m \in \Gamma(x)$ for all $x \in \mathcal{K}$. Substituting these two trajectories implies (15) for $t \in [0, \tau_{\max}(x_0))$. We now establish that $\tau_{\max}(x_0) = +\infty$. To this end, note that the bound (15) implies that the trajectory ϕ_{x_0} remains in the compact set $\{x \in \mathbb{R}^n \mid \|x\|_P \leq \|x_0\|_P\}$ for $t \in [0, \tau_{\max}(x_0))$ for which it is defined. Since this set is compact, the trajectory cannot escape in a finite amount of time, meaning that the trajectory is forward complete. This reasoning proves statements (i) and (ii). Statement (iii) is a consequence of (ii). ■

The key insight in Theorem 3 is that LTI systems with parametric projection controllers are a type of state-dependent Lur'e system $\dot{x} = Ax + B\,\mathrm{Proj}_{\Gamma(x)}(Kx)$ and that the virtual system method transforms them into standard time-varying ones with cocoercive nonlinearities, where Theorem 1 applies. Specifically, this analysis method allows us to treat general convex constraints in a unifying manner. We remark that one could also use classical results such as the circle criterion to establish stability of (10) but we elect to use the virtual system method to highlight its utility.

In the case that Assumptions (A1)-(A4) hold, $K = 0$, and the inequality (14) holds, the dynamics simply become $\dot{x} = Ax$ since $\mathbb{0}_n \in \Gamma(x)$ for all $x \in \mathcal{K}$. In other words, $K = 0$ always ensures global exponential stability under the stated assumptions since (14) implies that A is Hurwitz. Theorem 3 provides a novel sufficient condition for the system (10) to be exponential stable with $K \neq 0$ which has been designed to make $A + BK$ stable, e.g., using LQR.

C. Stability Analysis for Single-Integrators

A special class of LTI systems for which we can ensure different results is the single integrator $\dot{x} = u^\star(x)$, namely the system (10) with $A = 0$, $B = I_n$. For this system, (14) will never hold with $\eta > 0$ since A is not Hurwitz.

Theorem 4 (Exponential stability for single integrators with parametric projection-based controllers): Consider the dynamics (10) with $A = 0$ and $B = I_n$ and suppose Assumptions (A1)-(A4) hold. Further suppose $K = K^\top \preceq -\eta I_n, \eta > 0$. Then from any $x_0 \in \mathcal{K}$,

(i) solutions to (10), ϕ_{x_0}, are forward-complete and

(ii) solutions asymptotically converge to the set of equilibria, $\mathcal{X}_{\mathrm{eq}} := \{x \in \mathbb{R}^n \mid \mathrm{Proj}_{\Gamma(x)}(Kx) = \mathbb{0}_n\}$.

Moreover, under the additional assumption that $\mathbb{0}_n \in \mathrm{int}(\Gamma(\mathbb{0}_n))$, the following statement holds:

(iii) if $\phi_{x_0}(t) \to \mathbb{0}_n$ as $t \to \infty$, then there exists $M(x_0) > 0$ such that

$$\|\phi_{x_0}(t)\|_2 \leq M(x_0)\mathrm{e}^{-\eta t}\|x_0\|_2. \quad (18)$$

Proof: Consider as a Lyapunov function candidate $V(x) = -\frac{1}{2}x^\top K x$. The Lie derivative of V along trajectories of the dynamical system (10) is

$$\begin{aligned}\dot{V}(x) &= -x^\top K\,\mathrm{Proj}_{\Gamma(x)}(Kx) \\ &= -(Kx - \mathbb{0}_n)^\top(\mathrm{Proj}_{\Gamma(x)}(Kx) - \mathrm{Proj}_{\Gamma(x)}(\mathbb{0})) \\ &\leq -\|\,\mathrm{Proj}_{\Gamma(x)}(Kx) - \mathrm{Proj}_{\Gamma(x)}(\mathbb{0}_n)\|_2^2 \leq 0,\end{aligned}$$

