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Low-complexity linear parameter-varying approximations of incompressible Navier-Stokes equations for truncated state-dependent Riccati feedback

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***Abstract*—Nonlinear feedback design via state-dependent Riccati equations is well established but unfeasible for large-scale systems because of computational costs. If the system can be embedded in the class of linear parameter-varying systems with the parameter dependency being affine-linear, then the nonlinear feedback law has a series expansion with constant and precomputable coefficients. In this work, we propose a general method to approximate nonlinear systems such that the series expansion is possible and efficient even for high-dimensional systems. We lay out the stabilization of incompressible Navier-Stokes equations as application, discuss the numerical solution of the involved matrix equations, and confirm the performance of the approach in a numerical example.**

***Index Terms*—Nonlinear systems, parameter-varying approximations, Riccati matrix equations, state-feedback control**

I. Introduction

NONLINEAR feedback design for large-scale systems is challenging, as both the complexity induced by nonlinearities and the computationally demanding tasks caused by the system's size have to be resolved. The commonly used methods of *backstepping* [22], feedback linearization [29, Ch. 5.3], or *sliding mode* control [16] require structural assumptions and, thus, may not be accessible to a general computational framework. The both holistic and general approach via the Hamilton-Jacobi-Bellman (HJB) equations, however, is only feasible for very moderate system sizes or calls for model order reduction; see, e.g., [14] for a relevant discussion and an application in fluid flow control. As an alternative to reducing the system's size, one may consider approximations to the solution of the HJB equations of lower complexity. For that, for example, truncated polynomial expansions [13] or suboptimal solutions via the so called state-dependent Riccati equation (SDRE) [2] are considered. Here, we will follow on recent developments [1] about series expansions of the SDRE approximation to the HJB solution that can mitigate the still high computational costs of repeatedly solving high-dimensional Riccati equations.

As the general setup, we consider the control-affine system

$$\dot v(t) = f(v(t)) + Bu(t), \quad y(t) = Cv(t), \tag{1}$$

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where for time $t > 0$, $v(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$ denote the input and output, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a possibly nonlinear function, and B and C are linear input and output operators. Under the mild condition that f is Lipschitz continuous and $f(0) = 0$, one can factorize the nonlinearity $f(v) = A(v)v$ with a state-dependent coefficient matrix $A(v)$ and bring the system (1) into state-dependent coefficient (SDC) form:

$$\dot v(t) = A(v(t))v(t) + Bu(t), \quad y(t) = Cv(t); \tag{2}$$

see, e.g., [8, Eq. (7)]. For such systems, one can define a feedback by

$$u(t) = -B^{\mathsf{T}} P(v(t))v(t),$$

where $P(v)$ solves the SDRE

$$A(v)^{\mathsf{T}} P(v) + P(v)A(v) - P(v)BB^{\mathsf{T}} P(v) = -C^{\mathsf{T}} C; \tag{3}$$

see [2] for general principles and [8] for a proof of performance beyond an asymptotic smallness condition. Because of its nonlinear and, possibly high-dimensional nature, a solve of the SDRE (3) comes at high costs that make the SDRE approach unfeasible for large systems; see [8] for an example illustrating how the effort grows with the system's dimension.

If, however, the factorization $f(v) = A(v)v$ is affine-linear with respect to a parametrization $\rho(v) \in \mathbb{R}^m$ of v, i.e., it can be represented as

$$A(v) = A_0 + \sum_{k=1}^{m} \rho_k(v) A_k, \tag{4}$$

then the solution $P(v)$ of the SDRE has a *first-order* approximation of the form

$$P(v) \approx P_0 + \sum_{k=1}^{m} \rho_k(v) L_k,$$

where P_0 and L_k, for $k = 1, \ldots, m$, can be precomputed by one Riccati and m Lyapunov equations; see [1], [4]. In this work we propose a general approach for controller design that bases on approximative representations as in (4) for which we employ

- an SDC representation of the nonlinear system,

- an approximative parametrization $\hat{\rho}(v) \in \mathbb{R}^r$ in the sense that r is chosen small such that an exact reconstruction of v from $\hat{\rho}(v)$ might not be possible, and
- an affine-linear approximation of the coefficient (4).

