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Model uncertainty-aware residual generators for SISO LTI systems based on kernel identification and randomized approaches

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***Abstract*— Robustness of residual signals to model uncertainties and noise in the measurements is of paramount importance in model-based fault diagnosis. Model uncertainty has been mainly represented in a structured way by considering known bounds on the model parameters, thus relying on prior knowledge about the plant structure and values of its physical parameters. When the plant is completely unknown, system identification techniques must be used for model-based diagnosis. In this work, we present a data-driven approach to represent the uncertainty in the identified model. This uncertainty is described in the frequency domain using kernel-based identification and robust control tools. The estimated model uncertainty region overlaps with the true uncertainty region with a probability specified by the user. The user choices are thus reduced to the selection of only some interpretable hyperparameters. Then, a residual generator robust to the estimated model uncertainty and measurements noise is designed by a standard H_∞ approach. Simulation results on SISO LTI systems show the effectiveness of the approach in producing a residual signal viable for the detection of additive faults.**

I. Introduction

Model-based fault diagnosis aims to detect, isolate, identify and estimate faults acting on dynamical plants [1]. Based on a model of such systems, a filtering scheme is designed to process inputs/outputs plant measurements, producing a set of residual signals. The residuals are nominally zero considering a perfect model of the plant, in absence of faults, disturbances and noises[1] [1]. When a fault is present, the residuals must become non-zero for correct detection of faults. However, residuals become nonzero also if modeling errors, disturbances and noises are present, hindering the faults presence and therefore their diagnosis. Thus, the generation of residual signals should be made robust to such sources of uncertainty, while not reducing their sensitivity to faults (*active* robustness) [2]. Robust residuals can be also complemented by adaptive thresholds at the decision-making stage (*passive* robustness) [3][2].

Many approaches have been considered for achieving robustness in model-based fault diagnosis, especially in a Linear Time Invariant (LTI) context [2][4, Chapter 8]. Model uncertainty is often the most problematic source of uncertainty [5, Chapter 1], as a perfect model for even a simple engineering plant is hardly available and most plant parameters are unknown. In these cases, system identification procedures must be employed, which however come with estimation errors due to limited, noisy data and model structure/complexity determination. A typical representation of the model uncertainty consists in restating modeling errors as additive unknown inputs, which could be perfectly or approximately decoupled from residuals by specific diagnostic schemes [2, Chapter 5]. This reframing process requires the use of prior information or assumptions about the model uncertainties. One of the most useful assumptions considers bounded errors on the model parameters, representing the model uncertainty as *structured* [6], and paving the way for the application of set-membership approaches [7], [8].

The use of uncertainty regions provided by system identification methods has been investigated in the context of robust fault diagnosis, but mainly in a bounded-error setting [9], [10]. Stochastic uncertainty in the estimated model parameters for fault diagnosis has been considered in [11], with small assumptions about the boundedness of the norm of some covariance matrices. However, both these recent contributions focused on a parametric/structured model uncertainty representation. On the contrary, an *unstructured* representation allows to describe the model uncertainty in the frequency domain without specifying an uncertainty bound for each model parameter, also providing an alternative understanding of the designed residuals generator filter.

In this paper, we propose a rationale to *design residual generators* that are *robust to the uncertainty* originating by two sources: (i) model estimation variance, (ii) measurements noise at the outputs, considering SISO LTI systems. To this end, we adopt low-bias regularized kernel-based identification methods that relax the user from model structure/complexity determination [13]. The uncertainty region is estimated from input/output data by means of a randomized approach [14], that guarantees with high probability the goodness of the uncertainty region estimate [15]. Thanks to the low-bias property of regularized models, the estimated uncertainty region contains with high probability the true plant. The overall model uncertainty quantification procedure relieves the user from the development of an ad-hoc model-error model [16], that requires several critical user choices, to quantify the bias and variance of the estimated parametric model. The use of low-bias kernel identification leaves, approximately, only the model variance to be taken into account in the residual generator design [17]. In the proposed approach, the user needs only to specify a set of configuration hyperparameters, as a bound on the output noise amplitude.

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[1]We refer to a “disturbance” as an unknown input such that its decoupling from residuals is possible and can be targeted in design of the residual generator. “Noises” are other unknown inputs that cannot be perfectly decoupled.

[2]Passive robustness can be employed to further compensate modeling errors with less conservatism in the choice of the fault detection threshold.

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Then, a residual generator with optimal tradeoff between rejection of model uncertainty (e.g. model estimation variance), measurements noise and sensitivity to additive faults is provided by a standard H_∞ optimization problem, solved with recent computational and numerically reliable tools [1], [18], thus providing *active robustness* to uncertainties. With the proposed rationale, an automatic (optionally adaptive) detection threshold is defined, so providing *passive robustness*.