where we have used that $\mathbb{0}_n \in \Gamma(x)$ for all x and cocoercivity of $\mathrm{Proj}_{\Gamma(x)}$, see, e.g., [20, Eq. (2)]. Since $\dot{V}(x) \leq 0$, we preclude finite escape time and thus conclude statement (i). To establish asymptotic convergence to $\mathcal{X}_{\mathrm{eq}}$, we invoke LaSalle's invariance principle and see that trajectories converge to the largest forward-invariant set in $\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$. However, since $\dot{V}(x) \leq -\|\,\mathrm{Proj}_{\Gamma(x)}(Kx)\|_2^2$, $\dot{V}(x) = 0$ if and only if $\mathrm{Proj}_{\Gamma(x)}(Kx) = \mathbb{0}_n$, i.e., $x \in \mathcal{X}_{\mathrm{eq}}$. This argument establishes statement (ii). To establish statement (iii), note that $\mathbb{0}_n \in \mathrm{int}(\Gamma(\mathbb{0}_n))$ implies, by continuity of g, that there exist an open neighborhood, $\mathcal{O}_x$ containing the origin such that $g(x, Kx) \ll \mathbb{0}_p$ for all $x \in \mathcal{O}_x$. In other words, inside this neighborhood, $u^\star(x) = Kx$. Thus, the dynamics (10) are locally exponentially stable inside this neighborhood. Since the trajectory is asymptotically converging to the origin and locally exponentially stable inside $\mathcal{O}_x$, we conclude (18). ■

To prove Theorem 4, one could alternatively use the virtual system method to establish that $\|\phi_{x_0}(t)\|_{-K} \leq \|x_0\|_{-K}$ and then invoke LaSalle's invariance principle. We opt to give a more direct Lyapunov proof for simplicity.

V. Applications

A. State-Dependent Saturation Control

For $v \in \mathbb{R}^n_{>0}$, define the saturation function $\mathrm{sat}_v : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathrm{sat}_v(x) := \max\{-v, \min\{v, x\}\}$, where the max and

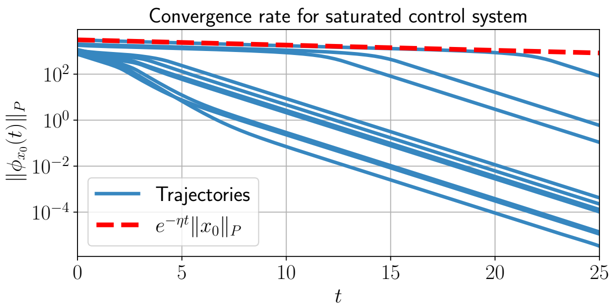

Fig. 1. The evolution of $\|\phi_{x_0}(t)\|_P$ for 10 different x_0 and P is chosen to maximize η in (14). We also plot $\mathrm{e}^{-\eta t}\|x_0\|_P$, where $\|x_0\|_P$ denotes the largest value of $\|x_0\|_P$ over all randomly generated initial conditions.

min are applied entrywise. An alternative characterization of the saturation function is via the minimization problem $\mathrm{sat}_v(x) = \arg\min_{u\in\mathbb{R}^n}\{\|u-x\|_2^2 \mid -v \le u \le v\}$. We consider the state-dependent saturated control system

$$\dot{x} = Ax + B\mathrm{sat}_{v(x)}(Kx), \tag{19}$$

where $v : \mathbb{R}^n \to \mathbb{R}^n_{>0}$ is a twice continuously differentiable map dictating actuation constraints as a function of the state.

When v is constant, one can use results from saturated control systems to assess the stability of (19). As v is state-dependent, one cannot apply these techniques here. It is straightforward to see that the system (19) is of the form (10) with $g(x, u) = (u-v(x), -u-v(x))$. Moreover, it is routine to establish that Assumptions (A1)-(A4) hold. Therefore, Theorem 3 may be applied to provide a sufficient condition for the global exponential stability of the system (19).

Example 1: We consider the system (19), where $n = 3, m = 2$, $A = -I_3 + N$, $B = \left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]^\top$, and $N \in \mathbb{R}^{3\times 3}$ is a random matrix with entries drawn from the standard normal distribution. We assume that $K \in \mathbb{R}^{2\times 3}$ is selected so that $u = Kx$ minimizes the objective $\int_0^\infty (x(t)^\top x(t) + u(t)^\top u(t))dt$ for $\dot{x} = Ax + Bu$. We let $v(x) = \mathrm{e}^{-\|x\|_2^2/2}\mathbb{1}_m$, where $\mathbb{1}_m$ is the all-ones vector. We find $\eta > 0, \lambda \ge 0, P \in \mathbb{R}^{3\times 3}$ satisfying (14) such that η is maximized and plot values of $\|\phi_{x_0}(t)\|_P$ for 10 different samples of x_0 from the multivariate normal distribution $\mathcal{N}(\mathbb{0}_3, 4I_3)$ in Figure 1.

