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# Local Static Anti-Windup Design With Sign-Indefinite Quadratic Forms

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***Abstract*— This note proposes static anti-windup gains design for closed-loop linear systems with saturating inputs providing maximized non-ellipsoidal estimates of the basin of attraction. The proposed design uses sign-indefinite quadratic forms leading to locally positive definite nonquadratic Lyapunov functions. An iterative algorithm that solves the bilinear matrix conditions inherent to this problem is proposed, based on a convex-concave decomposition. A numerical application is presented to illustrate the effectiveness of the proposed method.**

## I. Introduction

Anti-windup techniques are widely used in error-feedback control systems with integral action to prevent them from inducing diverging selections when the actuator saturates and the controller experiences the so-called "windup" phenomenon (see, for example, [4], [14]). As defined and explained in [1], anti-windup techniques limit the control effort when saturation is detected, inducing closed-loop recovery and resolving the "windup" phenomenon. This is particularly important in systems with aggressive controllers or fast dynamics, where poor performance or instability may produce catastrophic effects [12].

The existing algorithms for designing anti-windup loops mainly use quadratic Lyapunov functions, associated with generalized sector properties of the deadzone function (built for the saturation map and corresponding to the identity minus the saturation). See, for example, [6], [7], [15], [13], [16], [17] and references therein. Although the anti-windup technique is well known to preserve certain performance requirements despite the presence of saturation, results based on quadratic Lyapunov functions may reveal some conservatism as discussed, for example, in [17, Section 4.4.1.1], where system theoretic feasibility conditions are presented in terms of quasi common quadratic Lyapunov functions.

With global exponential guarantees, which are only achievable with exponentially stable plants, the conservatism was well overcome by the nonquadratic global design of [10]. Here we extend the construction to the regional case, so that the open-loop plant is not required to be exponentially stable. With nonglobal results, the size of the basin of attraction becomes an important performance indicator, which is indirectly taken into account in our novel design through suitable nonquadratic estimates. Applying the sign-indefinite quadratic form presented in [11], our approach allows developing potentially less conservative conditions than the existing designs, such as [5], [7], [8], [9], which are based on quadratic Lyapunov functions and whose conservativeness is discussed in [17, Section 4.4.1.1]. Paralleling [10], we exploit a convex-concave decomposition [3] to obtain a novel sequential LMI-based algorithm with a new selection of an initial feasible solution. Our algorithm performs an offline design of two anti-windup strategies, one with and one without a nonlinear algebraic loop. In contrast to [10], our design induces regional exponential stability and maximizes some proxy of the volume of a nonquadratic estimate of the basin of attraction, by imposing a set inclusion similar to the one in [11, Section III.c].

The paper is organized as follows. Section II describes the system under consideration and the anti-windup design goal. Section III introduces results based on sign-indefinite quadratic forms. The iterative algorithm for the static anti-windup design and its proof of convergence are given in Section IV. Section V discusses numerical results.

*Notation:* $\mathbb{R}^{m(\times n)}$ is the $m(\times n)$-dimensional Euclidean space. $\mathbb{S}^n_{>0}$ (respectively $\mathbb{S}^n_{\geq 0}$) is the set of symmetric positive definite (respectively, positive semi-definite) matrices of dimension $n$. $\mathbb{D}^n_{>0}$ (respectively $\mathbb{D}^n_{\geq 0}$) is the set of diagonal positive definite (respectively, positive semi-definite) matrices of dimension $n$. Given a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $\lambda(P)$, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote, respectively, the set of eigenvalues, minimum real eigenvalue and maximum real eigenvalue of $P$. $I_n$ is the identity matrix of size $n \times n$ and 0 the null matrix of appropriate dimensions. Finally, let $\mathrm{He}\,(A) = A + A^{\mathsf{T}}$, where $A^{\mathsf{T}}$ is the transpose of $A$.

## II. System definition

Consider the dynamic output feedback linear control system subject to input saturation

$$\left|\begin{array}{rcl} \dot{x}_p &=& A_p x_p + B_p \mathrm{sat}\,(u) \\ y &=& C_p x_p + D_p \mathrm{sat}\,(u) \\ \dot{x}_c &=& A_c x_c + B_c y + \nu_1 \\ u &=& C_c x_c + D_c y + \nu_2 \end{array}\right. , \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$, $y \in \mathbb{R}^p$ are the plant states and output and $x_c \in \mathbb{R}^{n_c}$, $u \in \mathbb{R}^m$ are the controller states and output, respectively. The decentralized symmetric saturation function is denoted as

$$\mathrm{sat}_i\,(u_i) := \max\left\{-\bar{u}_i, \min\left\{\bar{u}_i, u_i\right\}\right\}, \tag{2}$$

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where $i = 1, ..., m$ and $\bar{u}_i > 0$ are the saturation limits. Linear feedback (1) is well-posed if and only if there exist the inverses

$$\Delta_u := (I_m - D_c D_p)^{-1}, \qquad \Delta_y := (I_p - D_p D_c)^{-1}. \quad (3)$$

*Assumption 1:* The closed loop (1), with $\nu_1 = 0$, $\nu_2 = 0$, $\text{sat}(u) = u$, is exponentially stable.