Also, we discuss how such an approach can be realized for flow control problems modeled by semi-discrete Navier-Stokes equations (NSEs). For this, we rely on

- the coordinates provided by a proper orthogonal decomposition (POD) of the velocity states; see [24] for an introduction,
- the quadratic structure of the nonlinearity in the incompressible NSEs,
- implicit treatment of the incompressibility constraint, and, importantly,
- low-rank representations of the solutions to the high-dimensional Riccati and Lyapunov equations and suitable solution methods.

We note that with this line of arguments, the nonlinear feedback design via truncated SDRE approximations becomes feasible for finite element approximations of general nonlinear partial differential equations (PDEs).

Apart from the proposed algorithmic advances and numerical insights into the feedback approximation, we expand here on the work of [1] insofar as the parametrization step lifts fundamental structural assumptions on the problem class. A related approach, though with updates that require the solutions of nonlinear matrix equations, can be found in [15] based on the expansion of nonlinear systems into Volterra series [27]. Furthermore, we note that, with the explicit low-complexity parametrization of the nonlinearity in an otherwise linear problem formulation, the difficulties of exponentially growing dimensions that come with tensor expansions for general nonlinearities are mitigated; see [23] for a recent discussion regarding model order reduction and note that the approach in [14] requires the computation of a feedback gain matrix that scales with the square of the state-space dimension, which limits its applicability to systems of dimension about 10^5. From this perspective, the effective novelty of our contribution lies in the numerical design of a performant nonlinear feedback law for large-scale systems with a computational effort that only scales linearly with the system size.

The rest of the paper is organized as follows. In Section II, we explain how a low-complexity linear parameter-varying approximation can be obtained by parametrizing the state of an SDC system. The formulas for expanding the SDRE solution and state the constituting equations for the coefficients of the expansion are recalled in Section III. Section IV lays out how POD can be used to realize a low-complexity affine-linear parameter-varying approximation of incompressible Navier-Stokes equations and in Section V, we briefly describe the concepts for solving the high-dimensional matrix equations. In Section VI, we provide the results of a numerical study to show the applicability of the approach and the improvement it brings compared to linear feedback design. The paper is concluded in Section VII.

II. Low-complexity linear parameter-varying approximations

In this section, we consider now systems in SDC form (2). If the system state $v(t)$ is encoded into time-varying parameters $\rho(t) = \mu(v(t)) \in \mathbb{R}^m$, with $m \leq n$, and $v(t) = \nu(\rho(t))$, where μ and ν are the corresponding encoding and decoding maps, then the SDC representation (2) can be formulated as a linear parameter-varying (LPV) system via

$$\dot{v}(t) = \widetilde{A}(\rho(t))v(t) + Bu(t), \quad y(t) = Cx(t), \tag{5}$$

where $\widetilde{A}(\rho) := A(\nu(\rho))$. Such an embedding of a nonlinear system into the class of LPV systems is typically called *quasi LPV* system; see, e.g., [21]. Here we will focus on affine-linear LPV representations, where $\widetilde{A}$ depends affine-linearly on ρ so that (5) can be realized as

$$\dot{v}(t) = \big(\widetilde{A}_0 + \sum_{k=1}^{m} \rho_k(t)\widetilde{A}_k\big)v(t) + Bu(t), \quad y(t) = Cv(t),$$

where ρ_k is the k-th component of ρ and where $\widetilde{A}_0, \widetilde{A}_k \in \mathbb{R}^{n\times n}$ are constant, for $k = 1, \ldots, m$.

