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Some Remarks on LQ Mean-Field Social Control Problems for Stochastic Input Delay Systems

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***Abstract*—In this study, we consider linear-quadratic (LQ) mean-field social control problems for a class of stochastic systems with ordinary control input and delay control input. We define a stabilization problem via a memoryless static output feedback (SOF) strategy and then solve the problem of minimizing the upper bound of the cost function using guaranteed cost control theory. It is found that the minimization of the upper bound of the cost function cannot be attained if only a delay control input exists. Futhermore, it is proved that it is impossible to implement a mean-field SOF strategy to solve the minimization problem, and the input matrix must have the same dimension as the state matrix. To solve this minimiztion problem, the necessary conditions for the sub-optimality are established via stochastic cross-coupled matrix equations (SCCMEs) using the Karush-Kuhn-Tucker condition and the state feedback strategy. Finally, the performance and usefulness of the proposed strategy are investigated using an order-reduced scheme based on the direct method.**

***Index Terms*—Input delay, mean-filed stochastic system, decomposition technique.**

I. Introduction

In mechanical, electrical, biological, and chemical plant control systems, state and input delays often complicate the design of control strategies with feedback [1]. In such situations, methods such as the use of memory of current information based on measurements have been introduced. However, the introduction of such methods is often difficult due to concerns about increasing the complexity and cost of control strategies. Delay-dependent control techniques are another well-known approach. Although delay-dependent control is very attractive, its implementation is subject to various limitations [2]. Therefore, memoryless feedback control structures and delay-independent control strategies have attracted the interest of researchers. Several researchers have focused extensively on design techniques to guarantee stability. Recently, various design algorithms based on Lyapunov-Krasovskii functions have been developed to determine feedback gains [3], [4], [5]. In particular, the problem of output feedback stabilization for a class of stochastic feedforward nonlinear systems with input and state delays has been studied [3]. However, for stochastic systems with delays only on the inputs, the stabilization problem of minimizing the upper bound of the LQ-type cost function has been little studied.

In recent years, several studies on differential game theory for delay systems have been conducted. In [6], Nash games are considered for a class of linear stochastic systems with a state delay using general-guaranteed cost control theory. In [7], a continuous-time model of interbank borrowing and lending systems by using finite player linear-quadratic (LQ) stochastic differential games based on open-loop Nash equilibrium is proposed. In [8], an LQ mean-field game (MFG) for a class of stochastic delay systems is studied. In addition, in [9], the optimal control problem of mean-field stochastic systems with state delay and state constraints is addressed using the maximum principle. Although stability problems based on infinite-time linear quadratic cost functions in stochastic state-delay systems have been widely studied, stability problems in stochastic input-delay systems have not been fully investigated.

In this paper, we study infinite-time LQ mean-field social control problems for a class of stochastic systems with input delays and quadratic cost functions. In particular, we make remarks on the optimization of cost functions for stochastic delay systems. It should be noted that although the state delay case has been considered in [10], the input delay was not investigated. First, it is shown that, when only control input delays exist, the minimization problem for the upper bound of the cost function satisfying the necessary conditions does not have a reliable solution set, and thus it is not meaningful to design a feedback strategy. Second, it is shown that the static output feedback (SOF) strategy set is not feasible when delay control input based on individual SOF strategies and ordinary control inputs based on mean-field SOF are considered. Furthermore, it is shown that the input matrix must have the same dimension as the state matrix. After these remarks, we investigate a class of state feedback strategies to solve the optimization of the upper bound of the cost function. Note that the Karush-Kuhn-Tucker (KKT) condition is satisfied in order to obtain strategies that satisfy the minimization problem. Specifically, we show that each set of strategies is derived by solving the stochastic cross-coupled matrix equation (SCCME). Finally, we investigate the performance and usefulness of the proposed strategies using a reduction scheme based on a direct method to avoid the higher-dimensional problem.

Notation: The following notations are usually defined: $\mathbb{E}[\cdot]$ denotes the mathematical expectation; for any $z \in \mathbb{R}^n$ and $Z = Z^T \geq 0$, $\|z\|_Z^2 = z^T Z z$; **Tr** denotes trace of a matrix;

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vec denotes an ordered stack of the columns of the matrix; **det** denotes determinant of a matrix; **block diag** denotes the block diagonal matrix; $\otimes$ denotes the Kronecker product of matrices; I_n denotes the identity matrix of size n; and $\boldsymbol{J}_n$ denotes the all-ones matrix of size $\mathbb{R}^{n\times n}$; for any matrix

$$\boldsymbol{X}_c,\ \boldsymbol{X}_c=\boldsymbol{X}_c(\varepsilon,\boldsymbol{X}_d,\boldsymbol{X}_o)=\begin{bmatrix}\boldsymbol{X}_d & \varepsilon\boldsymbol{X}_o & \cdots & \varepsilon\boldsymbol{X}_o\\ \varepsilon\boldsymbol{X}_o & \boldsymbol{X}_d & \cdots & \varepsilon\boldsymbol{X}_o\\ \vdots & \vdots & \ddots & \vdots\\ \varepsilon\boldsymbol{X}_o & \varepsilon\boldsymbol{X}_o & \cdots & \boldsymbol{X}_d\end{bmatrix}\in\mathbb{R}^{nN\times nN},$$

$\boldsymbol{X}_d=\boldsymbol{X}_d^T\in\mathbb{R}^{n\times n}$, $\boldsymbol{X}_o=\boldsymbol{X}_o^T\in\mathbb{R}^{n\times n}$.