Define now the deadzone function $\text{dz}(u) := u - \text{sat}(u)$ and select anti-windup inputs $\nu_1$ and $\nu_2$ as

$$\nu_1 := E_c \text{dz}(u), \qquad \nu_2 := F_c \text{dz}(u), \quad (4)$$

where $E_c \in \mathbb{R}^{n_c \times m}$ and $F_c \in \mathbb{R}^{m \times m}$ are the anti-windup gains to be designed. Note that Assumption 1 is necessary for the algorithm design to make sense because the anti-windup inputs $\nu_1$ and $\nu_2$ are zero in a neighborhood of the origin. When combined with the anti-windup action (4) and definitions in (3), dynamics (1) becomes

$$\left| \begin{array}{lcl} \dot{x} & = & A_{cl} x + \left( B_0 + B_{aw} \begin{bmatrix} E_c \\ F_c \end{bmatrix} \right) \text{dz}(u) \\ u & = & K_1 x + (D_0 + \Delta_u F_c) \text{dz}(u) \end{array} \right. , \quad (5)$$

with $x := \begin{bmatrix} x_p^\mathsf{T} & x_c^\mathsf{T} \end{bmatrix}^\mathsf{T}$ and $A_{cl}$, $B_0$, $B_{aw}$, $K_1$, $D_0$, $D_{aw}$ defined in equation (6), at the bottom of this page. Additionally, for compact notation, define

$$B_{cl} := B_0 + B_{aw} \begin{bmatrix} E_c \\ F_c \end{bmatrix}, \quad K_2 := D_0 + \Delta_u F_c. \quad (7)$$

As discussed in [17, Chapter 2], when $D_c D_p \neq 0$, the linear well-posedness in (3) does not ensure well-posedness of the nonlinear algebraic loop in (5). Then, the proposed design ensures well-posedness via the action of $F_c$. Nevertheless, for the case where $D_c D_p = 0$, a simplified anti-windup design acting only on the dynamics of the controller is presented, which structurally leads to nonlinear well-possedness because imposing $F_c = 0$ induces $u = K_1 x$.

## III. Stability Certificates Via Sign-Indefinite Quadratic Forms

Denote by $x \to u(x)$ the solution of the nonlinear algebraic loop in the second equation of (5). The static linear anti-windup gains synthesis of this paper makes use of the quadratic form

$$\begin{aligned} V(x) &:= \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}^\mathsf{T} P \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix} \\ &= \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}^\mathsf{T} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\mathsf{T} & P_{22} \end{bmatrix} \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}, \end{aligned} \quad (8)$$

which is not constrained to be globally positive because, while $P_{11} > 0$ is enforced so that locally $V$ is quadratic positive definite, $P_{22}$ may be sign-indefinite, thus including a broad class of nonquadratic Lyapunov functions [11] and reducing conservativeness as compared to solutions using common quadratic forms $x^\mathsf{T} Q x$, as in [5], [8], which require positive-definiteness of $Q$ and the stringent feasibility conditions discussed in [17, Section 4.4.1]. With function $V$, non-global analysis results in [11, Theorem 3] provide nontrivial estimates of the basin of attraction of the origin for dynamics (5) by proceeding as follows. First, define the function $x \to h(x) \in \mathbb{R}^m$ as

$$h(x) := H_1 x + H_2 \text{dz}(u(x)), \quad (9)$$

where $H_1 \in \mathbb{R}^{m \times n}$ and $H_2 \in \mathbb{R}^{m \times m}$ are arbitrary design parameters. Then, define the following subset of $\mathbb{R}^n$

$$S_h := \{x \in \mathbb{R}^n : |h(x)|_\infty \leq 1\}, \quad (10)$$

and choose the Lyapunov function candidate

$$W(x) := \begin{cases} \min\{V(x), 1\} & \text{if } x \in S_h \\ 1 & \text{otherwise} \end{cases}, \quad (11)$$

which is shown to lead to a continuous $W$ by the fact that $V$ in (8) and $h$ in (9) are designed in such a way that

$$S(W, 1) = S(V, 1) \cap S_h. \quad (12)$$

With these selections, an estimate of the basin of attraction is given by the open sublevel set

$$S(W, 1) := \{x \in \mathbb{R}^n : W(x) < 1\}. \quad (13)$$

Moreover, in order to enlarge as much as possible the set $S(W, 1)$, introduce

$$\mathcal{E}(\hat{P}, 1) := \{x \in \mathbb{R}^n : x^\mathsf{T} \hat{P} x < 1\} \quad (14)$$

for some $\hat{P} > 0$, and impose

$$\mathcal{E}(\hat{P}, 1) \subset S(W, 1) \quad (15)$$

while maximizing the volume of (14), or similarly, by minimizing the trace of $\hat{P}$. The above steps are necessary to build the next theorem, which is an adaptation of [11, Theorem 3] and provides an estimate of the basin of attraction of the origin for nonlinear system (5).