If the state v is not exactly parametrized but only approximated with less degrees of freedom in $\hat{\rho}(t) = \hat{\mu}(v(t)) \in \mathbb{R}^r$ such that the dimension of $\hat{\rho}(t)$ is much smaller than the dimension of $\rho(t)$, i.e., $r \ll m$, with an inexact reconstruction

$$v(t) \approx \tilde{v}(t) = \hat{\nu}(\hat{\rho}(t)) = \hat{\nu}(\hat{\mu}(v(t))), \tag{6}$$

then an LPV approximation of (1) is given by

$$\dot{\hat{v}}(t) = \widehat{A}(\hat{\rho}(t))\hat{v}(t) + Bu(t), \quad \hat{y}(t) = C\hat{v}(t), \tag{7}$$

with the approximated system matrix $\widehat{A}(\hat{\rho}) := A(\hat{\nu}(\hat{\rho}))$ and the new system state $\hat{v}(t) \in \mathbb{R}^n$. Note that the state $\hat{v}(t)$ is of *full* dimension n, as our reduction efforts will target the structure of the model rather than the dimension of the states. Nonetheless, the techniques of approximate low-dimensional parametrizations of states and estimates on approximation errors of standard model order reduction (MOR) schemes readily apply.

III. Series expansions of state-dependent Riccati equations

The theory of first-order approximations to the SDRE for a single parameter dependency [4] has been extended in [1] to the multivariate case. We briefly recall the relevant formulas.

To prepare the argument, we assume that v is parametrized through ρ and consider the dependency of the SDRE solution P on the current value of ρ, i.e., $P(\cdot) = P(\rho(\cdot))$. Then, the multivariate Taylor expansion of P about $\rho_0 = 0$ up to order K reads

$$P(\rho) \approx P(0) + \sum_{1\leq|\beta|\leq K} \rho^{(\beta)} P_\beta, \tag{8}$$

where $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}_0^m := (\mathbb{N} \cup \{0\})^m$ is a multiindex with $|\beta| := \sum_{i=1}^m \beta_i$, where $\rho^{(\beta)} := \rho_1^{\beta_1}\rho_2^{\beta_2}\cdots\rho_m^{\beta_m}$, and where, importantly, P_β are constant matrices:

$$P_\beta := \tfrac{1}{\beta_1!\beta_2!\cdots\beta_m!}\tfrac{\partial^{|\beta|}}{\partial_{\rho_1}^{\beta_1}\partial_{\rho_2}^{\beta_2}\cdots\partial_{\rho_m}^{\beta_m}}P(0).$$

In particular, the expansion up to order one (i.e., the associated first-order approximation) writes

$$P(\rho) \approx P(0) + \sum_{|\beta|=1} \rho^{(\beta)} P_\beta =: P_0 + \sum_{k=1}^{m} \rho_k L_k. \quad (9)$$

Substituting P in the SDRE (3) by its series expansion (8) and considering the affine-linear dependency of A on ρ yields

$$\begin{aligned}(\sum_{k=0}^{m} \rho_k A_k)^{\mathsf T} (\sum_{|\beta|\le K} \rho^{(\beta)} P_\beta) + (\sum_{|\beta|\le K} \rho^{(\beta)} P_\beta)(\sum_{k=0}^{m} \rho_k A_k) \\ - (\sum_{|\beta|\le K} \rho^{(\beta)} P_\beta) B B^{\mathsf T} (\sum_{|\beta|\le K} \rho^{(\beta)} P_\beta) = -C^{\mathsf T} C, \quad (10)\end{aligned}$$

where, for compactness of the expression, we introduce $\rho_0 = 1$ and use the relevant conventions for $\beta = 0 \in \mathbb{N}_0^m$. By matching the coefficients for $\rho^{(\beta)}$ in (10), we obtain equations for the matrices of the first-order approximation (9):

$$A_0^{\mathsf T} P_0 + P_0 A_0 - P_0 B B^{\mathsf T} P_0 = -C^{\mathsf T} C, \quad (11)$$

for P_0 and

$$\begin{aligned}&\left(A_0 - BB^{\mathsf T} P_0\right)^{\mathsf T} L_k + L_k \left(A_0 - BB^{\mathsf T} P_0\right) \\ &= -\left(A_k^{\mathsf T} P_0 + P_0 A_k\right), \quad (12)\end{aligned}$$

for L_k, with $k = 1, \ldots, m$; see also [1].