II. Preliminary

Let us consider the mean-field social control problem for a stochastic system with input delay:

$$dx_i(t)=\Big[Ax_i(t)+Dx^{(N)}(t)+Bu_i(t)+B_hu_i(t-h)\Big]dt$$
$$+A_px_i(t)dw_i(t),\ x_i(\tau)=\phi(\tau),\tau\in[-h,\ 0],\tag{1a}$$
$$y_i(t)=Cx_i(t),\tag{1b}$$

where $i=1,\dots,N$, $x_i(t)\in\mathbb{R}^n$ denotes the i-th player's state, $u_i(t)\in\mathbb{R}^m$ denotes the i-th player's input, and $y_i(t)\in\mathbb{R}^p$ denotes the i-th output; $w_i(t)\in\mathbb{R}^1$ denotes a one-dimensional standard Wiener process defined in the filtered probability space [11]. The coefficient matrices, A, D, B, B_h, A_p, and C, of the stochastic delay system (1) must have the appropriate dimensions. It is assumed that delay h is fixed positive finite number and $\phi(t)$ is a continuous $\mathscr{F}_0$-measurable random variable such that $\mathbb{E}[\sup_{\tau\in[-h,0]}\|\phi(\tau)\|^2]<\infty$. It should be noted that the mean-field term is defined as $x^{(N)}(t)\in\mathbb{R}^n$ and $y^{(N)}(t)\in\mathbb{R}^p$, given by

$$x^{(N)}(t)=\frac{1}{N}\sum_{k=1}^{N}x_k(t)\in\mathbb{R}^n,\ y^{(N)}(t)=Cx^{(N)}(t)\in\mathbb{R}^p.\tag{2}$$

It should also be noted that the considered delay systems in the control part are appeared in the networked control systems such as time-delay, packet dropout, and signal attenuation [12] and the optimal management of combined water based on model predictive control [13]. Furthermore, it has been handled for a long time [14] and expresses generalization by using the mean-field term.

The cost functional is defined as follows:

$$J_{\rm soc}(\boldsymbol{u},\boldsymbol{x}(0))=\sum_{k=1}^{N}J_k(u_k,x_k(0)),\tag{3}$$

where

$$J_i(u_i,x_i(0))=\mathbb{E}\left[\int_0^\infty\left\{\|y_i(t)-\Phi y^{(N)}(t)\|_Q^2+\|u_i(t)\|_R^2\right\}dt\right],$$
$$\boldsymbol{x}(t)=\begin{bmatrix}x_1(t)\\ \vdots\\ x_N(t)\end{bmatrix},\ \boldsymbol{u}(t)=\begin{bmatrix}u_1(t)\\ \vdots\\ u_N(t)\end{bmatrix},Q=Q^T>0,\ R=R^T>0.$$

Because there are many players, minimizing the social cost as the sum of individual costs containing mean-field coupling is introduced as a cooperative game rather than a non-cooperative game. This problem is known as the mean-field linear quadratic social control problem [15], [16]. To better handle the mean-field social control problem under consideration, the centralized form of (4) is introduced, instead of the stochastic delay system (1):

$$d\boldsymbol{x}(t)=\big[A_c\boldsymbol{x}(t)+B_d\boldsymbol{u}(t)+B_{hd}\boldsymbol{u}(t-h)\big]dt$$
$$+\sum_{k=1}^{N}A_{pck}\boldsymbol{x}(t)dw_k(t),\tag{4a}$$
$$J_{\rm soc}(\boldsymbol{u},\boldsymbol{x}(0))=\mathbb{E}\left[\int_0^\infty\left\{\boldsymbol{x}^T(t)Q_{c\varepsilon}\boldsymbol{x}(t)+\boldsymbol{u}^T(t)R_d\boldsymbol{u}(t)\right\}dt\right],\tag{4b}$$
$$\boldsymbol{u}(t)=\mathbb{F}(F,C)\boldsymbol{x}(t),\ u_i(t)=\mathbb{F}_i(F,C)\boldsymbol{x}(t),\ i=1,\dots,N,\tag{4c}$$

where

$$A_c=I_N\otimes A+\varepsilon\boldsymbol{J}_N\otimes D,$$
$$A_{pci}=\textbf{block diag}\begin{pmatrix}0 & \cdots & A_p & \cdots & 0\end{pmatrix}\in\mathbb{R}^{nN\times nN},$$
$$F_d=\textbf{block diag}\begin{pmatrix}F & \cdots & F\end{pmatrix}\in\mathbb{R}^{mN\times pN},\ F\in\mathbb{R}^{m\times p},$$
$$\varepsilon=\frac{1}{N},\ Q_{c\varepsilon}=\begin{bmatrix}(1-\varepsilon)Q & \cdots & -\varepsilon Q\\ \vdots & \ddots & \vdots\\ -\varepsilon Q & \cdots & (1-\varepsilon)Q\end{bmatrix},\ R_d=I_N\otimes R.$$

It should be noted that each control $u_i(t)=FC\boldsymbol{x}(t)$ can be permitted as the mean-field control structure as the special form of $u_i(t)=Fy^{(N)}(t)=FC\boldsymbol{x}(t)=\varepsilon FC\begin{bmatrix}I_n & \cdots & I_n\end{bmatrix}\boldsymbol{x}(t)$. It is assumed that the number of players, N, is a huge value, and we aim at developing a scheme that can design a set of strategies even if N is sufficiently large.

Based on the mean-square stabilizability of stochastic delay system (4), the following result is considered.

Theorem 1: Consider the following closed-loop stochastic delay system (5) with the stabilizing control input $\boldsymbol{u}(t)$:

$$d\boldsymbol{x}(t)=\Big[\big(A_c+B_d\mathbb{F}(F,C)\big)\boldsymbol{x}(t)+B_{hd}\mathbb{F}(F,C)\boldsymbol{x}(t-h)\Big]dt$$
$$+\sum_{k=1}^{N}A_{pck}\boldsymbol{x}(t)dw_k(t),\tag{5a}$$
$$x_i(\tau)=\phi(\tau),\ \tau\in[-h,\ 0],\ i=1,\dots,N.\tag{5b}$$

Furthermore, the following cost functional is introduced:

$$J_{\rm soc}(\boldsymbol{u},\boldsymbol{x}(0))=J_{\rm soc}(\mathbb{F}(F,C)\boldsymbol{x},\boldsymbol{x}(0))$$
$$=\mathbb{E}\left[\int_0^\infty\boldsymbol{x}^T(t)\Big[Q_{c\varepsilon}+\mathbb{F}^T(F,C)R\mathbb{F}(F,C)\Big]\boldsymbol{x}(t)dt\right].\tag{6}$$

The stochastic delay system in (5) is stabilizable if there exist matrices $P_c>0$ and F satisfying the following matrix inequality:

$$\Lambda:=\begin{bmatrix}\Lambda_{11} & P_cB_{hd}\mathbb{F}(F,C)\\ \mathbb{F}^T(F,C)B_{hd}^TP_c & -H_c\end{bmatrix}\le 0,\tag{7}$$

where $\Lambda_{11}:=P_c\big(A_c+B_d\mathbb{F}(F,C)\big)+\big(A_c+B_d\mathbb{F}(F,C)\big)^TP_c+\sum_{k=1}^{N}A_{pck}^TP_cA_{pck}+(h+1)H_c+Q_{c\varepsilon}+\mathbb{F}^T(F,C)R\mathbb{F}(F,C)$.