*Theorem 1:* If there exist matrices $\hat{P} \in \mathbb{S}^n_{>0}$, $P_{11} \in \mathbb{S}^n_{>0}$, $P_{12} \in \mathbb{R}^{n \times m}$, $P_{22} \in \mathbb{S}^m$, $\hat{T} \in \mathbb{D}^m_{>0}$, $T_1 \in \mathbb{D}^m_{>0}$, $H_1 \in \mathbb{R}^{m \times n}$, $H_2 \in \mathbb{R}^{m \times m}$, $E_c \in \mathbb{R}^{n_c \times m}$ and $F_c \in \mathbb{R}^{m \times m}$ such that

$$\begin{aligned} \Psi_1 := &\begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12}^\mathsf{T} & P_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} + \\ &\text{He} \left( \begin{bmatrix} 0 & H_1^\mathsf{T} T_1 - K_1^\mathsf{T} T_1 & 0 \\ 0 & T_1 H_2 + T_1 - T_1 K_2 & 0 \\ H_{1\,i} & H_{2\,i} & 0 \end{bmatrix} \right) > 0 \end{aligned} \quad (16)$$

$$\left[ \begin{array}{c|c|c} A_{cl} & B_0 & B_{aw} \\ \hline K_1 & D_0 & D_{aw} \end{array} \right] := \left[ \begin{array}{cc|c|cc} A_p + B_p \Delta_u D_c C_p & B_p \Delta_u C_c & -B_p \Delta_u & 0 & B_p \Delta_u \\ B_c \Delta_y C_p & A_c + B_c \Delta_y D_p C_c & -B_c \Delta_y D_p & I_{n_c} & B_c \Delta_y D_p \\ \hline \Delta_u D_c C_p & \Delta_u C_c & I_m - \Delta_u & 0 & \Delta_u \end{array} \right] \quad (6)$$

with $i = 1, ..., m$ being the $i^{\text{th}}$ row of $H_1$ and $H_2$, and

$$\Psi_2 := \begin{bmatrix} \hat{P} & 0 \\ 0 & 0 \end{bmatrix} - P + \text{He}\left(\begin{bmatrix} 0 & -K_1^\mathsf{T}\hat{T} \\ 0 & \hat{T}\left(I_m - K_2\right) \end{bmatrix}\right) > 0 \quad (17)$$

with $K_2$ as in (7), hold and, additionally, there exist a scalar $\alpha \geq 0$ and matrices $T_2 \in \mathbb{D}^m_{>0}$, $T_3 \in \mathbb{D}^m_{>0}$ and $T_4 \in \mathbb{D}^m$ satisfying equation (18), at the bottom of this page, with $B_{cl}$ as in (7), then the nonlinear algebraic loop in (5) is well-posed and the origin of (5) is locally exponentially stable from $S(W, 1)$.

*Remark 1:* Due to the hypotheses in Theorem 1, $W$ in (11) is Lipschitz continuous, inclusions $S(W, 1) \subseteq S_h$ and (15) hold, and for all $x \in S_h$, there exist positive scalars $\beta_1$, $\beta_2$ and $\beta_3$ satisfying

$$\beta_1 |x|^2 \leq W(x) \leq \beta_2 |x|^2, \quad (19)$$

$$\dot{W}(x) := \left\langle \nabla W(x), A_{cl}x + \left(B_0 + B_{aw}\begin{bmatrix} E_c \\ F_c \end{bmatrix}\right) \text{dz}(u) \right\rangle \leq -\beta_3 |x|^2. \quad (20)$$

*Proof of Theorem 1:* The main elements used to prove Theorem 1 are given in [11]. Start by recalling a few useful sector-like conditions from [11, Facts 2-5]. First, for any $T \in \mathbb{D}^m_{\geq 0}$, it holds that

$$\text{dz}(u(x))^\mathsf{T} T\left(u(x) - \text{dz}(u(x))\right) \geq 0 \quad \forall x \in \mathbb{R}^n. \quad (21)$$

Moreover, for any $T \in \mathbb{D}^m$ and for almost all $x \in \mathbb{R}^n$

$$\text{dz}(u(x))^\mathsf{T} T\left(\dot{u}(x) - \dot{\text{dz}}(u(x))\right) \equiv 0, \quad (22)$$

$$\dot{\text{dz}}(u(x))^\mathsf{T} T\left(\dot{u}(x) - \dot{\text{dz}}(u(x))\right) \equiv 0, \quad (23)$$

and, finally, for any $T \in \mathbb{D}^m_{\geq 0}$, it holds that

$$\text{dz}(u(x))^\mathsf{T} T\left(u(x) - \text{dz}(u(x)) - h(x)\right) \geq 0, \ \forall x \in S_h. \quad (24)$$

Then, thanks to the results in [2, Proposition 1] and the definition of $T_3$, system (5) is well-posed under condition (18). Under well-posedness, Lipschitz continuity of $W$, inclusion $S(W, 1) \subseteq S_h$ and the existence of bounds $\beta_1$ and $\beta_2$ in (19) are proven in [11, Section III]. Additionally, notice that it can be stated that $W(x) \leq V(x)$ in $S_h$ from definition (11). Then, by hypothesis (17) and inequality (21),

$$\begin{aligned} x^\mathsf{T}\hat{P}x - V(x) &\geq x^\mathsf{T}\hat{P}x - V(x) \\ &\quad - 2\text{dz}(u(x))^\mathsf{T}\hat{T}\left(u(x) - \text{dz}(u(x))\right) \\ &= \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}^\mathsf{T} \Psi_2 \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix} > 0. \end{aligned}$$