IV. Approximations of Navier-Stokes equations through linear affine parametrizations

As the standard model for incompressible flows we consider the spatially discretized Navier-Stokes equations (NSEs). After a shift of variables that eliminates constant nonzero Dirichlet boundary conditions, the semi-discrete NSEs in the variables of the velocity $v(t) \in \mathbb{R}^n$ and pressure $p(t) \in \mathbb{R}^{n_p}$ with control input $u(t)$ writes

$$M\dot v = \widetilde N(v, v) + \widetilde A v + J^{\mathsf T} p + \widetilde B u, \quad (13a)$$

$$0 = Jv. \quad (13b)$$

with M, $\widetilde A \in \mathbb{R}^{n\times n}$ being the mass and stiffness matrix, with $J \in \mathbb{R}^{n_p \times n}$ being the discretized divergence, with $\widetilde N$ accounting for the convection and with B modelling the effect of the boundary control; see [5] for technical details. For theoretical considerations, the incompressibility constraint (13b) can be resolved and the velocity v can be determined by the equivalent projected equations

$$M\dot v = N(v, v) + Av + Bu, \quad (14)$$

where $N(v, v) = \Pi^{\mathsf T} \widetilde N(v, v)$, A and B denote $\Pi^{\mathsf T} \widetilde A$ and $\Pi^{\mathsf T} \widetilde B$, respectively, and

$$\Pi := I_n - M^{-1} J^{\mathsf T} \left(J M^{-1} J^{\mathsf T}\right)^{-1} J \quad (15)$$

is the so-called *discrete Leray projector*; see [20] for properties of Π and formulations in the coordinates of the subspace spanned by $\Pi^{\mathsf T}$ and see [7] where we have proven that the SDRE feedback based on (14) is equivalent to that of (13).

By the homogeneity in the boundary conditions, the nonlinearity $N(v, v)$ that models the convection is linear in both arguments (see, e.g., [5] for explicit formulas of $N(\cdot, \cdot)$ in a spatial discretization) so that both $N_1(v)\colon w \mapsto N(v, w)$ and $N_2(v)\colon w \mapsto N(w, v)$can be realized as state-dependent coefficient matrix and so that for any blending parameter $\lambda \in \mathbb{R}$, an SDC representation is given as

$$N(v, v) = \lambda N_1(v)v + (1 - \lambda) N_2(v) v =: N_\lambda(v) v. \quad (16)$$

Even more, if in an approximative parametrization as in (6), the decoding is linear, then the induced LPV approximation is *affine linear*; see [18, Rem. 2].

In fact, let $V_r \in \mathbb{R}^{n\times r}$ be the matrix of r POD modes designed to best approximate the velocity in an r-dimensional subspace of $\mathbb{R}^n$, then with

$$\hat\rho(t) := V_r^{\mathsf T} v(t) \quad \text{and} \quad \tilde v(t) := V_r \hat\rho(t)$$

and the approximation property of the POD basis, we obtain

$$v(t) \approx \tilde v(t) = V_r \hat\rho(v(t)) = \sum_{i=1}^{r} \hat\rho_i(v(t)) \hat v_i,$$

where $\hat v_i$, for $i = 1, \ldots, r$, are the columns of $V_r \in \mathbb{R}^{n\times r}$. By the linearity of $v \to N_\lambda(v)$, the SDC representation (16) is readily approximated by

$$N(v)v \approx N(\tilde v) v = \left(\sum_{i=1}^{r} \hat\rho_i \widehat N_i\right) v \quad (17)$$

with $\widehat N_i := N_\lambda(\hat v_i)$, for $i = 1, \ldots, r$.