In this case, for the upper bound of cost functional $J_{\rm soc}(\boldsymbol{u},\boldsymbol{x}(0))$ given in (6), the following inequality holds:

$$J_{\rm soc}(\boldsymbol{u},\boldsymbol{x}(0))\le\sum_{k=1}^{3}V_k(0,\boldsymbol{x}(0)),\tag{8}$$

where

$$V_1(t,\boldsymbol{x}(t))=\boldsymbol{x}^T(t)P_c\boldsymbol{x}(t),\ V_2(t,\boldsymbol{x}(t))=\int_{t-h}^{t}\boldsymbol{x}^T(\alpha)H_c\boldsymbol{x}(\alpha)d\alpha,$$

$$V_3(t,\boldsymbol{x}(t)) = \int_{-h}^{0}\int_{t+\beta}^{t} \boldsymbol{x}^T(\alpha) H_c \boldsymbol{x}(\alpha) d\alpha d\beta.$$

Proof: Define the following Lyapunov function candidate:

$$V(t,\boldsymbol{x}(t)) = \sum_{k=1}^{3} V_k(t,\boldsymbol{x}(t)). \tag{9}$$

Then, the stochastic differentiation of $V(t,\boldsymbol{x}(t))$ along with the trajectory of stochastic delay system (5) can be calculated based on the Ito's formula. Here, we define a new state variable $\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{x}^T(t) & \boldsymbol{x}^T(t-h) \end{bmatrix}^T \in \mathbb{R}^{2n}$. In this case, the following inequality can be derived:

$$\mathscr{L}V(t,\boldsymbol{x}(t)) \le \boldsymbol{z}^T(t) \begin{bmatrix} \Phi_{11} & P_c B_{hd}\mathbb{F}(F,C) \\ \mathbb{F}^T(F,C) B_{hd}^T P_c & -H_c \end{bmatrix} \boldsymbol{z}(t), \tag{10}$$

where $\Phi_{11} := P_c\big(A_c + B_d\mathbb{F}(F,C)\big) + \big(A_c + B_d\mathbb{F}(F,C)\big)^T P_c + \sum_{k=1}^{N} A_{pck}^T P_c A_{pck} + (h+1)H_c$.

Therefore, we have the following inequality:

$$\mathscr{L}V(t,\boldsymbol{x}(t)) + \boldsymbol{x}^T(t)\big(Q_{c\varepsilon} + \mathbb{F}^T(F,C)R\mathbb{F}(F,C)\big)\boldsymbol{x}(t) \le -\min[-\lambda_{\min}\Lambda]\boldsymbol{x}^T(t)\boldsymbol{x}(t) \le 0. \tag{11}$$

If inequality $\Lambda \le 0$ is satisfied, $\mathscr{L}V(t,\boldsymbol{x}(t)) < 0$ in (11) because $Q_{c\varepsilon} > 0$ is satisfied, and $V(t,\boldsymbol{x}(t))$ will be a Lyapunov function. Therefore, it is shown that stochastic delay system (5) is mean-square stable. Furthermore, if both sides of inequality (11) are integrated from 0 to $T > 0$, the following is obtained:

$$\mathbb{E}[V(T,\boldsymbol{x}(T))] - \mathbb{E}[V(0,\boldsymbol{x}(0))] \le -\mathbb{E}\left[\int_0^T \boldsymbol{x}^T(t)\big(Q_{c\varepsilon} + \mathbb{F}^T(F,C)R\mathbb{F}(F,C)\big)\boldsymbol{x}(t)dt\right]. \tag{12}$$

Finally, since the stochastic delay system in (5) is mean-square stable, i.e., $\mathbb{E}[V(t,\boldsymbol{x}(T))] \to 0$ when $T \to \infty$, we have $\mathbb{E}[\boldsymbol{x}^T(T)\boldsymbol{x}(T)] \to 0$. From the preceding discussion, the desired results are achieved. ■

Next, the minimization problem of the bound in (8) of cost functional (6) is solved such that the following inequality that is equivalent to (7) is satisfied:

$$\Omega_0(P_c, H_c, F) \le 0, \tag{13}$$

where $\Omega_0(P_c,H_c,F) := \Lambda_{11} + P_c B_{hd}\mathbb{F}(F,C)H_c^{-1}\mathbb{F}^T(F,C)B_{hd}^T P_c$.

Furthermore, some remarks on this problem are discussed as the main contributions of this study.