Thus,

$$x^\mathsf{T}\hat{P}x > W(x), \qquad \forall x \in S_h \quad (25)$$

and, consequently, $x^\mathsf{T}\hat{P}x < 1$ implies $W(x) < 1$ for all $x \in S_h$, which implies in turn inclusion (15). Furthermore, according to [11, Lemma 1], local exponential stability holds for almost all $x \in S(W, 1)$ if (20) is satisfied. Then, by exploiting (22), (23), (24) and recalling that $W$ coincides with $V$ for all $x \in S(W, 1)$, then, for almost all $x \in S(W, 1)$,

$$\begin{aligned} \dot{W}(x) &\leq \dot{W}(x) + 2\dot{\text{dz}}(u(x))^\mathsf{T} T_2\left(u(x) - \text{dz}(u(x)) - h(x)\right) \\ &\quad + 2\dot{\text{dz}}(u(x))^\mathsf{T} T_3\left(\dot{u}(x) - \dot{\text{dz}}(u(x))\right) \\ &\quad + 2\text{dz}(u(x))^\mathsf{T} T_4\left(\dot{u}(x) - \dot{\text{dz}}(u(x))\right) = \eta^\mathsf{T}\bar{\Psi}_3\eta, \quad (26) \end{aligned}$$

where $\eta = \begin{bmatrix} x^\mathsf{T} & \text{dz}(u(x))^\mathsf{T} & \dot{\text{dz}}(u(x))^\mathsf{T} \end{bmatrix}^\mathsf{T}$ is an extended state vector. Through extensive derivations it can be shown that

$$\bar{\Psi}_3 = \Psi_3 - \text{He}\left(\begin{bmatrix} P_{11} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \alpha I_n & 0 & 0 \end{bmatrix}\right) \quad (27)$$

and, by definitions of $P_{11}$ and $\alpha$ and by hypothesis (18),

$$\bar{\Psi}_3 \leq \Psi_3 < 0. \quad (28)$$

Consider now the inequalities

$$-\eta^\mathsf{T}\bar{\Psi}_3\eta \geq \lambda_{\min}\left(-\bar{\Psi}_3\right)|\eta|^2 \geq \lambda_{\min}\left(-\bar{\Psi}_3\right)|x|^2,$$

which can be combined with (26) to prove (20) with $\beta_3 = \lambda_{\min}\left(-\bar{\Psi}_3\right) > 0$, thus ensuring local exponential stability of (5) with basin of attraction containing $S(W, 1)$. □

## IV. Iterative Algorithm Based on Convex-concave Decomposition

The sufficient conditions (16), (17), (18) of Theorem 1 are BMI conditions in the decision variables. Therefore, we propose here an iterative approach starting from a non-stabilizing initial condition that aims at designing gains $(E_c, F_c)$ through iterative steps. The algorithm, presented in Section IV-C, is executed offline starting from the feasible initial condition constructed in Section IV-A and uses the convex-concave decomposition presented in Section IV-B.

### A. Feasible Initial Conditions

A feasible initial solution for which BMIs (16), (17) and (18) hold corresponds to

$$E_c = 0, \quad (29)$$

$$F_c = D_c D_p, \quad (30)$$

$$\Psi_3 := \text{He}\left(\begin{bmatrix} P_{11}\left(A_{cl} + \alpha I_n\right) & P_{11}B_{cl} & P_{12} \\ P_{12}^\mathsf{T}A_{cl} & P_{12}^\mathsf{T}B_{cl} & P_{22} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ T_2K_1 - T_2H_1 + T_4K_1A_{cl} & T_2K_2 - T_2 - T_2H_2 + T_4K_1B_{cl} & T_4K_2 - T_4 \\ T_3K_1A_{cl} & T_3K_1B_{cl} & T_3K_2 - T_3 \end{bmatrix}\right) < 0 \quad (18)$$

$$P_{11} = I_n, \quad P_{12} = 0, \quad P_{22} = 0, \quad H_1 = 0, \quad H_2 = 0,$$
$$\alpha = -\frac{1}{2}\lambda_{\max}\left(A_{cl} + A_{cl}^\mathsf{T} + I_n + \left(B_{cl} + K_1^\mathsf{T}\right)\left(B_{cl}^\mathsf{T} + K_1\right)\right),$$
$$T_1 = \lambda_{\max}\left(K_1 K_1^\mathsf{T}\right)^{-1} I_m, \quad T_2 = I_m,$$
$$T_3 = \lambda_{\max}\left(K_1\left[A_{cl}B_{cl}\right]\begin{bmatrix} A_{cl}^\mathsf{T} \\ B_{cl}^\mathsf{T}\end{bmatrix} K_1^\mathsf{T}\right)^{-1} I_m, \quad T_4 = 0,$$
$$\hat{T} = \lambda_{\max}\left(K_1 K_1^\mathsf{T}\right)^{-1} I_m, \quad \hat{P} = 2I_n. \tag{31}$$

Selection (31) generally has $\alpha < 0$, which does not satisfy the hypotheses of Theorem 1, but allows suitably initializing the iterative algorithm proposed in Section IV-C.

*Proposition 1:* Inequalities (16), (17) and (18) hold with selections (29), (30), (31).