From the orthogonality of the POD basis it follows that $\tilde v(t) = V_r V_r^{\mathsf T} v(t) \to v(t)$ uniformly with $r \to n$, $r \le n$, which can be translated into convergence of the LPV approximations

$$\left(\sum_{i=1}^{\hat r} \hat\rho_i \widehat N_i\right) \to \left(\sum_{i=1}^{r} \hat\rho_i \widehat N_i\right) \to \left(\sum_{i=1}^{n} \hat\rho_i \widehat N_i\right) = N(v),$$

for $\hat r \to r \to n$ and $\hat r \le r \le n$. Practically, in view of computing approximations to the SDRE solution and the associated feedback law, this means that the series expansion in (9) can be augmented or reduced by simply adding or discarding parameters and the corresponding factors L_k.

V. Handling of high-dimensional matrix equations

The solution of the arising matrix equations poses several challenges as outlined in the following. First, we note that for systems like the spatially discretized NSEs (14), a mass matrix M needs to be incorporated. Since M is typically positive definite and, thus, invertible, such systems are readily transformed into the standard form of (2). In practice, however, it is beneficial to consider formulations of the Riccati and Lyapunov equations (11) and (12) without the explicit inversion:

$$A_0^{\mathsf T} P_0 M + M^{\mathsf T} P_0 A_0 - M^{\mathsf T} P_0 B B^{\mathsf T} P_0 M = -C^{\mathsf T} C, \quad (18)$$

and

$$A_{0,\mathsf{cl}}^{\mathsf T} L_k M + M^{\mathsf T} L_k A_{0,\mathsf{cl}} = -[M^{\mathsf T} P_0 A_k + A_k^{\mathsf T} P_0 M], \quad (19)$$

for $k = 1, \ldots, m$, where $A_{0,\text{cl}} := A_0 - BB^\mathsf{T} P_0 M$ is the closed-loop system matrix corresponding to (18); see, e.g., [6]. Generally, these formulations cover problems where the mass matrix is not invertible as in differential-algebraic equations like the Navier-Stokes equations in the original formulation (13), for the presentation we refer to the projected system (14). In practice, the system matrices in (14) and the involved projection Π from (15) are realized implicitly during the application of iterative matrix equation solvers for (18) and (19); see [9].

Other than that, the solution of these large-scale matrix equations is a nontrivial task because of the memory requirement of the solutions $P_0 \in \mathbb{R}^{n \times n}$ and $L_k \in \mathbb{R}^{n \times n}$, $k = 1, \ldots, m$. State-of-the-art solvers address this issue by computing approximative low-rank factorizations. In fact, the stabilizing solution of the Riccati equation (18) namely P_0 is positive semi-definite such that, if the dimensions of inputs and outputs are small, it can be well represented by a low-rank Cholesky factorization, i.e., $P_0 \approx Z_0 Z_0^\mathsf{T}$ with $Z_0 \in \mathbb{R}^{n \times \ell_0}$ and ℓ_0 being smaller than n by several orders of magnitude; see, for example, [30] for an analysis of the low-rank structure of solutions to Riccati equations. One can find various numerical methods in the literature to efficiently compute these low-rank factors of Riccati equations without ever forming the full solution P_0; see [10] for an overview. In our numerical experiments, we rely on the low-rank Newton-Kleinman-ADI method [11].

The solution of the equations (19) for the L_k's comes with the additional challenge that the solutions are symmetric but indefinite, so that the standard approach of computing Cholesky factors does no more apply. Here, we have to rely on the low-rank factorization $P_0 \approx Z_0 Z_0^\mathsf{T}$ to define the low-rank indefinite factorization of the right hand sides as

$$M^\mathsf{T} P_0 A_k + A_k^\mathsf{T} P_0 M \approx \begin{bmatrix} Z_0^\mathsf{T} M \\ Z_0^\mathsf{T} A_k \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & I_{\ell_0} \\ I_{\ell_0} & 0 \end{bmatrix} \begin{bmatrix} Z_0^\mathsf{T} M \\ Z_0^\mathsf{T} A_k \end{bmatrix}$$

that enables the approximation of the solution to (19) as $L_k \approx Z_k D_k Z_k^\mathsf{T}$, for $k = 1, \ldots, m$, where D_k is a symmetric but possibly indefinite matrix. To compute the factors, we are using the LDL^T-factorized low-rank ADI method from [25]. However, as the size of the factors in the right-hand side is two times that of the Z_0 factors, which is comparatively large, intermediate compression steps are needed to facilitate the computation.