We see that all trajectories converge exponentially to the origin and that the exponential convergence rate from Theorem 3 is $\eta = 0.0525$. Empirically, we see that when trajectories are far from $\mathbb{0}_3$, this rate is tight since $v(x) \approx \mathbb{0}_m$ for x far from $\mathbb{0}_3$ and that the rate is very loose when trajectories are close to $\mathbb{0}_3$ since $v(x) \approx \mathbb{1}_m$ for $x \approx \mathbb{0}_3$.

Although we have focused on state-dependent saturation control in this section, we would like to note that many structured parametric optimization problems with convex constraints may be handled. Specifically, our methods are not simply restricted to saturations and can be applied to richer classes of examples in a methodological manner.

B. Stability with Control Barrier Functions

Consider the nonlinear control-affine system

$$\dot{x} = F(x) + G(x)u, \tag{20}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n, G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz.

Let $\mathcal{C} \subseteq \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a sufficiently smooth function such that $\mathcal{C} = \{x \in \mathbb{R}^n \mid h(x) \ge 0\}$. The set $\mathcal{C}$ is referred to as the "safe set".

Definition 5 (Control Barrier Function [3, Definition 3]): The function h is a control barrier function (CBF) for $\mathcal{C}$ if there exists a locally Lipschitz and strictly increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that for all $x \in \mathcal{C}$, there exists $u \in \mathbb{R}^m$ with

$$\nabla h(x)^\top F(x) + \nabla h(x)^\top G(x)u + \alpha(h(x)) \ge 0. \tag{21}$$

A continuous controller $u : \mathbb{R}^n \to \mathbb{R}^m$ which strictly satisfies (21) for all $x \in \mathcal{C}$ renders $\mathcal{C}$ forward-invariant under the dynamics (20) [21, Theorem 4].

A common way to synthesize controllers that render $\mathcal{C}$ forward invariant is via a parametric QP [3]. To this end, we consider a single-integrator being driven by the CBF constraint (21) and actuator constraints:

$$\begin{aligned} \dot{x} &= u^\star(x), \\ u^\star(x) := \arg\min_{u\in\mathbb{R}^m} \quad & \frac{1}{2}\|u - Kx\|_2^2 \\ \text{s.t.} \quad & -\nabla h(x)^\top u \le \alpha(h(x)), \\ & -\bar{u}\mathbb{1}_n \le u \le \bar{u}\mathbb{1}_n, \end{aligned} \tag{22}$$

where $\bar{u} > 0$. While safety of these systems was previously studied in, e.g., [3], we aim to also study their stability properties. To study convergence of (22), we can check for conditions under which the hypotheses of Theorem 4 hold.

Corollary 6 (Exponential stability for single integrators with CBF-based controllers): Consider the dynamics (22) and suppose (i) h is a CBF for $\mathcal{C}$, (ii) $K = K^\top \preceq -\eta I_n$, (iii) $\mathbb{0}_n \in \mathrm{int}(\mathcal{C})$, and (iv) h and α are thrice and twice continuously differentiable, respectively. Then from any $x_0 \in \mathrm{int}(\mathcal{C})$,

(i) solutions to (22), ϕ_{x_0}, are forward-complete,
(ii) solutions remain in $\mathcal{C}$ for all $t \ge 0$,
(iii) solutions converge to the set of equilibria and
(iv) if $\phi_{x_0}(t) \to \mathbb{0}_n$ as $t \to \infty$, then there exists $M(x_0) > 0$ such that

$$\|\phi_{x_0}(t)\|_2 \le M(x_0)\mathrm{e}^{-\eta t}\|x_0\|_2. \tag{23}$$

Proof: It is straightforward to verify Assumptions (A1)-(A4) and that $\mathbb{0}_n \in \mathrm{int}(\Gamma(\mathbb{0}_n))$. ■

In general, we cannot conclude uniqueness of equilibria. As we will see in the following example, the dynamics may have multiple equilibria, even in the case of simple CBFs. These results agree with the theory presented in [16], [17].