The remainder of the paper is as follows. Section II presents the problem statement and reviews the basics of kernel-based identification. Section III describes the rationale to design a residual generator robust to noise and model uncertainties, while being sensitive to faults. Section IV provides extensive simulation examples to validate the method. Section V is devoted to concluding remarks.

II. Preliminaries

This section introduces the problem statement and the adopted representation of model uncertainty. Kernel methods are briefly reviewed since they will form the basis for estimating the model uncertainty region.

Notation. The output signal $y(t)$ filtered by a SISO LTI discrete transfer function $G(z)$ fed by input $u(t)$, is described as $y(t) = G(q)u(t)$, where q is the the lag operator so that $y(t-1) = q^{-1} \cdot y(t)$, where z is the $\mathcal{Z}$-transform variable and $t \in \mathbb{N}$ is a discrete time index.

A. Problem statement

Consider an *unknown* stable LTI SISO plant $G_u^0(z)$ subject to control, fault and noise inputs[3]

$$\mathcal{S}: \quad y(t) = G_u^0(q)u(t) + G_w(q)w(t) + G_f(q)f(t), \quad (1)$$

where $u(t) \in \mathbb{R}$, $f(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}$ denote the (known) control input and (unknown) fault and noise inputs, respectively. The transfer functions $G_u^0(z), G_w(z), G_f(z)$ in (1) relate the inputs to the output measurements $y(t) \in \mathbb{R}$. We impose the following assumptions on the additive noise.

Assumption 1 (Output error noise): The noise $w(t)$ in (1) is a measurement output error, so that $G_w(z) = \sigma$ with $\sigma \in \mathbb{R}_{>0}$ a *known* constant positive value, denoting the noise standard deviation.

Assumption 2 (Bounded noise): Define the noise signal $e(t) \coloneqq G_w(q)w(t) = \sigma \cdot w(t)$ The noise $e(t)$ is possibly stochastic and norm-bounded with $|e(t)| < \bar{\delta}_e$, $\forall t$. It follows that $|w(t)| < \bar{\delta}_e/\sigma$, $\forall t$.

Model-based design of a residuals generator $\boldsymbol{Q}(z;\mathcal{S})$ for the system (2) aims to find $\boldsymbol{Q}(z;\mathcal{S})$ so that a residual signal $r(t) \in \mathbb{R}$ can be computed by the *computational form*

$$r(t) \coloneqq \boldsymbol{Q}(q;\mathcal{S}) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} Q_y(q) & Q_u(q) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \quad (2)$$

where $\boldsymbol{Q}(z;\mathcal{S})$ is a 1×2 transfer functions matrix (TFM). As represented in (2), the output $y(t)$ and control input $u(t)$ signals are filtered, respectively, by the SISO LTI filters $Q_y(z)$ and $Q_u(z)$ to produce the residual $r(t)$.

[3]In this work, we do not consider perfect decoupling of disturbances. In such cases, distrubances can be considered as noise inputs.

The residual $r(t)$ in (2) depends on all plant inputs $u(t)$, $w(t)$, and $f(t)$ through $y(t)$. The *internal form* of the residual generator is obtained by substituting (1) into (2) as

$$r(t) = \underbrace{\begin{bmatrix} R_u(q;\mathcal{S},G_u^0) & R_w(q) & R_f(q) \end{bmatrix}}_{\boldsymbol{R}(q;\mathcal{S})} \begin{bmatrix} u(t) \\ w(t) \\ f(t) \end{bmatrix}, \quad (3)$$

with $\boldsymbol{R}(z;\mathcal{S})$ a 1×3 TFM defined by

$$R_u(z;\mathcal{S},G_u^0) \coloneqq \boldsymbol{Q}(z;\mathcal{S}) \begin{bmatrix} G_u^0(z) \\ 1 \end{bmatrix}, \quad (4)$$

$$R_w(z) \coloneqq \boldsymbol{Q}(z;\mathcal{S}) \begin{bmatrix} G_w(z) \\ 0 \end{bmatrix}, \quad R_f(z) \coloneqq \boldsymbol{Q}(z;\mathcal{S}) \begin{bmatrix} G_f(z) \\ 0 \end{bmatrix}.$$

The TFM $\boldsymbol{R}(z;\mathcal{S})$ in (3) maps all plant inputs to the residual signal, fulfilling, whenever possible, specific fault detection and isolation requirements [1, Chapter 3]. The design of $\boldsymbol{Q}(z;\mathcal{S})$ in (2)-(4) relies on the perfect knowledge of the system model (1). When this is not the case, the residual signal $r(t)$ should be made robust to model uncertainty.

A bounded model uncertainty can be represented following a robust control description [21]. To this end, let $\Delta(z)$ be a stable transfer function that satisfies the bounded real condition $\|\Delta(z)\|_\infty \leq 1$ and consider a multiplicative output uncertainty model set [19]

$$\Pi: \quad G_p(z) \coloneqq \big(1 + \Delta(z)W(z)\big)G_u^0(z), \quad (5)$$

where $G_p(z)$ is a particular perturbed SISO plant model in Π and $\Delta(z)$ describes a normalized bounded frequency-domain unstructured uncertainty with $W(z)$ its frequency magnitude [20, Chapter 7]. Descriptions of model uncertainty as (5) are used to represent *measurement output errors* and *neglected high-frequency dynamics* [21, Chapter 9].

