

Equilibration of Coordinating Imitation and Best-response Dynamics

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Abstract—Decision-making individuals are often considered to be either *imitators* who copy the action of their most successful neighbors or *best-responders* who maximize their benefit based on the frequency of their neighbors. In the context of *coordination games*, where individuals earn more if they take the same action as those of their neighbors, by means of potential functions, it was shown that populations of all imitators and populations of all best-responders equilibrate in finite time when they become active to update their decisions sequentially. However, for mixed populations of the two, the equilibration was shown only for specific finite activation sequences. It is therefore, unknown, whether a potential function also exists for mixed populations or if there actually exists a counter example where an activation sequence prevents the population from reaching an equilibrium. We show that the number of consecutive individuals who have taken the same action in a *path network* serves as a potential function, leading to equilibration, and that this result can be extended to *sparse trees*. The existence of a potential function for other types of networks remains an open problem.

I. INTRODUCTION

Evolutionary game theory has been successfully applied in different applications ranging from cancer and epidemiology to finance and rumour propagation [1]–[4]. In the context of decision-making, individuals are modeled as game-playing agents who choose from a number of available strategies and accordingly earn payoffs against their matched opponents. The agents revise their decisions according to some update rules, the most common being (*myopic*) *best-response* and *imitation*. An agent following best-response, called a *best-responder*, chooses the strategy that maximizes its payoff against its neighbors given that they would not change their strategies. On the other hand, an agent following imitation, known as an *imitator* simply imitates a neighbor with a higher payoff. The wide use of best response by human has been confirmed in experimental studies [5]. Similarly, imitation behavior emerges in several real-world scenarios, such as the supervisor’s role in “sweethearting behavior” among employees [6], the role of imitation in building cultural intelligence [7], and training language models [8].

Researchers have explored the existence and convergence towards an equilibrium point in both imitation and best-response dynamics [9]–[12]. In the anti-coordination context where the highest-earning decision is the opposite of the opponent’s, a population of best-responders converges to an equilibrium state [13]. The same situation is true for a population of best-responders in the coordination context, where the highest-earning strategy matches the opponent’s [13]. The convergence to an equilibrium is, however, guaranteed only for a coordination population of imitators [14, Theorem 1]. All of these studies used a potential function to

prove equilibrium convergence. Clearly, a mixed population of imitators and best-responder may not equilibrate and undergo perpetual fluctuations. The outcome is known for the anti-coordination case: equilibration takes place if and only if there exists an equilibrium [15]. What about a mixed population of imitators and best-responders in the coordination context? The existence of an activation sequence was established in [16] that would drive any such mixed population to an equilibrium state. It however remains open whether a potential function exists for such populations, or if there is a counter example where an activation sequence can prevent a mixed population from equilibration.

We start tackling this problem for the simple path network and find that the number of so-called *sections* (consecutive number of same-strategy playing agents) serves as a potential function. We then extend the results to a ring network. Next, we proceed to *starlike* networks that is a tree with only one agent with a degree greater than two, which has several “branches.” We show that there always exists a branch where the number of sections in that branch will again be a potential function, leading to the convergence of starlikes. Finally, we generalize the idea to *sparse tree* networks that are trees, where the distance between each two branching nodes is at least three.

II. MODEL

Consider an undirected network \mathcal{G} over a finite set $\mathcal{N} = \{1, 2, \dots, n\}$ of agents who decide between strategies A and B over time $t = 0, 1, 2, \dots$. For each agent $i \in \mathcal{N}$, the network defines a set of *neighbors* $\mathcal{N}_i \subseteq \mathcal{N} \setminus \{i\}$ that are linked to agent i . At every time step, each agent $i \in \mathcal{N}$ plays a two-player *coordination game* with each of its neighbors $j \in \mathcal{N}_i$ and earns a payoff according to their strategies and its payoff matrix

$$\pi^{ij} = \begin{bmatrix} R_{ij} & S_{ij} \\ T_{ij} & P_{ij} \end{bmatrix}, R_{ij}, P_{ij} > T_{ij}, S_{ij} \quad (1)$$