VI. Numerical Experiments

The code, raw data and results of the presented numerical experiments are available at [19]. For the solution of Riccati and Lyapunov equations in MATLAB 9.9.0.1467703 (R2020b), we used the solver implementations from MORLAB version 5.0 [12] and M-M.E.S.S. version 2.2 [28]. For the simulation part, we resort to our Python interface [17] between *SciPy* and the finite element toolbox FEniCS [26].

We consider the stabilization of the flow in the wake of a 2D cylinder through two control inlets at the periphery of the cylinder. Measurement outputs are defined as averaged velocities over a small neighborhood of three sensor points in the wake; see [5] on technical details, the detailed control setup, how the Dirichlet control is relaxed as penalized Robin boundary conditions and how the coefficients of the system (13) are exported for the controller design.

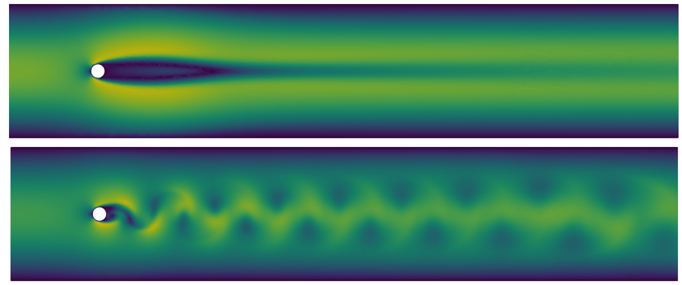

Figure 1. Snapshots of the flow in the unstable steady state and in the fully developed periodic vortex shedding regime.

As for the numerical setup, we consider here the Reynoldsnumber $Re = 60$ and start from the associated non-zero steady state, which is to be stabilized; see Fig. 1 for the basic geometry of the example and snapshots of the steady state and periodic regime that develops if no stabilization is employed. For the spatial discretization, we use quadratic-linear *Taylor-Hood* finite elements on a nonuniform mesh that leads to a system of size 57 000. For the time integration, we use an implicit-explicit Euler time stepping method that in particular treats the linear part and the incompressibility constraint implicitly, whereas the nonlinear part and the feedback is treated explicitly in time. Generally, we are concerned with a system of type (13) with $\begin{bmatrix} v(t)^\mathsf{T} & p(t)^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{57\,000}$ with the input $u(t) \in \mathbb{R}^2$ and the output $y(t) \in \mathbb{R}^6$ extracted from the velocity state by a linear output operator C as $y(t) = Cv(t)$.

The procedure of the simulations comprises the following steps:

(0) Compute the steady state for $u = 0$ to be used as reference for the stabilization, for the shift of the system that removes nonzero boundary conditions, and as the starting value for the closed-loop simulations.
(1) Perform open-loop simulations to collect data for the POD basis for the affine-linear LPV approximation (17).
(2) Compute the Riccati solution P_0 and the Lyapunov solutions L_k via (18) and (19).
(3) Close the control loop with the nonlinear feedback law

$$u(t) = -B^\mathsf{T}\big(P_0 + \sum_{k=1}^{r} \hat{\rho}_k(v(t)) L_k\big) M v(t). \tag{20}$$