Since $G_u^0(z)$ is assumed to be *unknown*, practical use of (2) for robust residual generation under the uncertainty description (5) requires the development of a plant model. Here we focus on the case where a data-driven model $\hat{G}_u(z)$ is identified from a set of n noisy input/output data $\mathcal{D} = \{u(t), y^0(t)\}_{t=1}^n$ collected from an open-loop, fault-free experiment on the plant, so that

$$y^0(t) = G_u^0(q)u(t) + G_w(q)w(t) = G_u^0(q)u(t) + e(t) \quad (6)$$

where n is the number of measurements.

The randomness in $e(t)$ influences the estimate of $\hat{G}_u(z)$, thereby acting as a source of model uncertainty. We describe such uncertainty as in (5), where the boundedness of $\Delta(z)W(z)$ derives from the bounded nature of $e(t)$.

Combining (1), (5), and $\hat{G}_u(z)$ leads to the the following uncertain model for plant $G_u^0(z)$ subject to noise and faults:

$$\mathcal{M}: \quad y(t) = \big(1 + \Delta(q)W(q)\big)\hat{G}_u(q)u(t) + G_w(q)w(t) + G_f(q)f(t). \quad (7)$$

The aim of this work is to provide and automatic procedure to design a residuals generator $\boldsymbol{Q}(z;\mathcal{M})$ as in (2), with $\hat{G}_u(z)$ in place of $G_u^0(z)$ and an estimation of the worst-case uncertainty $\hat{W}(z)$ (i.e. when $\|\Delta(z)\|_\infty = 1$), so that the residual $r(t)$ is robust to measurements noise and to the model uncertainty endowed in the identification process under Assumption 2, described by the uncertain model (5), see Figure 1.

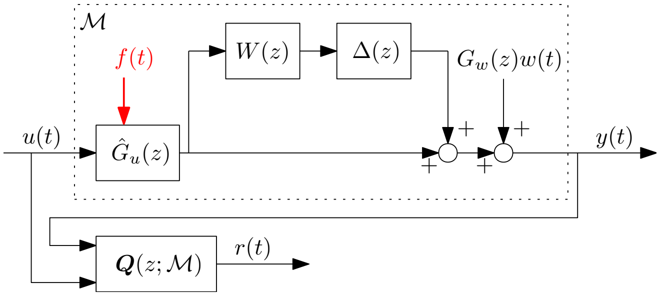

Fig. 1. Open-loop plant with multiplicative output uncertainty and identified model $\hat{G}_u(z)$ for the design of the residual generator $\boldsymbol{Q}(z;\mathcal{M})$. The term $\Delta(z)W(z)$ represents the uncertainty in the identification of $\hat{G}_u(z)$.

While Assumption 2 calls for a set-membership identification approach, in this framework there is no guarantee that $\hat{G}_u(z)$ is close to $G_0(z)$ in L_2-norm [16]. Thus, modeling bias may jeopardize the quality of the residual signal $r(t)$. So, we employ low-bias kernel-based methods, where $G_u^0(z) \approx \hat{G}_u(z)$ if the model is flexible enough. When using stochastic identification approaches, the Assumption 2 on bounded noise is not leveraged. However, the price paid for ignoring this information is way lower than the price paid by using a model with high bias in robust residuals generator design, as the true system may well be outside the defined uncertainty region. This happens also when wrong modeling choices are performed in a model-error model design to characterize the model uncertainty region.

B. Kernel-based identification review

Consider (6) with $w(t) \sim \mathcal{N}(0,1)$ and $G_w(z) = \sigma$ the noise standard deviation, and the FIR model of order n_g

$$G(q,\boldsymbol{\theta}) = \sum_{i=1}^{n_g} g_i q^{-i}, \quad \boldsymbol{\theta} = \left[g_1\, g_2\, \ldots,\, g_{n_g}\right]^\top. \tag{8}$$