where R_{ij}, S_{ij}, T_{ij} , and P_{ij} are agent i ’s payoffs when agents i and j play strategy pairs (A, A), (A, B), (B, A), and (B, B). The inequality $R_{ij}, P_{ij} > T_{ij}, S_{ij}$ represents the coordination setting of the game, implying that each player earns more if it plays the same strategy as its neighbor. Then agent i ’s *utility* u_i is the accumulated payoff earned against all of its neighbors:

$$u_i(\mathbf{x}) = \sum_{j \in \mathcal{N}_i} \pi_{x_i, x_j}^{ij}$$

where x_k is the strategy of agent k , the *state* $\mathbf{x} = [x_k]$ is the vector of all agents’ strategies, and \mathbf{X}_{pq} denotes the

entry of matrix \mathbf{X} at row p and column q . Agents update their strategies based on the type of *update rule* they follow, which is either *best response*, that is to choose the strategy that maximizes its utility, or *imitation*, that is to copy the strategy of its highest earning neighbor. The updates happen asynchronously over time, i.e., at each time step, a single agent becomes active to update her strategy at the next time step. More specifically, agent i active at time t updates its strategy at time $t + 1$ to the following if it is an *imitator*:

$$x_i(t + 1) = x_k(t), \quad k = \arg \max_{j \in \mathcal{N}_i} u_j(t). \quad (2)$$

and to the following if it is a *best-responder*:

$$x_i(t + 1) = \arg \max_{x \in \{A, B\}} u_i(x_{i=x}(t)) \quad (3)$$

where $x_{i=x}$ is the vector x where the i^{th} entry is fixed to strategy X . In the case where both strategies A and B maximize the utilities in (2) or (3), agent i does not switch strategies, i.e., $x_i(t + 1) = x_i(t)$.

Example 1. [Programming languages] Given the required effort to master a new programming language, programmers have to decide between two options each time they program an application: (i) the comfort of working in the already experienced language and (ii) the benefit of learning a new language. Some base their decisions on the prevalence of the language, because commonly used languages are supported by a community of peers who can smoothen the learning experience via online forums. Others may base their decisions on how successful other programmers were in terms of, e.g., their salaries or reputation of developed applications.

The agents here are the community of App developers who interact via online networks. The programming languages Python and Java may be considered as the strategies. An example of the payoff matrix for a particular programmer is

$$\begin{array}{cc} & \begin{array}{c} \text{Python} \\ \text{Java} \end{array} \\ \begin{array}{c} \text{Python} \\ \text{Java} \end{array} & \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}, \quad (4)$$

indicating that he would benefit more from his peers if they use the same language.

Example 2. [Social media] Telegram and WhatsApp are two social media applications. Individuals choosing one of them as their main communication stream may decide based on the frequency or satisfaction of their friends on each platform, implying the best response and imitation update rules respectively. The individuals also have personal preferences over the apps because of their features, resulting in different payoff matrices.

Define the agents' *activation sequence* as the sequence $\langle a_t \rangle_{t=0}^{\infty}$, where a_t is the active agent at time t . An activation sequence together with update rules (2) and (3) govern the population state $\mathbf{x}(t)$ and define the *decision-making dynamics*, which we refer to as the *coordinating best-response and*

imitation dynamics. A state $\mathbf{x}^* \in \{A, B\}^n$ is an *equilibrium* of the dynamics if under every activation sequence,

$$\mathbf{x}(0) = \mathbf{x}^* \Rightarrow \mathbf{x}(t) = \mathbf{x}^* \forall t \geq 0. \quad (5)$$

We are interested in determining the asymptotic behaviour of the dynamics.