*Proof:* First, notice that with $E_c = 0$ and $F_c = D_c D_p$, from (7), $K_2 = I_m - \Delta_u + \Delta_u D_c D_p = 0$. Therefore,

$$\Psi_1 = \begin{bmatrix} I_n & -K_1^\mathsf{T} T_1 & 0 \\ -T_1 K_1 & 2T_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is positive definite if and only if the upper left block satisfies

$$\begin{bmatrix} I_n & -K_1^\mathsf{T} T_1 \\ -T_1 K_1 & 2T_1 \end{bmatrix} > 0. \tag{32}$$

To show (32), applying Schur complement, observe that $2T_1 - T_1 K_1 K_1^\mathsf{T} T_1 > 0$, or $2T_1^{-1} - K_1 K_1^\mathsf{T} > 0$. Therefore, $T_1 = \lambda_{\max}\left(K_1 K_1^\mathsf{T}\right)^{-1} I_m$ ensures (32) and leads to $\Psi_1 > 0$. Recalling selection (31) and that $K_2 = 0$, notice that

$$\Psi_2 = \begin{bmatrix} I_n & -K_1^\mathsf{T}\hat{T} \\ -\hat{T}K_1 & 2\hat{T} \end{bmatrix}.$$

To show $\Psi_2 > 0$, apply again Schur complement and follow the same derivations as those for (32). Consider now $\Psi_3$ in (18) with the selections in (31), which reduces to

$$\Psi_3 = \text{He}\left(\begin{bmatrix} A_{cl} + \alpha I_n & B_{cl} & 0 \\ K_1 & -I_m & 0 \\ T_3 K_1 A_{cl} & T_3 K_1 B_{cl} & -T_3 \end{bmatrix}\right). \tag{33}$$

Consider its first two rows and columns, namely

$$\text{He}\left(\begin{bmatrix} A_{cl} + \alpha I_n & B_{cl} \\ K_1 & -I_m \end{bmatrix}\right) + I_{m+n} - I_{m+n}$$
$$= \begin{bmatrix} A_{cl} + A_{cl}^\mathsf{T} + 2\alpha I_n + I_n & B_{cl} + K_1^\mathsf{T} \\ B_{cl}^\mathsf{T} + K_1 & -I_m \end{bmatrix} - I_{m+n} \tag{34}$$

and notice that, with the selection of $\alpha$ in (31),

$$A_{cl} + A_{cl}^\mathsf{T} + 2\alpha I_n + I_n + \left(B_{cl} + K_1^\mathsf{T}\right)\left(B_{cl}^\mathsf{T} + K_1\right) < 0$$

which implies, with a Schur complement on the left matrix at the second row of (34), $\Psi_3 < -I_{m+n}$. Exploiting this last inequality and defining $\tilde{K} := K_1\begin{bmatrix} A_{cl} & B_{cl}\end{bmatrix}$, observe that

$$-\Psi_3 > \begin{bmatrix} I_{m+n} & -\tilde{K}^\mathsf{T} T_3 \\ -T_3\tilde{K} & T_3 \end{bmatrix}.$$

Proceeding with a Schur complement as in (32), the positivity of the matrix at the right side is proven, thus $-\Psi_3 > 0$ and the proof is complete. □

Consider now the static anti-windup solution without $F_c$ action. For this case, another proposition of feasible initial solutions for BMIs (16), (17), (18) can be stated.

*Proposition 2:* When $D_c D_p = 0$, expressions (16), (17), (18) hold with selections (29), (31) and $F_c = 0$.

*Proof:* Notice that $D_c D_p = 0$ leads to $\Delta_u = I_m$. Then, imposing $F_c = 0$ suppresses the algebraic loop, because $K_2 = 0$. By using selections (29) and (31), the same reasoning as in the proof of Proposition 1 applies. □

### *B. Convex-Concave Decomposition*

Since (16), (17) and (18) are all BMIs in the decision variables, we may apply a concave-convex decomposition as presented in [3]. First, focus on (16) and introduce

$$\left[\, X_1 \,\middle|\, -Y_{1\,i} \,\middle|\, \bar{Y}_1 \,\right] :=$$
$$\left[\begin{array}{ccc|ccc|ccc} 0 & T_1 & 0 & H_1 & H_2 - \Delta_u F_c & 0 & -K_1 & I_m - D_0 & 0 \\ 0 & 0 & 1 & H_{1\,i} & H_{2\,i} & 0 & 0 & 0 & 0 \end{array}\right]. \tag{35}$$

Then, regarding (17), define

$$\left[\, X_2 \,\middle|\, -Y_2 \,\middle|\, \bar{Y}_2 \,\right] :=$$
$$\left[\begin{array}{cc|cc|cc} 0 & \hat{T} & 0 & -\Delta_u F_c & -K_1 & I_m - D_0 \end{array}\right]. \tag{36}$$