In the presented numerical study, the relevant steps were realized as follows. To acquire the data for the POD basis, we take 401 snapshots of the velocity equally distributed on the time interval $[0, 0.5]$ for the test signal

$$u(t) = \begin{bmatrix} \sin(t) & 0 \end{bmatrix}^\mathsf{T}, \tag{21}$$

and define V_r as the matrix of the r leading left singular vectors with respect to the weighted inner-product induced by the mass matrix M of the finite element discretization; see [3]. Then, the LPV approximation (16) is computed for the NSEs with $\lambda = 0.75$. In the following, we let xSDRE-r denote the feedback definition by the truncated SDRE series approximation (20) of parameter dimension r and let LQR denote the *linear-quadratic regulator*, which is readily defined as

$$u(t) = -B^{\mathsf{T}} P_0 M v(t).$$

Note that the LQR resembles xSDRE-r for $r = 0$; cf. (20).

To trigger the instabilities in the closed-loop simulations, we apply the test signal (21) on a short time $[0, t_{\mathsf{c}}]$ before we *switch on* the feedback at $t = t_{\mathsf{c}}$. In this way, the system will deviate from the linearization point. For t_{c} too large, the state may have left the region of attraction for which the linear LQR or the SDRE-based controller will stabilize the nonlinear system.

In our experiments, we employed *Tikhonov regularization* in the form of an $\alpha \in [10^{-2}, 10^{3}]$, which is included by replacing the original input matrix B by the scaled version $\breve{B} := \frac{1}{\sqrt{\alpha}} B$ in the definition of the SDRE (3) as well as in the solved Riccati and Lyapunov equations (18) and (19). Consequently, the corresponding feedback needs to be scaled as well, e.g., in the LQR case as $u(t) = -\frac{1}{\alpha} B^{\mathsf{T}} \breve{P}_0 M v(t)$, where $\breve{P}_0$ solves the Riccati equation with $\breve{B}$.

For a reliable estimate on the potential improvements by the inclusion of nonlinear feedback relations, we discretized the parameter domain of (α, t_{c}) and recorded the results of the closed-loop simulations for those parameters values that appeared to be at the border of the domain of attraction; see Fig. 2. Generally, we can confirm that a smaller regularization parameter α, increases the performance in the sense that the controller stabilizes the flow after larger idle times t_{c}. However the domain of attraction does not cover arbitrarily large t_{c} such that numerical issues with less regularized feedback actions (for $\alpha < 10^{-2}$) are observed, too.

From our simulations (see Fig. 2), we can say that the inclusion of nonlinear modes in the feedback gain reliably increases the domain of attraction. In particular for a fixed idle time t_{c} during which the controller is not active and the flow evolves exponentially away from the linearization point, the xSDRE-r approach generally allows for larger values of α, which is preferable from a numerical (more regularization) and potentially practical (less input energy is used) perspective.

This widening of the domain of attraction becomes evident for parameter values, where the LQR-feedback fails and the xSDRE-r approach succeeds. The effect of the nonlinear feedback can be clearly seen in Fig. 3 and 4, where the norm of the input signals over time for the LQR feedback and different truncation orders in the xSDRE-r feedback are shown. In Fig. 3 and 4 we can see a decay in the input signal norms for xSDRE-r (indicating stabilization) and a growth for the LQR feedback (indicating that no stabilization was achieved). Moreover, for the parameter setup of $\alpha = 100$ and $t_{\mathsf{c}} = 0.2114$, a continuous performance improvement with the truncation order r can be observed in Fig. 3. The effects of less regularization can be observed in Fig. 4, which shows the less regular feedback action. Nonetheless, unlike LQR, the xSDRE-r approach with $r = 10$ achieves stabilization.

Overall, we can state that the truncated SDRE approach, with an almost negligible overhead in the simulation phase, is capable of improving the feedback performance if compared to the classical linear but provably robust and performant LQR feedback.