In view of the coordinating condition (1), update rule (3) for best-responders can be simplified as

$$x_i(t + 1) = \begin{cases} A & \text{if } n_i^A(\mathbf{x}) > n_i^*, \\ B & \text{if } n_i^A(\mathbf{x}) < n_i^*, \\ x_i(t) & \text{otherwise,} \end{cases} \quad (6)$$

where n_i^A is the number of A-playing neighbors of agent i , and n_i is agent i 's *temper* defined by $n_i^* = |\mathcal{N}_i| \frac{P_{ij} - S_{ij}}{R_{ij} - T_{ij} + P_{ij} - S_{ij}}$. It follows that if agent i tends to play A at some state, so does it at any other state with more A-playing neighbors. In a more restrictive sense, it can be also shown that if an imitator tends to play A at some state, so does it at any other state where all of its A-playing neighbors still play A. This property is referred to as *A-coordinating* [16, Definition 2], based on which, the existence of an activation sequence that would drive the dynamics from a given initial condition to an equilibrium state was shown in [16, Lemma 1, Theorem 2]. However, it remains open whether the dynamics equilibrate under an arbitrary activation sequence. We investigate the problem for the simple path network structure, and then extend the results to a ring, a starlike, and finally a "sparse" tree.

III. PATH NETWORKS

Consider network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with edge set \mathcal{E} is $\{(i, i + 1) \mid i = 1, \dots, n - 1\}$, called a *path*. We call agent i a *right-border* (resp. *left-border*) if it has a different strategy compared to agent $i + 1$ (resp. $i - 1$). A single agent with a strategy different from those of its two neighbors is both a right and left border agent. Agent i is a *border* if it is right or left-border (or both). We consider the most "left" (resp. "right") agent, i.e., agent 1 (resp. n), as a left (resp. right) border agent. Two consecutive borders with the same strategy form a *section* (Figure 1).

Definition 1 (Section). A section in a path network at a given strategy state is a set of consecutive same-strategy playing agents $p, p + 1, \dots, q$, where $q \geq p$ and agents p and q are borders. The size of the section is defined as $q - p + 1$.

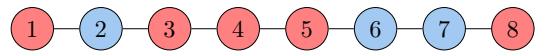


Fig. 1: A path network with three sections. The sections in this path are $\{1\}$, $\{2\}$, $\{3, 4, 5\}$, $\{6, 7\}$, and $\{8\}$. Blue and red are used for strategies A and B, respectively. Agents 1, 2, and 8 are each both left and right borders. Agents 3, and 6 are only left-borders, while agents 5 and 7 are right-borders. Agent 4 is a non-border agent.

The special case of $p = q$ results in a size-one section consisting of a single agent. The number of sections appears

to serve as a potential function according to the following lemma. The key idea of the proof is that the emergence of a new section requires the sequence (A, A, A) (resp. (B, B, B)) to turn into (A, B, A) (resp. (B, A, B)), which is impossible due to the coordinating nature of the dynamics.

Lemma 1. *The number of sections in a path network does not increase under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof: A change in the population state takes place only if a border agent is active because other agents play the same strategy as their neighbors and hence do not switch strategies according to update rules (6) and (2). So the number of sections change at time t only if some border agent i becomes active at time $t-1$ and switches its strategy at time t to s . At least one neighbor of the border agent plays s at time $t-1$ as otherwise, the agent is not border. We have the following two cases:

Case 1. Agent i has two neighbors, i.e., $i \notin \{1, n\}$. If both neighbors play s , then the border agent itself forms a section at time $t-1$, which disappears at time t . Since no other sections are generated, this results in a reduction in the number of sections. If only one neighbor plays s , then the number of sections does not change after the switch.

Case 2. Agent i has one neighbor, i.e., $i \in \{1, n\}$. Then the neighbor plays s at time $t-1$, implying that agent i itself again forms a section, which disappears at time t , resulting in a reduction.

The proof is complete as the number of sections does not increase in both cases. ■

As the number of sections are finite, in view of Lemma 1, there exists some time $T > 0$ when the number of sections becomes fixed and no longer changes. The sections may still expand or shrink though, preventing equilibration. However, one can show that once a section expands from a certain direction, say left (i.e., in the descending order of the agents' labels), then it may no longer shrink from left. Namely, if the left border of a section "moves" left after time T , it never "moves" right in the future. This idea is rigorously captured in the following lemma.