Finally, considering $\Psi_3$ in (18), define $X_3$, $Y_3$ and $\bar{Y}_3$ as in equation (37) at the bottom of this page. Then, the decompositions of BMIs (16), (17) and (18) are

$$-\Psi_1 = \Phi_1 + X_1^\mathsf{T} Y_{1\,i} - Y_{1\,i}^\mathsf{T} X_1 = \Phi_1 + M_{1\,i} - N_{1\,i} < 0,$$
$$-\Psi_2 = \Phi_2 + X_2^\mathsf{T} Y_2 - Y_2^\mathsf{T} X_2 = \Phi_2 + M_2 - N_2 < 0,$$
$$\Psi_3 = \Phi_3 + X_3^\mathsf{T} Y_3 - Y_3^\mathsf{T} X_3 = \Phi_3 + M_3 - N_3 < 0, \tag{38}$$

for $i = 1, ..., m$ and with

$$\Phi_1 := -\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} - X_1^\mathsf{T}\bar{Y}_1 - \bar{Y}_1^\mathsf{T} X_1,$$
$$\Phi_2 := -\begin{bmatrix} \hat{P} & 0 \\ 0 & 0 \end{bmatrix} + P - X_2^\mathsf{T}\bar{Y}_2 - \bar{Y}_2^\mathsf{T} X_2,$$
$$\Phi_3 := X_3^\mathsf{T}\bar{Y}_3 + \bar{Y}_3^\mathsf{T} X_3,$$

$$\left[\, X_3 \,\middle|\, Y_3 \,\middle|\, \bar{Y}_3 \,\right] := \left[\begin{array}{ccc|ccc|ccc} P_{11} & 0 & 0 & \alpha I_n & 0 & 0 & 0 & 0 & 0 \\ P_{11} & P_{12} & 0 & 0 & B_{aw}\begin{bmatrix} E_c \\ F_c \end{bmatrix} & 0 & A_{cl} & B_0 & 0 \\ P_{12}^\mathsf{T} & P_{22} & 0 & 0 & 0 & 0 & 0 & 0 & I_m \\ 0 & T_2 & 0 & -H_1 & \Delta_u F_c - H_2 & 0 & K_1 & D_0 - I_m & 0 \\ 0 & T_4 & T_3 & 0 & K_1 B_{aw}\begin{bmatrix} E_c \\ F_c \end{bmatrix} & \Delta_u F_c & K_1 A_{cl} & K_1 B_0 & D_0 - I_m \end{array}\right] \tag{37}$$

$$M_j := \frac{1}{2}\left(X_j + Y_j\right)^{\mathsf{T}}\left(X_j + Y_j\right),$$
$$N_j := \frac{1}{2}\left(X_j - Y_j\right)^{\mathsf{T}}\left(X_j - Y_j\right), \quad \forall j \in \{1\,i, 2, 3\}. \quad (39)$$

Observe that $\Phi_1$ is linear in the decision variables $P_{11}$, $P_{12}$, $P_{22}$ and $T_1$; $\Phi_2$ is linear in $\hat{P}$, $P_{11}$, $P_{12}$, $P_{22}$ and $\hat{T}$; $\Phi_3$ is linear in $P_{11}, P_{12}, P_{22}, T_2, T_3$ and $T_4$ and $M_j, -N_j$ are convex and concave, respectively. Additionally, resort to $-\Psi_1 < 0$ and $-\Psi_2 < 0$ instead of $\Psi_1 > 0$ and $\Psi_2 > 0$ in (38) for a simplified notation with respect to the convex semidefinite program presented in [3, Section IV], where the main LMI requires negative-semidefiniteness.

### C. Iterative Algorithm For Anti-Windup Design

Algorithm 1 designs the anti-windup gains $(E_c, F_c)$ both for $F_c = 0$ (this is only possible when $D_c D_p = 0$) and with generic $F_c$. In the first design (where $F_c = 0$), the variable $F_c$ (indicated in square brackets) must not be considered in problems (45), (47) and $F_c$ should be set to zero in (35), (36) and (37). The algorithm involves two optimization loops. In the first loop, starting from the (non-necessarily stabilizing) selection of Section IV-A, $\alpha$ is maximized for seeking a stabilizing solution. In the second loop, starting from the stabilizing solution, the ellipsoid $\mathcal{E}\big(\hat{P}, 1\big)$ in (14) included in the basin of attraction (through (15)) is enlarged.

First, exploit (38) and (39) to define

$$N_j(k) := \tfrac{1}{2}\big(X_j(k) - Y_j(k)\big)^{\mathsf{T}}\big(X_j(k) - Y_j(k)\big),$$
$$\tilde{N}_j(k) := \tfrac{1}{2}\mathrm{He}\Big(\big(X_j(k) - Y_j(k)\big)^{\mathsf{T}}\big(X_j - X_j(k) - Y_j + Y_j(k)\big)\Big),$$

where $k$ is the iteration index of the algorithm. As explained in [3], any solution of the convexified problem $\Phi_j + M_j - \Big(N_j(k) + \tilde{N}_j(k)\Big) < 0$ is also feasible for the original nonconvex constraints $\Psi_1 > 0$, $\Psi_2 > 0$ and $\Psi_3 < 0$, because concavity of $-N_j$ implies, for all $j \in \{1\,(i), 2, 3\}$,

$$\Phi_j + M_j - N_j \le \Phi_j + M_j - \Big(N_j(k) + \tilde{N}_j(k)\Big) < 0, \quad (40)$$

with $i = 1, ..., m$. Applying a Schur complement, define

$$\hat{\Psi}_1 := \begin{bmatrix} N_{1\,i}(k) + \tilde{N}_{1\,i}(k) - \Phi_1 & (X_1 + Y_{1\,i})^{\mathsf{T}} \\ (X_1 + Y_{1\,i}) & 2I_{m+1} \end{bmatrix} > 0, \quad (41)$$

$$\hat{\Psi}_2 := \begin{bmatrix} N_2(k) + \tilde{N}_2(k) - \Phi_2 & (X_2 + Y_2)^{\mathsf{T}} \\ (X_2 + Y_2) & 2I_m \end{bmatrix} > 0, \quad (42)$$

$$\hat{\Psi}_3 := \begin{bmatrix} N_3(k) + \tilde{N}_3(k) - \Phi_3 & (X_3 + Y_3)^{\mathsf{T}} \\ (X_3 + Y_3) & 2I_{2n+3m} \end{bmatrix} > 0. \quad (43)$$