A. Optimization for Cost Bound

To avoid the dependence on the initial condition, the condition $\mathbb{E}[x(0)x^T(0)] = I_n$ is assumed without loss of generality. Furthermore, the following constants are specified:

$$\int_{-h}^{0} \phi^T(\alpha)\phi(\alpha)d\alpha = M,\ \int_{-h}^{0}\int_{\beta}^{0} \phi^T(\alpha)\phi(\alpha)d\alpha d\beta = L,$$

$$M_c = I_N \otimes M,\ L_c = I_N \otimes L.$$

At this time, the optimization problem with constraints to be solved is as follows:

$$\min_{P_c, H_c}\Big[\mathbf{Tr}[P_c] + \mathbf{Tr}[M_c H_c] + \mathbf{Tr}[L_c H_c]\Big],\ \text{s.t. (13)}. \tag{14}$$

Conjecture 1: Consider the stochastic delay system in (1). If there exist $P_c = P_c^T \ge 0$, $S_c = S_c^T \ge 0$, $H_c = H_c^T$, and F such that the equation set of the SCCMEs in (15) are satisfied, then the closed-loop stochastic delay system is mean-square stable, and the optimization problem has a solution set of the following SCCMEs:

$$0 = \Omega_0(P_c, H_c, F), \tag{15a}$$
$$0 = \Omega_1(P_c, S_c, H_c, F), \tag{15b}$$
$$0 = \Omega_2(P_c, S_c, H_c, F), \tag{15c}$$
$$0 = \Omega_3(P_c, S_c, F), \tag{15d}$$

where

$$\Omega_1(P_c,S_c,H_c,F) = S_c\tilde{A}_c^T + \tilde{A}_c S_c + \sum_{k=1}^{N} A_{pck} S_c A_{pck}^T + I_{nN},$$

$$\Omega_2(P_c,S_c,H_c,F) = H_c[M_c + L_c + (h+1)S_c]H_c - \mathbb{F}^T(F,C)B_{hd}^T P_c S_c P_c B_{hd}\mathbb{F}(F,C),$$

$$\Omega_3(P_c,S_c,F) = 2\frac{\partial}{\partial F}\mathbf{Tr}\Big[S_c P_c\big(A_c + B_d\mathbb{F}(F,C)\big)\Big] + \frac{\partial}{\partial F}\mathbf{Tr}\Big[S_c P_c B_{hd}\mathbb{F}(F,C)H_c^{-1}\mathbb{F}^T(F,C)B_{hd}^T P_c\Big],$$

$$\tilde{A}_c = A_c + B_d\mathbb{F}(F,C) + B_{hd}\mathbb{F}(F,C)H_c^{-1}\mathbb{F}^T(F,C)B_{hd}^T P_c.$$

Proof: Let us define the following Lagrange function:

$$\mathscr{L}(P_c,S_c,H_c,F) = \mathbf{Tr}[P_c] + \mathbf{Tr}[M_c H_c] + \mathbf{Tr}[L_c H_c] + \mathbf{Tr}[S_c\Omega_0(P_c,H_c,F)]. \tag{16}$$

Then, performing the partial derivative with respect to the variables, the following SCCMEs can be obtained by means of the KKT condition:

$$\frac{\partial\mathscr{L}}{\partial P_c} = \Omega_1(P_c,S_c,H_c,F) = 0, \tag{17a}$$
$$\frac{\partial\mathscr{L}}{\partial H_c} = H_c^{-1}\Omega_2(P_c,S_c,H_c,F)H_c^{-1} = 0, \tag{17b}$$
$$\frac{1}{2}\cdot\frac{\partial\mathscr{L}}{\partial F} = \Omega_3(P_c,S_c,F) = 0. \tag{17c}$$

Therefore, the results of this theorem can be proved to be the necessary conditions. ■

It should be noted that if the control laws have the following forms (18), the necessary conditions of (17c) can be represented appropriately.

(i) $\text{mean}-\text{field form}: \boldsymbol{u}(t) = \mathbb{F}(F,C)\boldsymbol{x}(t) = \varepsilon F_d C_c \boldsymbol{x}(t) \Rightarrow$

$$\Omega_3(P_c,S_c,F) = \varepsilon\bar{B}^T P_c S_c \bar{C}^T + \varepsilon^2 RF\bar{C}S_c\bar{C}^T + \sum_{k=1}^{N} e_{1k}B_{hd}^T P_c S_c P_c B_{hd} F_d C_d H_c^{-1} C_d^T e_{2k}^T, \tag{18a}$$

(ii) $\boldsymbol{u}(t) = \mathbb{F}(F,C)\boldsymbol{x}(t) = F_d C_d \boldsymbol{x}(t) \Rightarrow$

$$\Omega_3(P_c,S_c,F) = RF\sum_{k=1}^{N} e_{1k}S_c e_{2k}^T + B^T\sum_{k=1}^{N} e_{1k}P_c S_c e_{2k}^T + \sum_{k=1}^{N} e_{1k}B_{hd}^T P_c S_c P_c B_{hd}\left(\sum_{\ell=1}^{N} e_{1\ell}^T F e_{2\ell}\right) C_d H_c^{-1} C_d^T e_{2k}^T, \tag{18b}$$

where $e_{1i} = \begin{bmatrix} 0 & \cdots & 0 & I_m & 0 & \cdots & 0 \end{bmatrix}$, $e_{2i} = \begin{bmatrix} 0 & \cdots & 0 & I_p & 0 & \cdots & 0 \end{bmatrix}$, $C_d = I_N \otimes C$, $C_c = \boldsymbol{J}_N \otimes C$, $\bar{C} = \begin{bmatrix} C & \cdots & C \end{bmatrix}$, $\bar{B}^T = \begin{bmatrix} B^T & \cdots & B^T \end{bmatrix}$.

In this paper, the control laws with the structures (18) are handled later. By tracing the above-mentioned proof, the following important remarks can be stated.

Remark 1: If there is no ordinarry control input $\boldsymbol{u}(t)$, the minimization of the optimization problem for the cost bound of (8) cannot be attained. On the other hand, to guarantee the necessary conditions based on SCCMEs (15), the following structure assumptions should be satisfied:

$$C,\ F,\ B,\ B_h \in \mathbb{R}^{n\times n},\ \mathbf{det}(B_h)\mathbf{det}(F)\mathbf{det}(C) \neq 0. \tag{19}$$

In other words, it is impossible to stabilize using the SOF strategy; instead, only stabilization by state feedback can be accomplished in this problem setting. Finally, the mean-field SOF strategy in (18a) is not realized in practice.