For every time $t \geq T$, there is the same number of sections which we label as $1, 2, \dots, S$ from left to right; that is in the ascending order of their left borders. Denote by $L_s(t)$ and $R_s(t)$ the left and right borders of section s at time $t \geq T$. Given a sequence of consecutive agents $p, p+1, \dots, q$, where $q \geq p$, we denote their strategies by $\mathbf{x}_{(p,p+1,\dots,q)}$.

Lemma 2. *Consider the time T when the number of sections in the path network is fixed. Then for every section s and any time $t_1 \geq T$,*

$$L_s(t_1 + 1) = L_s(t_1) - 1 \Rightarrow \forall t \geq t_1 \ L_s(t + 1) \leq L_s(t), \quad (7)$$

$$R_s(t_1 + 1) = R_s(t_1) + 1 \Rightarrow \forall t \geq t_1 \ R_s(t + 1) \geq R_s(t).$$

Proof: We prove the first equation by contradiction; the proof of the second equation is similar. Assume the contrary and let $t_3 > t_1$ be the first time (7) is violated, i.e., $L_s(t_3 +$

$1) = L_s(t_3) + 1$. Let $t_2 \in [t_1, t_3 - 1]$ be the last time that the left border of s decreased, i.e., $L_s(t_2 + 1) = L_s(t_2) - 1$. Let agent i be the left border of section s at time t_2 , i.e., $i = L_s(t_2)$. Then

$$L_s(t_2) = i, \quad (8)$$

$$L_s(t) = i - 1 \quad \forall t \in [t_2 + 1, t_3], \quad (9)$$

$$L_s(t_3 + 1) = i. \quad (10)$$

Without loss of generality, assume that $x_i(t_2) = B$. It is straightforward to show that if the agents of section s play a strategy, say B, at time T , then the agents of section s will play B at every future time step as well. Therefore, since agent i is the left border of section s at time t_2 and plays B at t_2 , it follows that all the agents in section s play B at every time $t \geq T$. Thus, in view of (8) to (10),

$$\begin{aligned} \mathbf{x}_{(i-2,i-1,i)}(t_2) &= (A, A, B), \\ \mathbf{x}_{(i-2,i-1,i)}(t_2 + 1) &= (A, B, B), \\ \mathbf{x}_{(i-2,i-1,i)}(t) &= (A, B, *) \quad \forall t \in [t_2 + 1, t_3 - 1] \\ \mathbf{x}_{(i-2,i-1,i)}(t_3) &= (A, B, B), \\ \mathbf{x}_{(i-2,i-1,i)}(t_3 + 1) &= (A, A, B). \end{aligned}$$

The reason why $x_{i-2}(t_2) = A$ is that otherwise a section would be removed at $t_2 + 1$, which is impossible. Similarly, $x_{i-2}(t_3) = A$ as otherwise a new section would be generated at time $t_3 + 1$.

Now we show that the two switches of strategies of agent $i-1$ at times $t_2 + 1$ and $t_3 + 1$ are in conflict. Note that at both times t_2 and t_3 agent i plays B but has at time t_2 at most and at time t_3 at least one other B-playing neighbor. So as the game is coordinating, i.e., in view of (1), $u_i(t_3) \geq u_i(t_2)$. We reach a contradiction in view of Lemma 6 and by letting $T = t_2$ and $T' = t_3$.

We are ready to prove the equilibration of paths. Consider a section s at time T . We say that *the left border of section s moves left at time $t \geq T + 1$* if $L_s(t) = L_s(t-1) - 1$ and *moves right* if $L_s(t) = L_s(t-1) + 1$. Similarly, the movement of the right border is defined.

Proposition 1. *A path network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof: Consider some section s at time T when the number of sections is fixed. If the left border of section s moves left at any future time, then it can only move left afterward according to Lemma 2. Since the path is constrained from left by agent 1, the left border of section s will be fixed at some time. Similarly, the right border will be fixed if it moves right at some point. So if the left border moves left at some time and the right border moves right, then the borders of section s will be fixed for all future times.