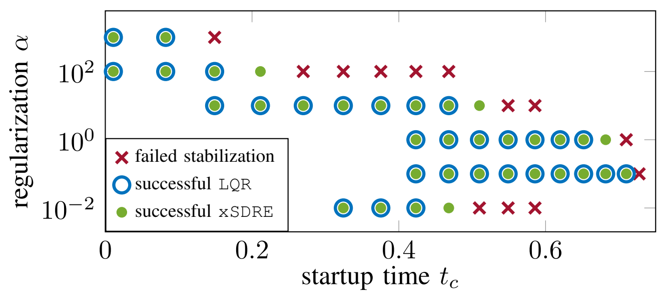

Figure 2. Overview about the experimental results with a sweep over the relevant part of the (α, t_{c}) parameter domain. For an illustration of the feedback action for parameter configurations $(\alpha, t_{\mathsf{c}}) = (100, 0.2114)$ and $(\alpha, t_{\mathsf{c}}) = (1, 0.6819)$ for which, in particular, xSDRE-r stabilizes the system while LQR fails to do so, see Fig. 3 and 4, respectively.

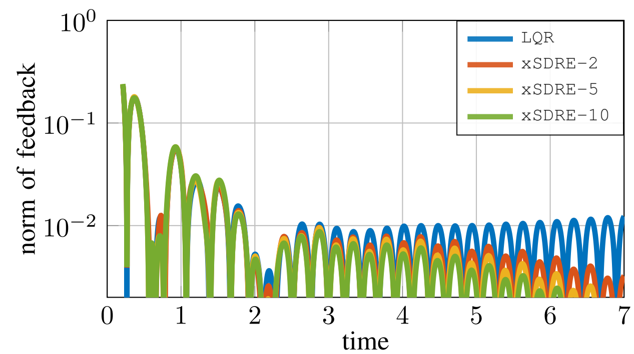

Figure 3. Norms of input signal for the LQR and different r in the xSDRE-r feedback for the case of $\alpha = 100$ and $t_{\mathsf{c}} = 0.2114$. Stabilization and a continuous performance improvement for xSDRE-r can be observed, while the classical LQR-feedback fails.

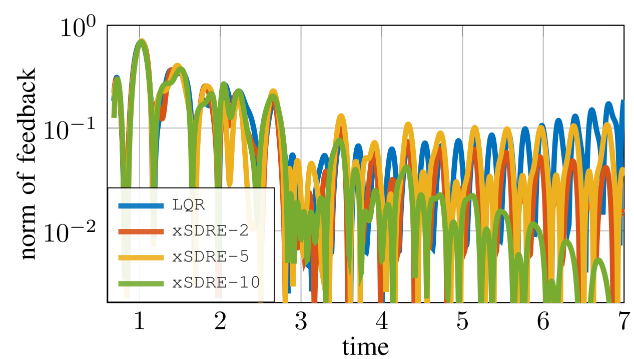

Figure 4. Norms of input signal for the LQR and different r in the xSDRE-r feedback for the case of $\alpha = 1$ and $t_{\mathsf{c}} = 0.6819$. The xSDRE-10-feedback clearly stabilizes the system in contrast to the classical LQR approach and xSDRE-r for smaller values of r.

VII. Conclusion

We have presented here a general framework that uses the embedding of nonlinear systems in the class of LPV systems, POD for reduction of the complexity of the parameter dimension, and the quadratic structure of the convection term in the Navier-Stokes equations to make the nonlinear feedback design through truncated expansions of the SDRE applicable. With state-of-the-art matrix equation solvers, computational feasibility of this approach was achieved, too. As illustrated by a numerical example, this generic nonlinear approach provides a measurable improvement over the closely related classical LQR approach with a small additional computational effort at runtime.

The potentials and needs for future work are manifold. Firstly, it would be interesting to investigate expansions of second order. Secondly, we have not paid particular attention to the definition of the coordinates for the affine LPV approximation and simply resorted to POD. Since a small dimension r is most important, in particular if one wants to consider a second-order expansions of the SDRE, other, possibly nonlinear parametrizations might be considered. For the analysis of the approximation, suitable measures for the residual, e.g., in the SDRE approximation, need to be derived together with formulas for feasible evaluations in the large-scale system case.

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