Assume that a prior distribution $\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, \mathbf{K})$ is placed on $\boldsymbol{\theta} \in \mathbb{R}^{n_g \times 1}$, where $\mathbf{K} \in \mathbb{R}^{n_g \times n_g}$ is a covariance (kernel) matrix encoding prior information on parameters $\boldsymbol{\theta}$. A common parametrization of $\mathbf{K}$ employs the tuned-correlated kernel [13], where the (i,j) element of $\mathbf{K}$ is defined as $K_{ij} := \lambda \cdot \alpha^{\max(i,j)}$ with $\lambda > 0, 0 \leq \alpha < 1$ are the kernel hyperparameters. The posterior distribution $\boldsymbol{\theta}|\boldsymbol{y}$ is then Gaussian with [13]

$$\boldsymbol{\theta}|\boldsymbol{y} \sim \mathcal{N}\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}\right), \tag{9a}$$

$$\hat{\boldsymbol{\theta}} = \left(\mathbf{K}\boldsymbol{\Phi}^\top\boldsymbol{\Phi} + \sigma^2 I_{n_g}\right)^{-1} \mathbf{K}\boldsymbol{\Phi}^\top \boldsymbol{y}, \tag{9b}$$

$$\hat{\boldsymbol{\Sigma}} = \mathbf{K} - \mathbf{K}\boldsymbol{\Phi}^\top\left(\boldsymbol{\Phi}\mathbf{K}\boldsymbol{\Phi}^\top + \sigma^2 I_n\right)^{-1} \boldsymbol{\Phi}\mathbf{K}, \tag{9c}$$

$$\text{where } \boldsymbol{\Phi} := \left[\boldsymbol{\varphi}(1)\, \boldsymbol{\varphi}(2)\, \ldots\, \boldsymbol{\varphi}(n)\right]^\top \in \mathbb{R}^{n \times n_g}, \tag{9d}$$

$$\boldsymbol{\varphi}(t) := \left[u(t-1)\, \ldots\, u(t-n_g)\right]^\top \mathbb{R}^{n_g \times 1} \tag{9e}$$

$$\boldsymbol{y} := \left[y(1)\, y(2)\, \ldots\, y(n)\right]^\top \in \mathbb{R}^{n \times 1}. \tag{9f}$$

The kernel's hyperparameters $\boldsymbol{\zeta} = [\alpha, \lambda]^\top$ (along with the noise variance σ^2) can be estimated by employing an Empirical Bayes scheme, by maximizing the log-marginal likelihood of the data [13]

$$\hat{\boldsymbol{\zeta}} = \arg\min_{\boldsymbol{\zeta}} \boldsymbol{y}^\top \mathbf{Z}(\boldsymbol{\zeta})^{-1} \boldsymbol{y} + \log\det \mathbf{Z}(\boldsymbol{\zeta}), \tag{10a}$$

$$\mathbf{Z}(\boldsymbol{\zeta}) := \boldsymbol{\Phi}\mathbf{K}(\boldsymbol{\zeta})\boldsymbol{\Phi}^\top + \sigma^2 I_n. \tag{10b}$$

At this stage, a model order reduction to order n_r can be performed, at the expense of introducing a bit of modeling bias.

The data-driven estimation of the model uncertainty region relies on (9b)-(9c) and it will be covered next.

III. Uncertainty-aware robust residual generators

This section reviews the procedure proposed in [15] for estimating the model uncertainty region $\Delta(z)W(z)$ in (5) in the worst-case scenario $\|\Delta(z)\|_\infty = 1$, by using kernel methods. Then, a robust residual generator is designed to perform fault detection in spite of modeling and noise uncertainties, using H_∞ tools and the estimate of the uncertainty region.

A. Estimation of the model uncertainty region

Let $\|\Delta(z)\|_\infty \leq 1$ and $G_p(z) := G(z, \boldsymbol{\theta}_p)$, with $\boldsymbol{\theta}_p$ a random sample drawn from (9a), denoting an impulse response. Then, the relation between the modules in (5) reads as

$$\left|\frac{G_p\left(e^{j\omega}\right)}{G_u^0\left(e^{j\omega}\right)} - 1\right| \leq \left|W(e^{j\omega})\right|, \quad \forall \omega \in [0, f_s\pi] \tag{11}$$

where ω is a specific pulse in rad/s and f_s is the sampling frequency. Define

$$\Omega\left(e^{j\omega}; G_u^0\right) := \max_p \left|\frac{G_p\left(e^{j\omega}\right)}{G_u^0\left(e^{j\omega}\right)} - 1\right|, \tag{12}$$

so that $\Omega\left(e^{j\omega}; G_u^0\right)$ denotes an amplitudes envelope. The magnitude of the least conservative $W(z)$ can be estimated by evaluating (12) in a discrete grid of n_m frequencies $\mathcal{F} = \{\omega_1, \omega_2, \ldots \omega_{n_m}\} \subseteq [0, \pi f_s]$ for a set of n_p samples, with f_s the sampling frequency of the measurements.

However, (12) can not be computed since $G_u^0(z)$ is unknown. Thus, by assuming that $G_u^0(z) \approx \hat{G}_u(z)$, where $\hat{G}_u(z) := G(z, \hat{\boldsymbol{\theta}})$ by (9b), a nonparametric sampled estimate of the magnitude of $W(z)$ is

$$\left|\hat{W}\left(e^{j\omega_m}\right)\right| = \Omega\left(e^{j\omega_m}; \hat{G}_u\right), \quad \forall \omega_m \in \mathcal{F}. \tag{13}$$

Remark 1 (Low-bias model): The assumption $\hat{G}_u(z) := G(z, \hat{\boldsymbol{\theta}})$ is not critical if low-bias regularized models of enough flexibility are employed.