Now if any of the borders, say right, becomes fixed but the left one only moves right after time T , then also the left border becomes fixed at some point as it cannot pass the fixed right border. On the other hand, if the right border only

moves left after time T and the left only moves right, again the two will become fixed as they cannot pass each other.

Therefore, the borders of section s will become fixed at some finite time. Since s was an arbitrary section, it holds that at some finite time, the borders of every section becomes fixed. This implies equilibration, completing the proof.

A. Extension to rings

A network $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with edge set \mathcal{E} is $\{(1, 2), (2, 3), \dots, (n, n + 1), (n + 1, 1)\}$ is a *ring*.

Proposition 2 (Rings). *A ring network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof: Following the same arguments used for the proof of the equilibration of a path, it can be shown that the number of sections in a ring may not increase, and hence will become fixed at some time T , and that if a section moves right at some time, it may never move left afterwards. So the only possibility for the non-equilibration of a ring is that some section s only and infinitely often moves right or only and infinitely often moves left. Consider the first case, i.e., moving right. Then for every agent i in the ring, there exists a time when it belongs to the section s and a time when it does not belong to the section. Hence, it will undergo the switches from $\mathbf{x}_{(i-1, i, i+1)} = (A, A, B)$ to (A, B, B) and from (B, B, A) to (B, A, A) . So the agent decides differently at two states with the same number of A and B-playing neighbors. Thus, in view of (6), agent i , and hence, the whole ring are imitators. In view of the convergence result in [14, Theorem 1] for arbitrary networks of all coordinating imitators, the proof is complete. ■

IV. STARLIKE POPULATIONS

We now proceed to a more general network: The *starlike* [17], that is a tree with at most one node having more than two neighbours, which we refer to as the *branching agent*. Define a *branch* as a path that begins from a neighbour of the branching agent and ends with a leaf but does not contain the branching agent itself (Figure 2-a). The following definition

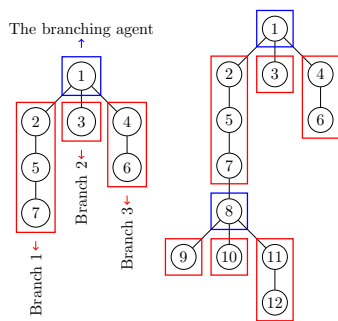


Fig. 2: a) Starlike graph. The agent on the top is the branching agent. The graph has three branches. **b) A sparse-tree population.** Each red section demonstrates a line in the population. Blue agents are branching agents of the population.

is motivated by the notion of *eventually periodic sequences* [18].

Definition 2 (Eventually periodic sequences). *The activation sequence $\langle a_t \rangle_{t=0}^\infty$ is eventually periodic if both the sequence and the resulting states of the decision-making dynamics become periodic after some finite time t_0 , i.e.,*

$$\exists T, t_0 \in \mathbb{N} : (a_{t_0+t+T} = a_{t_0+t}, \mathbf{x}_{t_0+t+T} = \mathbf{x}_{t_0+t} \quad \forall t \in \mathbb{N}),$$

where T as the periodicity of both the states and activation sequence after time t_0 , and time interval $[t_0, \infty)$ as the periodic part of the activation sequence.

The following result holds for general decision-making dynamics but is framed here according to Section II.

Lemma 3. *If the imitation and best-response dynamics do not equilibrate under some persistent activation sequence, then there also exists a persistent eventually periodic activation sequence under which the population does not equilibrate.*

Lemma 4. *A starlike network equilibrates under the coordinating best-response and imitation dynamics, if the branching fixes does not switches strategies infinitely many times.*

Proof: It is straightforward to show that Lemmas 1 and 2 and hence proposition 1 hold if any of the end nodes in a line network fix their strategies. Consequently, every branch together with the branching agent forms a path in the starlike that will equilibrate, leading to the equilibration of the whole starlike. ■

The equilibration of starlike networks is established in Proposition 3. The idea of the proof is to focus on the times when the branching agent has the maximum number of same strategy, say B,-playing, neighbors. The moment one of these neighbors, referred to as the “special agent” switches, the number of sections in the branch containing this special agent, referred to as the “special branch”, will decrease, and this decrement will never be compensated in the future. So the number of sections in the the spacial branch is an energy-like function (see (11)). If before any of the neighbors switch, the branching agent itself switches, then the branching agent must be an imitator and the neighbor with the maximum utility will serve as the special agent. Given a path P , denote the number of sections in P by $n(P)$, and more specifically by $n(P, t)$ to denote the number at time t .