Recall then

$$\hat{P} > 0,\ P_{11} > 0,\ \hat{T} > 0,\ T_1 > 0,\ T_2 > 0,\ T_3 > 0, \quad (44)$$

and the optimization to be solved for the stability of (5) is

$$\max_{\substack{P_{11}, P_{12}, P_{22}, \hat{P}, \\ \hat{T}, T_1, T_2, T_3, T_4, \\ H_1, H_2, E_c, [F_c], \alpha}} \alpha, \text{ s.t. } (41),(42),(43),(44). \quad (45)$$

Convex optimization (45) ensures the stability of (5) with the estimate $S(W, 1)$ whenever an

$$\alpha \ge 0 \quad (46)$$

**Algorithm 1** Anti-windup design without [or with] $F_c$ action

**Input:** $\alpha_0$, $E_{c\,0}$, $[F_{c\,0}]$, $P_0$, $\hat{P}_0$, $\hat{T}_0$, $T_{1\,0}$, $T_{2\,0}$, $T_{3\,0}$, $T_{4\,0}$, ... $H_{1\,0}$, $H_{2\,0}$
**Parameters:** $A_p$, $B_p$, $C_p$, $D_p$, $A_c$, $B_c$, $C_c$, $D_c$
Construct $X_1(0)$, $Y_{1\,i}(0)$, $X_2(0)$, $Y_2(0)$, $X_3(0)$, $Y_3(0)$ ... from (29), (30), (31)
Set $k = 0$
*Optimization Loop 1*
**do**
  Solve (45) for $X_1$, $Y_{1\,i}$, $X_2$, $Y_2$, $X_3$, $Y_3$
  $k \leftarrow k + 1$
  Set $X_1(k) = X_1$, $Y_{1\,i}(k) = Y_{1\,i}$, $X_2(k) = X_2$, ... $Y_2(k) = Y_2$, $X_3(k) = X_3$, $Y_3(k) = Y_3$, $\alpha(k) = \alpha$
**while** $|\alpha(k) - \alpha(k-1)| > \epsilon$
**if** $\alpha(k) \ge 0$ **then**
*Optimization Loop 2*
  **do**
    Solve (47) for $X_1$, $Y_{1\,i}$, $X_2$, $Y_2$, $X_3$, $Y_3$
    $k \leftarrow k + 1$
    Set $X_1(k) = X_1$, $Y_{1\,i}(k) = Y_{1\,i}$, $X_2(k) = X_2$, ... $Y_2(k) = Y_2$, $X_3(k) = X_3$, $Y_3(k) = Y_3$, ... $\mathrm{trace}\{\hat{P}\}(k) = \mathrm{trace}\{\hat{P}\}$
  **while** $\Big|\mathrm{trace}\{\hat{P}\}(k) - \mathrm{trace}\{\hat{P}\}(k-1)\Big| > \epsilon$
  **return** $E_c$, $[F_c]$, $P$, $H_1$, $H_2$
**else**
  **return** No stabilizing solution found
**end**

is returned and, in order to enlarge this estimate, recalling inclusion (15), it is useful to solve

$$\min_{\substack{P_{11}, P_{12}, P_{22}, \hat{P}, \\ \hat{T}, T_1, T_2, T_3, T_4, \\ H_1, H_2, E_c, [F_c], \alpha}} \mathrm{trace}\big\{\hat{P}\big\}, \text{ s.t. } (41),(42),(43),(44),(46), \quad (47)$$

which motivates including two do-while loops in Algorithm 1, namely Optimization Loops 1 and 2. Furthermore, the values of $P$, $H_1$ and $H_2$ returned by Algorithm 1 allow computing the sublevel set $S(W, 1)$, which corresponds to an inner approximation or estimate of the basin of attraction of the origin of the closed loop. Therefore, for each $x \in S(W, 1)$ uniform convergence to zero is guaranteed.

*Proposition 3:* Algorithm 1 terminates in a finite number of iterations and feasibility of the ensuing optimizations is guaranteed at each iteration.