A stable and proper parametric model $\hat{W}(z)$ of fixed order n_w can then be obtained by fitting a *parametric model* to the magnitude frequency points (13), taking care that the magnitude of the fitted model lies above (or is equal to)

(13). The number n_p of dynamical systems drawn from (9a) can be selected relying on the following result[4].

Proposition 1 (Uncertainty bound reconstruction): Define a fixed confidence level $\delta \in (0,1)$ and a fixed accuracy level $\varepsilon \in (0,1)$. Let

$$n_p \geq \frac{1}{2\varepsilon^2} \ln\left(\frac{2}{\delta}\right). \tag{14}$$

Then, with probability $\geq 1-\delta$, it holds that

$$\left|\hat{W}\left(e^{j\omega_m}\right) - W\left(e^{j\omega}\right)\right| < \varepsilon, \quad \forall \omega, \omega_m \in \mathcal{F}.$$

Proof: The proof follows from the Chernoff's bound [14, Chapter 8]. ■

Implementation details for the uncertainty region estimation are described in [15]. The model of the uncertainty region $\hat{W}(z)$ will be used to design a residual generator with robustness properties against modeling uncertainty and noise.

B. Design of robust residual generators

With $\hat{W}(z)$ as a description for the model uncertainty region representing $\Delta(z)W(z)$, the uncertain model in (7) can be replaced by the *robust diagnostic model*

$$\begin{aligned}\mathcal{M}_v:\ y(t) &= \big(1+\hat{W}(q)\big)\hat{G}_u(q)u(t)\\ &\qquad + G_w(q)w(t) + G_f(q)f(t)\\ &= \hat{G}_u(q)u(t) + \boldsymbol{G}_v(q)\boldsymbol{v}(t) + G_f(q)f(z)\end{aligned} \tag{15}$$

where

$$\boldsymbol{G}_v(z) := \begin{bmatrix}\hat{W}(z)\hat{G}_u(z) & G_w(z)\end{bmatrix} \tag{16a}$$

$$\boldsymbol{v}(t) := \begin{bmatrix}u(t)\\ w(t)\end{bmatrix}, \tag{16b}$$

so that the effects of the known input $u(t)$ and uncertain or unknown terms in $\boldsymbol{v}(t)$ have been separated.

Similar to (3), the internal form of a residual generator $\boldsymbol{Q}(z;\mathcal{M}_v)$ for the diagnostic model (15) reads as

$$r(t) = \begin{bmatrix}R_u(q;\mathcal{M}_v,\hat{G}_u) & \boldsymbol{R}_v(q) & R_f(q)\end{bmatrix}\begin{bmatrix}u(t)\\ \boldsymbol{v}(t)\\ f(t)\end{bmatrix}, \tag{17}$$

with $\boldsymbol{R}_v(z) := \boldsymbol{Q}(z;\mathcal{M}_v)\begin{bmatrix}\boldsymbol{G}_v(z)\\ \boldsymbol{0}\end{bmatrix}$. The aim is now to design $\boldsymbol{Q}(z;\mathcal{M}_v)$ requiring

$$R_u(z;\mathcal{M}_v,\hat{G}_u) = 0, \tag{18a}$$

$$\boldsymbol{R}_v(z) \approx \boldsymbol{0}, \tag{18b}$$

$$R_f(z) \neq 0, \tag{18c}$$

$$\boldsymbol{R}_v(z), R_f(z) \text{ stable}, \tag{18d}$$

so that the effect of uncertainties $\boldsymbol{v}(t)$ is minimized while maintaining the faults effect on the residual. The design of $\boldsymbol{Q}(z;\mathcal{M}_v)$ satisfying (18) can be performed by the nullspace-based synthesis approach [1, Chapter 5]. The solution is a residual generator of the form

$$\begin{aligned}\boldsymbol{Q}(z;\mathcal{M}_v) &= \bar{Q}(z)\boldsymbol{Q}_1(z)\\ &= \bar{Q}(z)\begin{bmatrix}Q_{1,y}(z) & Q_{1,u}(z)\end{bmatrix} = \begin{bmatrix}Q_y(z) & Q_u(z)\end{bmatrix},\end{aligned} \tag{19}$$

[4]The choice of n_m is not critical. It suffices for the frequency grid to be resolute enough.