Proof: Because Cases (i) and (ii) are identical under $B \equiv 0$ condition in Equation (1a), the controls of (i) are only discussed as a special case. For the first part, if $B_d\boldsymbol{u}(t) = B_dF_dC_d\boldsymbol{x}(t) \equiv 0$, the optimization problem changes as follows:

$$\min_{P_c,H_c}\Big[\mathbf{Tr}[P_c] + \mathbf{Tr}[M_cH_c] + \mathbf{Tr}[L_cH_c]\Big],$$
$$\text{s.t. } P_cA_c + A_c^TP_c + \sum_{k=1}^{N} A_{pck}^TP_cA_{pck} + (h+1)H_c + Q_{c\varepsilon}$$
$$+ C_d^TF_d^TRF_dC_d + P_cB_{hd}F_dC_dH_c^{-1}C_d^TF_d^TB_{hd}^TP_c \leq 0. \tag{20}$$

In this case, using the same procedure, the following relation holds:

$$\frac{1}{2}\cdot\frac{\partial\mathscr{L}}{\partial F} = RF\sum_{k=1}^{N} e_{1k}S_ce_{2k}^T + \sum_{k=1}^{N} e_{1k}B_{hd}^TP_cS_cP_cB_{hd} \times\left(\sum_{k=1}^{N} e_{1k}^TFe_{2k}\right)C_dH_c^{-1}C_d^Te_{2k}^T = 0. \tag{21}$$

From (21), it is obvious that $F \equiv 0$ is the trivial solution, the minimization cannot be attained. That is, the optimal feedback gain does not exist.

As the second part, equation (15c) plays an important role. The determinant of equation (15c) is given below:

$$\mathbf{det}([M_c + L_c + (h+1)S_c])[\mathbf{det}(H_c)]^2 = \mathbf{det}(P_cS_cP_c)[\mathbf{det}(B_{hd}F_dC_d)]^2. \tag{22}$$

From (22), if $m < n$, it is clear that $\mathbf{det}(B_{hd}F_dC_d) = 0$ or equivalent to $\mathbf{det}(B_hFC) = 0$. This means that H_c is a singular matrix. This is contradiction for the existence of H_c^{-1}. Finally, $m = n$ must be satisfied and the result of $\mathbf{det}(B_hFC) = \mathbf{det}(B_h)\mathbf{det}(F)\mathbf{det}(C) \neq 0$ holds.

In the last part of this remark, mean-field control is impossible because the non-singularity condition for H_c in (15c) is not guaranteed. ■

III. Decentralized Strategy Set

In this section, to avoid the large dimensional computation, the order-reduced method is considered under condition (19). Without loss of generality, $C = I_n$ in (18b) is assumed because of the similarity transformation. First, the following partitioned matrices are assumed as simple simulation results show the following structures:

$$P_c = P_c(\varepsilon, P_d, P_o),\ S_c = S_c(\varepsilon, S_d, S_o),$$
$$H_c = H_c(\varepsilon, H_d, H_o),\ X_c = X_c(\varepsilon, X_d, X_o),\ X_c = H_c^{-1}. \tag{23}$$

To avoid the computational difficulty of inverse matrix, auxiliary matrix $H_e^{-1} = X_e$ is introduced. Substituting the matrix set of (23) into the SCCMEs in (15) yields the following reduced-order SCCMEs:

$$0 = \Xi_1 = \Xi_1(\varepsilon, P_d, P_o, H_d, X_d, X_o, F), \tag{24a}$$
$$0 = \Xi_2 = \Xi_2(\varepsilon, P_d, P_o, H_o, X_d, X_o, F), \tag{24b}$$
$$0 = \Xi_3 = \Xi_3(\varepsilon, P_d, P_o, S_d, S_o, X_d, X_o, F), \tag{24c}$$
$$0 = \Xi_4 = \Xi_4(\varepsilon, P_d, P_o, S_d, S_o, X_d, X_o, F), \tag{24d}$$
$$0 = \Xi_5 = \Xi_5(\varepsilon, P_d, P_o, S_d, S_o, H_d, H_o), \tag{24e}$$
$$0 = \Xi_6 = \Xi_6(\varepsilon, P_d, P_o, S_d, S_o, H_d, H_o), \tag{24f}$$
$$0 = \Xi_7 = \Xi_7(\varepsilon, P_d, P_o, S_d, S_o, F), \tag{24g}$$
$$0 = \Xi_8 = \Xi_8(\varepsilon, H_d, H_o, X_d, X_o), \tag{24h}$$
$$0 = \Xi_9 = \Xi_9(\varepsilon, H_d, H_o, X_d, X_o), \tag{24i}$$