Proposition 3 (Starlike). *A starlike network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof: We prove by contradiction. By assuming the contrary, it follows from Lemma 3, the existence of a persistent eventually periodic activation sequence denoted by $\langle b_t \rangle_{t=0}^\infty$ with periodic part $[t_0, \infty)$. The branching agent, say i , switches strategies under $\langle b_t \rangle_{t=0}^\infty$ infinitely often; otherwise, the network equilibrates due to Lemma 4. Let $t_B \geq t_0$ be the first time agent i plays B and has the maximum number of B-playing neighbours during $[t_0, \infty)$. Denote by

t_1 the first moment after time t_B when either agent i or one of its neighbors switches strategies, resulting in the following two cases:

Case 1: An agent i 's neighbour, say agent j , changes its strategy at time t_1 . Consider the branch P (referred to as the “special” branch) including agent j (referred to as the “special” agent). Denote the times when agent j switches strategies after t_1 by t_2, t_3, \dots . Out of these time steps, let $\langle t_k^B \rangle_{k=1}^\infty \subset \langle t_k \rangle_{k=1}^\infty$ be those time steps such that agent i had its maximum number of B-playing neighbors at each time $t_k^B - 1$. Clearly, $t_1^B = t_1$. We show that

$$\forall k \geq 1 \quad n(P, t_{k+1}^B) - n(P, t_k^B) \leq -1. \quad (11)$$

At every time t_k^B agent j switches from B to A; otherwise, agent i will have more B-neighbors at time t_k^B compared to t_B . Thus, as the dynamics are coordinating, agent j has at least one A-playing neighbour at t_k^B , who is not agent i . Because the network is starlike, agent j has at most two neighbors, so it has exactly one other neighbor, say agent k , who plays A at t_k^B . So $n(P)$ reduces by one at time t_k^B . In view of Lemma 7, $n(P)$ does not increase if any agent other than j switches strategies. So $n(P)$ may increase in the future, only at times when agent j switches strategies, i.e., t_{k+1}, t_{k+2}, \dots

We show that $n(P)$ decreases at each time $T = t_{2r}$ for an arbitrary $r \in \mathbb{N}$. Agent j switches from A to B at time t_{2r} . If neighbor k plays A at time $T - 1$, then agent i plays B at the same time; otherwise, agent j does not tend to switch. So $\mathbf{x}_{(k,j,i)}(T - 1) = (A, A, B)$. Having the maximum number of B-playing neighbors at time t_1 , agent i 's utility at time $T - 1$ is no more than at time $t_1 - 1$: $u_i(T - 1) \leq u_i(t_1 - 1)$. But this is impossible according to Lemma 6. So neighbor k plays B at time $T - 1$. Then $n(P)$ reduces by the switch of agent j at time T .

On the other hand, $n(P)$ may increase by at most one at each time $t_{2r+1}, r \in \mathbb{N}$. Therefore, there is no finite time $T' > t_k^B$ when $n(P)$ equals its value at t_k^B , proving (11), a contradiction.

Case 2: Agent i switches from B to A at time t_1 . There exists time $t_2 > t_1$ when agent i tends to switch back to B. However, the number of agent i 's B-playing neighbors is maximized at time t_1 , when it switched to A. Hence, because of the coordinating dynamics, agent i is an imitator.