where $\boldsymbol{Q}_1(z)$ is a 1×2 proper and stable TFM that satisfies (18a) and (18c) by filtering plant outputs and inputs trough $Q_{1,y}(z)$ and $Q_{1,u}(z)$ respectively, and $\bar{Q}(z)$ acts as a *post-filter* for enhancing the robustness of the residual signal, thus enforcing (18b). The synthesis procedure is composed by two stages. In the first stage, $\boldsymbol{Q}_1(z)$ is designed by solving for (18a). To this end, define

$$\bar{\boldsymbol{R}}_v(z) := \boldsymbol{Q}_1(z)\begin{bmatrix}\boldsymbol{G}_v(z)\\ \boldsymbol{0}\end{bmatrix}, \quad \bar{R}_f(z) := \boldsymbol{Q}_1(z)\begin{bmatrix}G_f(z)\\ 0\end{bmatrix}.$$

Let $\gamma \geq 0$ be a given admissible level for the effect of the uncertainties signal $\boldsymbol{v}(t)$ on the residual $r(t)$. Then, in the second stage of the synthesis procedure, the post-filter $\bar{Q}(z)$ is found by solving the H_∞ problem

$$\begin{aligned}\beta = \max_{\bar{Q}(z)} &\left\|\bar{Q}(z)\bar{R}_f(z)\right\|_\infty\\ \text{s.t.}\quad &\left\|\bar{Q}(z)\bar{\boldsymbol{R}}_v(z)\right\|_\infty < \gamma,\end{aligned} \tag{20}$$

with $\beta > 0$ the fault sensitivity level. The fault/uncertainties gap $\eta = \beta/\gamma$ can be interpreted as a measure of the quality of fault detection (the higher, the better).

Remark 2: When an estimate of the uncertainty region $\Delta(z)W(z)$ is not available, the uncertain model in (7) can be employed for residual generator design by defining the diagnostic model

$$\mathcal{M}'_v:\ y(t) = \hat{G}_u(q)u(t) + \boldsymbol{G}'_v(q)\boldsymbol{v}'(t) + G_f(q)f(t) \tag{21}$$

with

$$\boldsymbol{G}'_v(z) := \begin{bmatrix}\hat{G}_u(z) & G_w(z)\end{bmatrix} \tag{22a}$$

$$\boldsymbol{v}'(t) := \begin{bmatrix}\Delta(z)W(z)u(t)\\ w(t)\end{bmatrix}, \tag{22b}$$

so that all the uncertainty is coupled in the noise term (22b). In our proposed approach, the modeling uncertainty is explicitly moved outside the noise signal as shown in (16), guiding the residual generator design in (20).

C. Automatic threshold determination

Considering (18)-(19), the residual (17) can be written as

$$r(t) = \bar{Q}(q)\begin{bmatrix}\bar{\boldsymbol{R}}_v(q) & \bar{R}_f(q)\end{bmatrix}\begin{bmatrix}\boldsymbol{v}(t)\\ f(t)\end{bmatrix}, \tag{23}$$

In a fault-free situation, $f(t) = 0\,\forall t$ and

$$\begin{aligned}\left\|r(e^{j\omega})\right\|_\infty &= \left\|\bar{Q}(e^{j\omega})\bar{\boldsymbol{R}}_v(e^{j\omega})\boldsymbol{v}(e^{j\omega})\right\|_\infty\\ &\leq \left\|\bar{Q}(e^{j\omega})\bar{\boldsymbol{R}}_v(e^{j\omega})\right\|_\infty \cdot \left\|\boldsymbol{v}(e^{j\omega})\right\|_\infty\\ &\leq \gamma\cdot\max\left\{\left\|u(e^{j\omega})\right\|_\infty, \left\|w(e^{j\omega})\right\|_\infty\right\}\\ &= \gamma\cdot\max\left\{\left\|u(e^{j\omega})\right\|_\infty, \bar{\delta}_e/\sigma\right\}.\end{aligned} \tag{24}$$

Therefore, an adaptive threshold $\tau(t)$ can be generated as

$$\tau(t) = \gamma\cdot\max\left\{|u(t)|, \bar{\delta}_w\right\}, \quad \bar{\delta}_w := \bar{\delta}_e/\sigma. \tag{25}$$

IV. EXAMPLE

Consider the following benchmark system [22]

$$G_u^0(z) = \frac{0.28261z + 0.50666}{D(z)}, \tag{26a}$$
$$y^0(t) = G_u^0(q)u(t) + G_w(q)w(t) = G_u^0(q)u(t) + e(t),$$

$D(z) = z^4 - 1.41833z^3 + 1.58939z^2 - 1.31608z + 0.88642$, sampled at $T_s = 1/f_s = 0.01$ s. For model identification, we simulated $n = 5000$ data from (26) using a Gaussian white noise input signal $u(t) \sim \text{GWN}(0, 1)$ and a bounded Gaussian white noise $e(t) \sim \text{GWN}(0, \sigma^2)$, $e(t) = G_w(q)w(t)$ with $\text{SNR} = \text{var}\left[G_u^0(q)u(t)\right]/\text{var}\left[e(t)\right] = 25$, with $\text{var}[\cdot]$ the variance operator. This lead to $G_w(z) = \sigma = 0.5151$. Noise boundaries were set to $\bar{\delta}_e = 2$ so that no saturation in $e(t)$ was present in the noise signal. The first 1000 samples are discarded to remove the transient effects from the data. The FIR order for kernel identification is set as $n_g = 200$, followed by model reduction to order $n_r = 10$. By selecting $\delta = 0.05$ and $\varepsilon = 0.01$ the number of sampled system is set as $n_p = 18445$ following (14), and the number of sampled frequencies is set to $n_m = 200$, logarithmically spaced in the range $\left[10^{-3}, \pi f_s\right]$ rad/s. The order of the parametric model $\hat{W}(z)$ for (13) is set to $n_w = 3$. As stated in Assumption 1, we assume to know the true value of the noise variance σ.[5]