where
$$\Xi_1(\varepsilon, P_d, P_o, H_d, X_d, X_o, F) = P_d(A + \varepsilon D + BF) + (A + \varepsilon D + BF)^TP_d + \varepsilon(1-\varepsilon)(P_oD + D^TP_o) + A_p^TP_dA_p + (1-\varepsilon)Q + F^TRF + (1+h)H_d + P_dB_hFX_dF^TB_h^TP_d + \varepsilon(1-\varepsilon)[P_oB_hFX_oF^TB_h^TP_d + P_dB_hFX_oF^TB_h^TP_o + P_oB_hFX_dF^TB_h^TP_o + (1-2\varepsilon)P_oB_hFX_oF^TB_h^TP_o],$$
$$\Xi_2(\varepsilon, P_d, P_o, H_o, X_d, X_o, F) = P_dD + D^TP_d + P_o[A + BF + (1-\varepsilon)D] + [A + BF + (1-\varepsilon)D]^TP_o - Q + (1+h)H_o + P_dB_hFX_dF^TB_h^TP_o + P_oB_hFX_dF^TB_h^TP_d + P_dB_hFX_oF^TB_h^TP_d + (3\varepsilon^2 - 3\varepsilon + 1)P_oB_hFX_oF^TB_h^TP_o + (1-2\varepsilon)(P_oB_hFX_oF^TB_h^TP_d + P_dB_hFX_oF^TB_h^TP_o + P_oB_hFX_dF^TB_h^TP_o),$$
$$\Xi_3(\varepsilon, P_d, P_o, S_d, S_o, X_d, X_o, F) = S_d\Psi_d^T + \Psi_dS_d + \varepsilon(1-\varepsilon)(S_o\Psi_o^T + \Psi_oS_o) + A_pS_dA_p^T + I_n,$$
$$\Xi_4(\varepsilon, P_d, P_o, S_d, S_o, X_d, X_o, F) = S_d\Psi_o^T + \Psi_oS_d + S_o\Upsilon_o^T + \Upsilon_oS_o,$$
$$\Xi_5(\varepsilon, P_d, P_o, S_d, S_o, H_d, H_o) = H_dZ_dH_d + \varepsilon(1-\varepsilon)(H_oZ_oH_d + H_dZ_oH_o + H_oZ_dH_o) + \varepsilon(1-\varepsilon)(1-2\varepsilon)H_oZ_oH_o - F^TB_h^TP_dS_dP_dB_hF - \varepsilon(1-\varepsilon)F^TB_h^T(P_oS_oP_d + P_dS_oP_o + P_oS_dP_o)B_hF - \varepsilon(1-\varepsilon)(1-2\varepsilon)F^TB_h^TP_oS_oP_oB_hF,$$
$$\Xi_6(\varepsilon, P_d, P_o, S_d, S_o, H_d, H_o) = H_dZ_dH_o + H_oZ_dH_d + H_dZ_oH_d + (3\varepsilon^2 - 3\varepsilon + 1)H_oZ_oH_o + (1-2\varepsilon)(H_oZ_oH_d + H_dZ_oH_o + H_oZ_dH_o) - (3\varepsilon^2 - 3\varepsilon + 1)F^TB_h^TP_oS_oP_oB_hF - F^TB_h^T(P_dS_dP_o + P_oS_dP_d + P_dS_oP_d)B_hF - (1-2\varepsilon)F^TB_h^T(P_oS_oP_d + P_dS_oP_o + P_oS_dP_o)B_hF,$$
$$\Xi_7(\varepsilon, P_d, P_o, S_d, S_o, X_d, X_o, F) = RFS_d + B^TP_dS_d + \varepsilon(1-\varepsilon)B^TP_oS_o + B_h^TP_dS_dP_dB_hFX_d + \varepsilon(1-\varepsilon)B_h^T[(P_dS_oP_o + P_oS_oP_d + P_oS_dP_o)B_hFX_d + (1-2\varepsilon)P_oS_oP_oB_hFX_d + (P_dS_oP_d + P_dS_dP_o + P_oS_dP_d)B_hFX_o + (1-2\varepsilon)(P_dS_oP_o + P_oS_oP_d + P_oS_dP_o)B_hFX_o + (3\varepsilon^2 - 3\varepsilon + 1)P_oS_oP_oB_hFX_o],$$
$$\Xi_8(\varepsilon, H_d, H_o, X_d, X_o) = H_dX_d + \varepsilon(1-\varepsilon)H_oX_o - I_n,$$
$$\Xi_9(\varepsilon, H_d, H_o, X_d, X_o) = H_oX_d + [H_d + (1-2\varepsilon)H_o]X_o,$$
$$\Psi_d = A + \varepsilon D + BF + B_hFX_dF^TB_h^TP_d + \varepsilon(1-\varepsilon)B_hFX_oF^TB_h^TP_o,$$
$$\Psi_o = D + B_hF(X_dF^TB_h^TP_o + X_oF^TB_h^TP_d) + (1-2\varepsilon)B_hFX_oF^TB_h^TP_o,\ \Upsilon_o = A + (1-\varepsilon)D + BF + B_hFX_dF^TB_h^TP_d + (1-2\varepsilon)B_hF(X_oF^TB_h^TP_d + X_dF^TB_h^TP_o) + (3\varepsilon^2 - 3\varepsilon + 1)B_hFX_oF^TB_h^TP_o,$$

$$Z_c = M_c + L_c + (h+1)S_c = Z_c(Z_d, Z_o).$$

If the value of N is sufficiently large, the following reduced-order parameter independent SCCMEs are convenient by letting $\varepsilon = 1/N \to 0$:

$$\bar{P}_d(A+B\bar{F})+(A+B\bar{F})^T\bar{P}_d+A_p^T\bar{P}_dA_p+Q \\ +\bar{F}^TR\bar{F}+(1+h)\bar{H}_d+\bar{P}_dB_h\bar{F}\bar{X}_d\bar{F}^TB_h^T\bar{P}_d=0, \tag{25a}$$

$$\bar{P}_dD+D^T\bar{P}_d+\bar{P}_o(A+B\bar{F}+D)+(A+B\bar{F}+D)^T\bar{P}_o-Q \\ +(1+h)\bar{H}_o+\bar{P}_dB_h\bar{F}\bar{X}_d\bar{F}^TB_h^T\bar{P}_o+\bar{P}_oB_h\bar{F}\bar{X}_d\bar{F}^TB_h^T\bar{P}_d \\ +\bar{P}_dB_h\bar{F}\bar{X}_o\bar{F}^TB_h^T\bar{P}_d+\bar{P}_oB_h\bar{F}\bar{X}_o\bar{F}^TB_h^T\bar{P}_o+\bar{P}_oB_h\bar{F}\bar{X}_o\bar{F}^TB_h^T\bar{P}_d \\ +\bar{P}_dB_h\bar{F}\bar{X}_o\bar{F}^TB_h^T\bar{P}_o+\bar{P}_oB_h\bar{F}\bar{X}_d\bar{F}^TB_h^T\bar{P}_o=0,$$

$$\bar{S}_d\bar{\Psi}_d^T+\bar{\Psi}_d\bar{S}_d+A_p\bar{S}_dA_p^T+I_n=0, \tag{25b}$$

$$\bar{S}_d\bar{\Psi}_o^T+\bar{\Psi}_o\bar{S}_d+\bar{S}_o\bar{\Upsilon}_o^T+\bar{\Upsilon}_o\bar{S}_o=0, \tag{25c}$$

$$\bar{H}_dZ_d\bar{H}_d-\bar{F}^TB_h^T\bar{P}_d\bar{S}_d\bar{P}_dB_h\bar{F}=0, \tag{25d}$$