Example 2: Consider a single integrator in $\mathbb{R}^2$ avoiding a disk-shaped obstacle centered at $(0, 4)$ with radius 2. The corresponding CBF is $h(x_1, x_2) = x_1^2 + (x_2 - 4)^2 - 4$ with $\alpha(r) = r$. We take $K = \left[\begin{smallmatrix}-2&-0.5\\-0.5&-1\end{smallmatrix}\right]$ and $\bar{u} = 1$. We plot numerical simulations of (22) with these parameters along with the corresponding convergence rate and value of CBF along trajectories in Figure 2.

We observe that trajectories converge to the set of equilibria and the majority of them converge to the origin with

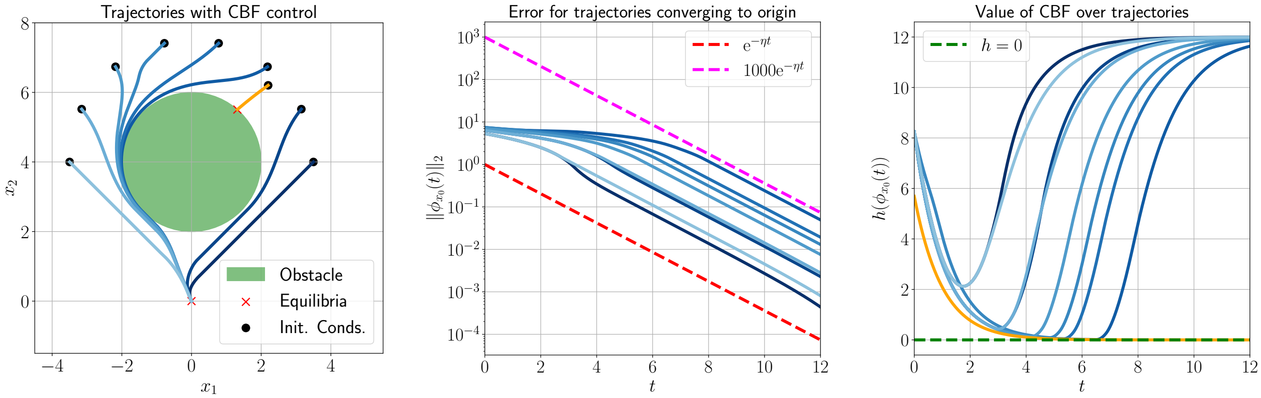

Fig. 2. The left figure shows plots of trajectories of (22). We can see that most trajectories, indicated by shades of blue, converge to the origin, while one converges to a point on the boundary of the safe set, shown in orange. The center figure plots the convergence rate of trajectories that converge to the origin. It also plots $\mathrm{e}^{-\eta t}$ and $1000\mathrm{e}^{-\eta t}$ and demonstrates that the exponential convergence rate in (23) cannot be improved in this instance and that $1000 > M(x_0)\|x_0\|_2$ for these initial conditions. The right figure plots the evolution of the CBF, h, along trajectories.

exponential convergence rate predicted in (23). While the equilibrium point on the boundary of the safe set is unavoidable in this example, we can numerically observe that there are no equilibria inside $\mathrm{int}(\mathcal{C})$ other than the origin. This result is in contrast with the CLF-CBF controllers studied in [16], [17], where there may exist additional equilibria in $\mathrm{int}(\mathcal{C})$. This numerical example provides evidence that, for exponential stability of the origin, one does not need to rely upon any CLF decrease condition, except on a measure zero set.

VI. Discussion and Future Work

In this letter we study LTI systems with controllers solving a special class of parametric programs, namely parametric projections. Using the virtual system method and a novel contractivity result for Lur'e systems, we provide sufficient conditions for the exponential stability of these systems. Separately, for single integrators, we prove convergence to the set of equilibria and an exponential convergence rate for trajectories converging to the origin. As applications, we consider state-dependent saturated control systems and CBF-based control.

We believe that there are many avenues for future research. First, it would be useful to explore relaxing Assumptions (A1)-(A4) to allow for a larger class of LTI systems, possibly using the tools in [22]. Second, it would be useful to characterize the set of trajectories converging to the origin in Theorem 4. Finally, it is important to study systems whose controllers only approximately solve the parametric program.

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