A. *Comparison for fixed estimated model* $\hat{G}_u(z)$

We compare three cases:

1) *Not robust* design: the residual generator $\boldsymbol{Q}(z; \mathcal{M}_v, \boldsymbol{G}_v = \boldsymbol{0})$ is designed for the system (15), using the data (6), by ignoring the information about the modeling uncertainty $\Delta(z)W(z)$ and noise $w(t)$ (i.e. $\boldsymbol{G}_v(z)$ is set to zero) and it satisfies only (18a) and (18c);
2) *Partially robust* design: the residual generator $\boldsymbol{Q}(z; \mathcal{M}_v')$ is designed for the system (21), using the data (6), by ignoring the information about the modeling uncertainty $\Delta(z)W(z)$, so that it satisfies (18a) and (18c), and *partially* (18b);
3) **(Proposed)** *Uncertainty-aware robust* design: the residual generator $\boldsymbol{Q}(z; \mathcal{M}_v)$ is designed for the system (15), using the data (6), so that it *fully* satisfies (18a)-(18c).

In all the three cases, (18d) is satisfied, and the estimated model $\hat{G}_u(z) = G(z, \hat{\boldsymbol{\theta}})$ is employed in place of the unknown plant $G_u^0(z)$ for fault detection. Moreover, $\gamma = 1$ is employed in cases 2) and 3). Healthy and failed data are collected from the true plant $G_u^0(z)$ in (26) for 40 s. A step-like input fault of amplitude 5 is injected after 20 s.

Figure 2 shows the estimation of the modeling uncertainty region $\hat{W}(z)$ as described in Section III-A. Figure 3 depicts the residual signals, with automatic threshold selection for the proposed design rationale, which also exhibit the highest fault/uncertainties gap η.

[5]This assumption is not critical since the noise variance can be estimated with good accuracy from data using (10).

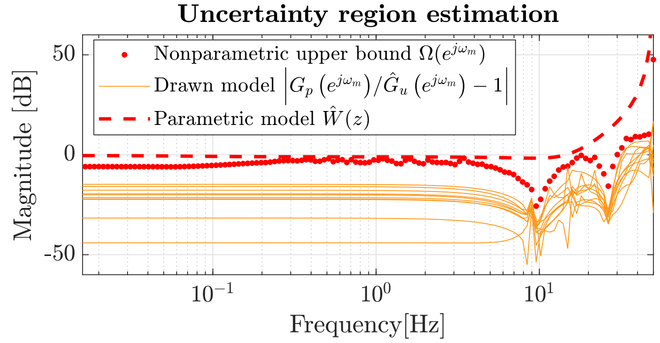

Fig. 2. Uncertainty region estimation. (Continuous line) Magnitude of a second-order parametric model $\hat{W}(z)$. (Dots) Nonparametric estimate $\Omega(e^{j\omega_m})$. (Dashed line) Evaluation of $\left|G_p(e^{j\omega_m})/\hat{G}_u(e^{j\omega_m}) - 1\right|$ in (12) for a sample of systems $G_p(z) := G(z, \boldsymbol{\theta}_p)$ drawn from the sampling distribution of the parameters estimate (9), in the frequency grid $\mathcal{F}$.

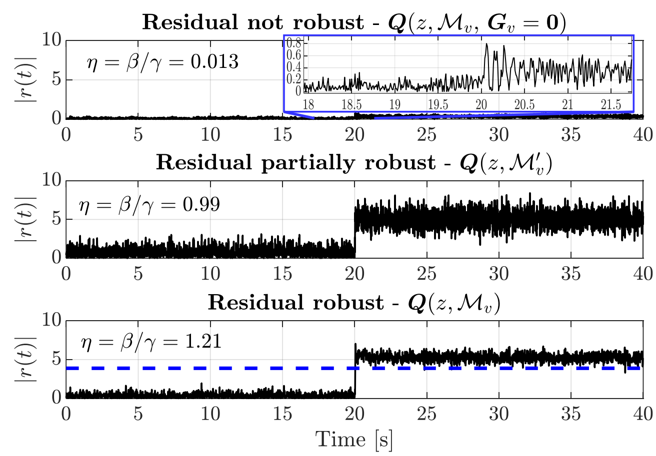

Fig. 3. (Continuous line) Residual signals. (Dashed line) Automatic threshold (25). The fault/uncertainties sensitivity gap η is reported.