$$\bar{H}_dZ_d\bar{H}_o+\bar{H}_oZ_d\bar{H}_d+\bar{H}_dZ_o\bar{H}_d+\bar{H}_oZ_o\bar{H}_o+\bar{H}_oZ_o\bar{H}_d+\bar{H}_dZ_o\bar{H}_o \\ +\bar{H}_oZ_d\bar{H}_o-\bar{F}^TB_h^T\bar{P}_o\bar{S}_o\bar{P}_oB_h\bar{F} \\ -\bar{F}^TB_h^T(\bar{P}_d\bar{S}_d\bar{P}_o+\bar{P}_o\bar{S}_d\bar{P}_d+\bar{P}_d\bar{S}_o\bar{P}_d)B_h\bar{F} \\ -\bar{F}^TB_h^T(\bar{P}_o\bar{S}_o\bar{P}_d+\bar{P}_d\bar{S}_o\bar{P}_o+\bar{P}_o\bar{S}_d\bar{P}_o)B_h\bar{F}=0, \tag{25e}$$

$$R\bar{F}\bar{S}_d+B^T\bar{P}_d\bar{S}_d+B_h^T\bar{P}_d\bar{S}_d\bar{P}_dB_h\bar{F}\bar{X}_d=0, \tag{25f}$$

where $\bar{\Psi}_d = A+B\bar{F}+B_h\bar{F}\bar{X}_d\bar{F}^TB_h^T\bar{P}_d$, $\bar{\Psi}_o = D +B_h\bar{F}(\bar{X}_d\bar{F}^TB_h^T\bar{P}_o+\bar{X}_o\bar{F}^TB_h^T\bar{P}_d+\bar{X}_o\bar{F}^TB_h^T\bar{P}_o)$, $\bar{\Upsilon}_o = A+D+B\bar{F}+B_h\bar{F}(\bar{X}_d\bar{F}^TB_h^T\bar{P}_d+\bar{X}_o\bar{F}^TB_h^T\bar{P}_d +\bar{X}_d\bar{F}^TB_h^T\bar{P}_o+\bar{X}_o\bar{F}^TB_h^T\bar{P}_o)$, $\bar{H}_d=\bar{X}_d^{-1}$, $\bar{H}_o=(\bar{H}_d+\bar{H}_o)\bar{X}_o\bar{H}_d^{-1}$, $\bar{Z}_d = M+L+(h+1)\bar{S}_d$, $\bar{Z}_o=(h+1)\bar{S}_o$.

Let us consider the following parameter independent decentralized strategy set:

$$\bar{\boldsymbol{u}}(t)=\begin{bmatrix}\bar{u}_1(t)\\ \vdots \\ \bar{u}_N(t)\end{bmatrix}=\begin{bmatrix}\bar{F}x_1(t)\\ \vdots \\ \bar{F}x_N(t)\end{bmatrix}=\bar{F}_d\boldsymbol{x}(t). \tag{26}$$

The following corollary shows the asymptotic structure of the feedback gain F and the feature for using approximate strategy set (26).

Theorem 2: Suppose that

$$\nabla(\bar{\boldsymbol{x}}) = \left.\frac{\partial \mathbf{vec}[\mathbb{F}(\boldsymbol{x})]}{\partial \mathbf{vec}[\boldsymbol{x}]^T}\right|_{(\varepsilon=0,\boldsymbol{x}=\bar{\boldsymbol{x}})} \neq 0, \tag{27}$$

where

$$\mathbb{F}(\boldsymbol{x})=\begin{bmatrix}\mathbf{vec}\Xi_1\\ \mathbf{vec}\Xi_2\\ \mathbf{vec}\Xi_3\\ \mathbf{vec}\Xi_4\\ \mathbf{vec}\Xi_5\\ \mathbf{vec}\Xi_6\\ \mathbf{vec}\Xi_7\end{bmatrix},\ \boldsymbol{x}=\begin{bmatrix}\mathbf{vec}[P_d]\\ \mathbf{vec}[P_o]\\ \mathbf{vec}[S_d]\\ \mathbf{vec}[S_o]\\ \mathbf{vec}[H_d]\\ \mathbf{vec}[H_o]\\ \mathbf{vec}[F]\end{bmatrix},\ \bar{\boldsymbol{x}}=\begin{bmatrix}\mathbf{vec}[\bar{P}_d]\\ \mathbf{vec}[\bar{P}_o]\\ \mathbf{vec}[\bar{S}_d]\\ \mathbf{vec}[\bar{S}_o]\\ \mathbf{vec}[\bar{H}_d]\\ \mathbf{vec}[\bar{H}_o]\\ \mathbf{vec}[\bar{F}]\end{bmatrix}.$$

In this case, there exists a solution F such that the following relation holds.

$$F=\bar{F}+O(\varepsilon). \tag{28}$$

Furthermore, consider approximate strategy set (26). Here, the following relation holds:

$$\frac{1}{N}|J_{\mathrm{soc}}(\boldsymbol{u},\boldsymbol{x}(0))-J_{\mathrm{soc}}(\bar{\boldsymbol{u}},\boldsymbol{x}(0))|=O(\varepsilon). \tag{29}$$

Proof: If the assumption (27) holds, the Jacobian matrix at $\varepsilon=0$ which relates to the following reduced-order SCCMEs (24) is nonsingular. By applying the implicit function theorem, the asymptotic structure (28) can be proven. On the other hand, using the results of Theorem 1, the following relation holds:

$$\frac{1}{N}|J_{\mathrm{soc}}(\boldsymbol{u},\boldsymbol{x}(0))-J_{\mathrm{soc}}(\bar{\boldsymbol{u}},\boldsymbol{x}(0))|=\varepsilon|\mathbf{Tr}[P_c-\boldsymbol{Z}_c]|, \tag{30}$$

where

$$\boldsymbol{Z}_c\big(A_c+B_d\bar{F}_d\big)+\big(A_c+B_d\bar{F}_d\big)^T\boldsymbol{Z}_c+\sum_{k=1}^{N}A_{pck}^T\boldsymbol{Z}_cA_{pck}+(h+1)H_c \\ +Q_{c\varepsilon}+\bar{F}_d^TR\bar{F}_d+\boldsymbol{Z}_cB_{hd}\bar{F}_dH_c^{-1}\bar{F}_d^TB_{hd}^T\boldsymbol{Z}_c=0. \tag{31}$$