Denote by $\langle T_r \rangle_{r=0}^\infty$ the time steps after t_0 that agent i changes its strategy, and let $\langle a_r \rangle_{r=0}^\infty$ be the corresponding neighbors imitated by agent i . Let a_j be an agent among $\langle a_r \rangle_{r=0}^\infty$ with the maximum utility, i.e.,

$$a_j = \arg \max_r u_{a_r}. \quad (12)$$

So the maximum utility among the agent i 's neighbors was earned by agent a_j at time T_j . Consider the branch P including agent a_j . We show that the number of sections in P decreases at least once after time T_j but never increases afterwards, which is in contradiction with T_j belonging to the periodic part of the activation sequence.

First, we prove the following:

Statement 1. At any time $T_r, r \geq 0$, when agent i switches

to A, agent a_j must also play A. At time $T_j - 1$, agent a_j plays B and has at most one B-playing neighbor. At time $T_r - 1$, agent i plays B, so agent a_j has at least one B-playing neighbor. So if agent a_j plays B at time T_r , it earns no less than at time T_j because of the coordinating dynamics, i.e., $u_{a_j}(T_r) \geq u_{a_j}(T_j)$. Hence, in view of (12), agent a_j is a maximum earner at time T_r . Since agent i does switch at time T_r , it has to switch to the strategy of agent a_j according to (2). This is, however, impossible since both agents a_j and i play the same strategy B at time T_r . This proves Statement 1.

Now we list and investigate the possible strategy states for the pair (i, a_j) starting from time T_j :

Case 2.1. $\mathbf{x}_{(i,a_j)}(t) = (B, B)$. Then at the next time $T_r \geq t$ when agent i changes strategies, it switches to A. Hence, according to Statement 1, $\mathbf{x}_{(i,a_j)}(T_r - 1) = (B, A)$. So according to Lemma 7, $n(P)$ reduces by at least 1 during $[t, T_r - 1]$ as agent i does not switch strategies in this interval. We reach Case 2.2 at time T_r as $\mathbf{x}_{(i,a_j)}(T_r) = (A, A)$.

Case 2.2. $\mathbf{x}_{(i,a_j)}(t) = (A, A)$. Then at the next time $T_s \geq t$ when agent i changes strategies, it switches to B. Now if agent a_j plays B at time $T_s - 1$, we have $\mathbf{x}_{(i,a_j)}(t) = (A, B)$. So again according to Lemma 7, $n(P)$ reduces by at least 1 during $[t, T_s - 1]$. We reach Case 1 at time T_s as $\mathbf{x}_{(i,a_j)}(T_s) = (B, B)$. Now if agent a_j plays A at time $T_s - 1$, we have $\mathbf{x}_{(i,a_j)}(t) = (A, A)$. So according to Lemma 7, $n(P)$ may not increase during $[t, T_s - 1]$. We reach Case 2.3 at time T_s as $\mathbf{x}_{(i,a_j)}(T_s) = (B, A)$.

Case 2.3. $\mathbf{x}_{(i,a_j)}(t) = (B, A)$. Then at the next time $T_p \geq t$ when agent i changes strategies, it switches to A. Hence, according to Statement 1, $\mathbf{x}_{(i,a_j)}(T_p - 1) = (B, A)$ which is the same as the state at time t in this case. So according to Lemma 7, $n(P)$ does not increase during $[t, T_p - 1]$. We reach Case 2.2 at time T_p as $\mathbf{x}_{(i,a_j)}(T_p) = (A, A)$.

At time T_j , the strategy state $\mathbf{x}_{(i,a_j)}$ matches Case 1, where $n(P)$ reduces. The proof is complete since it does not increase afterwards in any of the above cases. ■

V. SPARSE TREE-STRUCTURED POPULATIONS

The *distance* of two nodes in a graph is the number of edges in the shortest path connecting the two. We call a tree *sparse* if the distance between every pair of its branching agents is greater than two. Equilibration of sparse trees is presented in the following theorem. The idea of the proof is to show that the “special branches” of two branching nodes will intersect, resulting in a “golden branch” (figure 2) which is guaranteed to equilibrate.