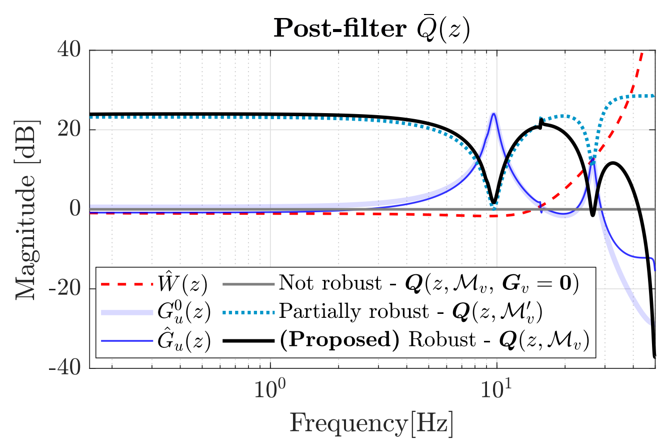

Fig. 4. Comparison of post-filter $\bar{Q}(z)$ from different design rationales. The estimated model uncertainty $\hat{W}(z)$, the frequency response of the estimated model $\hat{G}(z)$ and frequency response of the true plant $G_u^0(z)$ are also depicted.

Figure 4 represents a comparison of the post-filter $\bar{Q}(z)$ from (20). In the not robust case, $\bar{Q}(z) = 1$. In both the partially robust and proposed uncertainty-aware robust designs, the post-filter counteracts the resonances of the true plant $G_u^0(z)$. However, in the proposed design, the high-frequency components of the model uncertainty are greatly penalized: where the uncertainty is greater, the attenuation performed by the post-filter is higher. This helps the detection of the fault by rejecting model and noise uncertainty contributions.

B. Comparison for different perturbed systems

We now compare the same three designs as in Section IV-A, by considering 50 systems drawn from the distribution

$$\boldsymbol{\theta}_V \sim \mathcal{N}(\boldsymbol{\theta}^0, \hat{\boldsymbol{\Sigma}}), \tag{27}$$

where each $\boldsymbol{\theta}_V$ is the impulse response of a perturbed system and $\boldsymbol{\theta}^0$ is the noiseless impulse response of the true plant (26), with $\hat{\boldsymbol{\Sigma}}$ derived in (9c). The idea is to evaluate the robustness of the proposed design scheme when the data-generating system and the model used for residual generator design are different. To this end, for each system $\boldsymbol{\theta}_V$, we generate the data as in Section IV-A. The residual generator, for the three design schemes, is built by using $G_u^0(z)$ in place of $\hat{G}_u(z)$, so to avoid any modeling bias *on average* with respect to the data-generating system. In this way, we are effectively evaluating the robustness against modeling variance and measurements noise. The results are shown in Figure 5, where again the proposed design scheme shows superior performance.

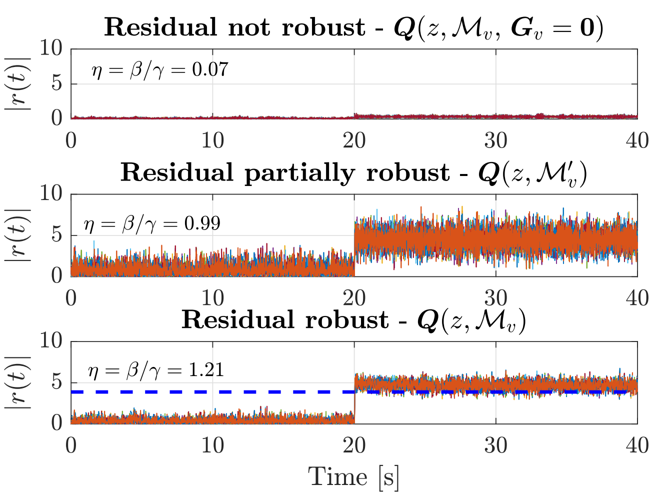

Fig. 5. (Continuous lines) Residual signals, for each of the 50 randomly drawn perturbed systems. (Dashed line) Automatic threshold (25). The fault/uncertainties sensitivity gap η is reported.

V. Conclusions

This paper presented a rationale for designing a residual generator for SISO LTI systems that is robust to model uncertainty and measurements noise. The model uncertainty region is estimated using a randomized approach with low-bias kernel identification methods, avoiding the explicit model-error modeling phase and neglecting the effect of identification bias on the estimated model. The proposed robust residual generator is also endowed with an automatic (eventually adaptive) threshold selection mechanism, assuming a known bound on output measurements noise. The approach is vastly data-driven and the user needs to select only several hyperparameters.

Fault diagnosis benefits from multiple independent output measurements. Thus, a future direction is the extension of the approach to MIMO systems.

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