*Proof:* Problem (45) is feasible at the initial step of Optimization Loop 1 since its constraints hold with selections (29), (30), (31). Under hypothesis of Theorem 1, feasibility of (47) in the first step of Optimization Loop 2 is only guaranteed if a stabilizing solution is found at the last iteration of Optimization Loop 1. Furthermore, feasibility of (45) and (47) at each iteration is ensured by (40), as it implies monotonicity of $\alpha$, with $\alpha \ge \alpha(k)$, and $\hat{P}$, with $\mathrm{trace}\{\hat{P}\} \ge \mathrm{trace}\{\hat{P}\}(k)$, at their corresponding Optimization Loops. Finally, since $\hat{P}$ and $\alpha$ are respectively bounded by expressions (42) and (43), Algorithm 1 stops in a finite number of iterations, thus completing the proof. □

## V. Numerical Application and Simulation

Algorithm 1 is applied to a numerical example inspired from [8, Example 1]. The matrices of the state-space dynamics model of the controller and the plant are

$$A_p = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathsf{T}}, D_p = 0,$$

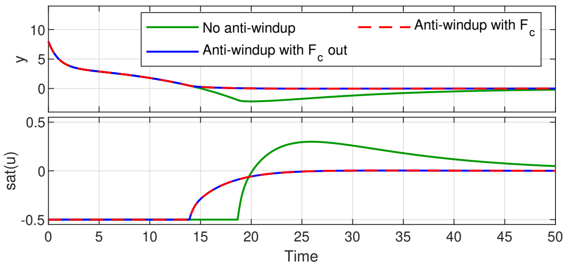


Fig. 1. Simulation of the response of system (48) from the initial state $x = \begin{bmatrix}4 & 4 & 0 & 0\end{bmatrix}^\mathsf{T}$ on top and its control input below.

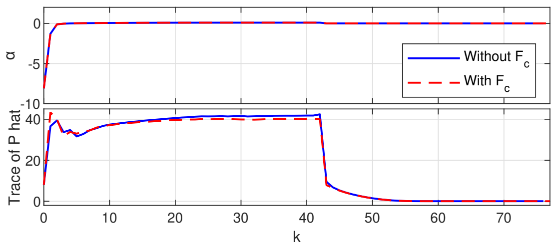


Fig. 2. Evolution of $\alpha$ (top) and of the trace of $\hat{P}$ (bottom) as a function of the index $k$ in the execution of Algorithm 1.

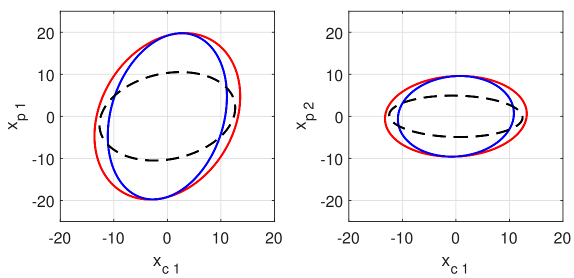


Fig. 3. Sections of set $S(W, 1)$ found with null and free $F_c$ (blue and red, respectively) and results obtained using the solution of [8] (dashed black).

$$A_c = \begin{bmatrix} -100 & 0 \\ 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, C_c = \begin{bmatrix} 11 \\ -1 \end{bmatrix}^\mathsf{T}, D_c = -2, \quad (48)$$

with $\bar{u} = 1$. Fix $\epsilon = 10^{-6}$. With non-restricted $F_c$, Algorithm 1 performs 77 iterations, to find $\alpha = 2.9689 \cdot 10^{-9}$, $E_c = \begin{bmatrix}-0.1776 & 0.1383\end{bmatrix}^\mathsf{T}$ and $F_c = -0.4619$, which guarantees stability. We also have $\lambda(P) = \{-0.0013, 0.0011, 0.0017, 0.0089, 0.0167\}$, $H_1 = \begin{bmatrix}0.0029 & -0.0983 & -0.0102 & 0.0018\end{bmatrix}$ and $H_2 = 0.0013$. For the case without $F_c$, we obtain $\alpha = 5.5616 \cdot 10^{-9}$ and $E_c = \begin{bmatrix}0.0127 & 0.0921\end{bmatrix}^\mathsf{T}$ after 76 iterations, which ensures stability as well. Moreover, we find $\lambda(P) = \{-0.0012, 0.0011, 0.0017, 0.0089, 0.0168\}$, $H_1 = \begin{bmatrix}0.0030 & -0.0982 & -0.0106 & 0.0018\end{bmatrix}$ and $H_2 = 0.0010$. Notice that in both cases $P$ is sign-indefinite, which is an allowable selection exploited by the optimizer.

Figure 1 shows the input-output response with initial states $x = \begin{bmatrix}4 & 4 & 0 & 0\end{bmatrix}^\mathsf{T}$. Both anti-windup solutions allow to eliminate the overshoot. In addition to this, Figure 2 reports on the monotonic evolution of the maximization of $\alpha$ and the minimization of the trace of $\hat{P}$ in their respective Optimization Loop, as established in Proposition 3. Furthermore, Figure 3 presents two sections of the estimates of the region of attraction $S(W, 1)$ with and without $F_c$ action, evincing an important enlargement of these estimates as compared to the results obtained with the method presented in [8].

## VI. Conclusion

We addressed the local static anti-windup design for dynamic output feedback linear control systems subject to input saturation by using Lyapunov functions comprising sign-indefinite quadratic forms. The bilinear design conditions are solved with a sequential LMI-based algorithm based on a convex-concave decomposition. Convergence of the algorithm is ensured by constructing a feasible initial solution to the BMI conditions.Algorithm 1 represents a straightforward and effective solution, but future work may include more efficient iterative methods to solve this nonconvex problem, together with the simultaneous design of linear dynamic output feedback controllers.

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