Theorem 5 (Sparse tree). *A sparse tree network equilibrates under the coordinating best-response and imitation dynamics with an arbitrary activation sequence.*

Proof: Equilibration of starlike networks were shown in Proposition 3. So here we consider the case with at least two branching agents. We prove by contradiction and consider a persistent eventually periodic activation sequence denoted by $\langle b_t \rangle_{t=0}^\infty$ with periodic part $[t_0, \infty)$. Similar to the proof of lemma 4, it can be shown that at least one branching

agent changes its strategy during the periodic part of the oscillation. We refer to the agents who change their strategy during $[t_0, \infty)$ a *settling* agent and otherwise *unsettling*. For each unsettling agent i , denote its special branch defined in the proof of Proposition 3 by P_i . Equilibration can be shown using Lemma 4 when there is no unsettling branching agent and similar to Proposition 3 when the special branches of no two branching agents overlap (no golden branch). So consider the case where there are two branching agents with the corresponding special neighbors i and j , and whose special branches intersect, denoted by P . In view of Lemma 7, $n(P)$ increases only at the time steps when either agent i or j switches. On the other hand, for both Case 1 and 2 in Proposition 3, it is guaranteed that there exists some infinite time series $\langle t_k^i \rangle_k^\infty$ (when agent i switches) such that $n(P, t_{k+1}^i) - n(P, t_k^i) \leq -1$ for all $k \geq 0$, and a some time series $\langle t_k^j \rangle_k^\infty$ (when agent j switches) such that $n(P, t_{k+1}^j) - n(P, t_k^j) \leq -1$ for all $k \geq 0$. This is a contradiction as then $n(P)$ is unbounded. ■

VI. CONCLUSION

We showed that every sparse tree network of coordinating imitators and best-responders equilibrates under any activation sequence. For the proof, we introduced the number of sections (consecutive same-strategy playing agents) in a path as a potential function and generalized it to the starlike and then sparse tree networks. Whether dense trees or general graphs equilibrate under every activation sequence remains an open problem.

APPENDIX

Lemma 6. *Consider a network governed by the coordinating best-response and imitation dynamics with an arbitrary activation sequence. Assume that the network includes neighboring agents $p - 1$ and p , each of degree two, and denote the other neighbor of agent p by $p + 1$. If there exists some time $T \geq 0$ when agent p tends to switch strategies and $\mathbf{x}_{(p-1,p,p+1)}(T) = (A, A, B)$, then agent p does not tend to switch strategies at any time T' when $\mathbf{x}_{(p-1,p,p+1)}(T') = (A, B, B)$ and when agent $p + 1$ earns non-less, i.e., $u_{p+1}(T') \geq u_{p+1}(T)$.*

Proof: Should agent p be a best-responder, its tendency to switch strategies at time T implies that it tends to play B if at least one of its neighbors plays B in view of (3) and (1). Therefore, agent p also tends to play B at time T' since it has a B-playing neighbor. So consider the case where agent p is an imitator. At time T agent p tends to imitate agent $p + 1$ who plays B and has at most one other B-playing neighbor. So agent $p + 1$ earns more than agent $p - 1$, i.e., $u_{p+1}(T) > u_{p-1}(T)$. We know that agent $p + 1$ earns at time T no more than at time T' , i.e., $u_{p+1}(T) \leq u_{p+1}(T')$. Moreover, at both times T and T' , agent $p - 1$ plays A but has at time T at least and at time T' at most one other A-playing neighbor, implying $u_{p-1}(T) \geq u_{p-1}(T')$. These inequalities result in $u_{p+1}(T') > u_{p-1}(T)$, which completes the proof.

We say that a network *admits a path* $(1, 2, \dots, m)$ if there is a link between node i and $i + 1$ for all $i = 1, \dots, m - 1$

and the degree of every node $2, \dots, m - 1$ is two. We refer to $(2, \dots, m - 1)$ as the *interior* of the path.

Lemma 7. *Consider a network admitting the path $(1, \dots, m)$ governed by the coordinating best-response and imitation dynamics. Then the number of sections in the interior of the path does not increase if each of the ending agents 1 and 2 either are a leaf or its strategy does not change under the activation sequence.*

Proof: The proof follows Lemma 1 and by choosing an activation sequence that does not activate the ending agents.

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