Subtracting stochastic matrix equation (SME) (15a) from SME (31) yields the following SME:

$$(P_c-\boldsymbol{Z}_c)\big(A_c+B_d\bar{F}_d\big)+\big(A_c+B_d\bar{F}_d\big)^T(P_c-\boldsymbol{Z}_c) \\ +\sum_{k=1}^{N}A_{pck}^T(P_c-\boldsymbol{Z}_c)A_{pck}+O(\varepsilon)=0. \tag{32}$$

If equation (27) is satisfied, the following relation holds:

$$P_c-\boldsymbol{Z}_c=O(\varepsilon). \tag{33}$$

Furthermore, it is assumed that $\boldsymbol{Z}_c$ has the following form:

$$\boldsymbol{Z}_c=\boldsymbol{Z}_c(\varepsilon,\boldsymbol{Z}_d,\boldsymbol{Z}_o)\in\mathbb{R}^{nN\times nN}, \tag{34}$$

and the following relation holds because of $P_d-\boldsymbol{Z}_d=O(\varepsilon)$:

$$\varepsilon\mathbf{Tr}[P_c-\boldsymbol{Z}_c]=\varepsilon N\mathbf{Trace}[P_d-\boldsymbol{Z}_d]=O(\varepsilon). \tag{35}$$

This is the desired result. ■

The concept of proposed decentralized approximate strategy set (26) is based on the direct method. The derivation process for the approximation error bound in Equation (32) can be evaluated within the limit of an infinite number of players.

Remark 2: It should be noted that a hypothetical heterogeneous case, where the structure is not assumed to have the uniform characteristics of mean-field systems, can be resolved by the weak couple systems theory [17]. In this case, Assumption (23) must be changed to a general symmetric matrix.

Remark 3: It should be noted that the state feedback LQ optimal control problem of stochastic systems with time delay has been investigated in [18]. According to this results, the original problem was well solved by presenting necessary and sufficient. On the other hand, our results are novel because it is shown that the SOF design is not possible under the GCC approach theoretically.

IV. Numerical Example

In this section, the numerical example is presented to show the reliability and effectiveness of the proposed strategy set. To design the control strategy, the following matrices are chosen:

$$N=10,\ \varepsilon=\frac{1}{N}=0.1,\ A=\begin{bmatrix}-4.5 & 0\\ 0.8 & -2\end{bmatrix},\ D=1.5\times A,$$

$$A_p=0.1\times A,\ B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ B_h=\begin{bmatrix}0 & 1\\ 0.1 & 0.5\end{bmatrix},$$

$$Q=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix},\ R=\begin{bmatrix}0.5 & 0\\ 0 & 2\end{bmatrix},\ h=1.0,\ \phi=\begin{bmatrix}0.1\\ -0.5\end{bmatrix}.$$

To obtain the solution set, the numerical algorithm based on the gradient method was used: $\boldsymbol{x}^{(\kappa+1)} = \boldsymbol{x}^{(\kappa)} + \tilde{\eta}\mathbb{F}(\boldsymbol{x}^{(\kappa)})$, $k = 0, 1, \ldots$, $\boldsymbol{x}^{(0)} = \textbf{block diag}(I_n, I_n, I_n, I_n, I_n, I_n, I_n, I_n)$, $\tilde{\eta} = \eta\textbf{diag}(1, 1, 1, 1, -1, -1, -1)$, where η is small positive parameter. It is well-known that although the convergence is not always guaranteed and the convergence speed is very slow, the implementation is easy and work well. From the centralized SCCMEs in (15), the following required gain F can be computed directly if the number N of players is small:

$$F = \begin{bmatrix} -5.8735 \times 10^{-1} & -1.5559 \\ 6.5432 \times 10^{-2} & 1.7399 \times 10^{-1} \end{bmatrix}.$$

On the other hand, in the case of $N = 1,000$, it is impossible to compute the gain F by using the same approach because of the memory constraint. Therefore, the required gain F can be obtained using the proposed decomposition technique based on SCCMEs (24). In fact, the exact feedback gain $F = F^*$ and approximated gain $F = \bar{F}$ are given as follows:

$$F^* = \begin{bmatrix} -6.1305 \times 10^{-1} & -1.6383 \\ 6.8179 \times 10^{-2} & 1.8311 \times 10^{-1} \end{bmatrix},$$
$$\bar{F} = \begin{bmatrix} -6.1281 \times 10^{-1} & -1.6378 \\ 6.8151 \times 10^{-2} & 1.8306 \times 10^{-1} \end{bmatrix}.$$

From the above results, these gains are very close and the proposed approximate strategy can work well if the value of N is sufficiently large.

Finally, Fig. 1 shows the trajectory of this obtained feedback gain. It can be seen that, even if there are state and input delays, the state values attain the mean-square stability. It should be noted that although Fig. 1 does not appear to be mean-square stable, since the state is included in the diffusion term as state-dependent noise, mean-square stability is guaranteed even if there is one Brownian motion.

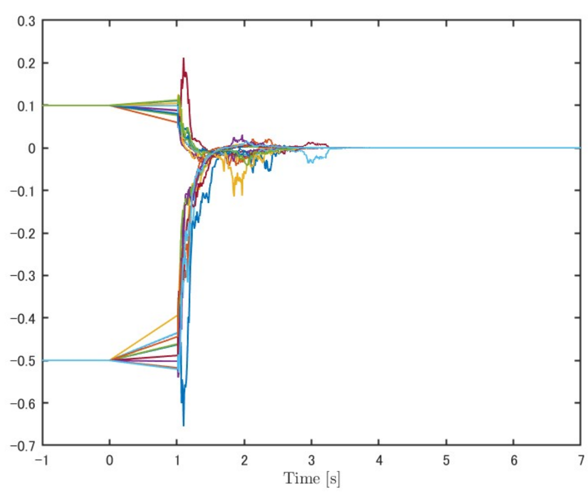

Fig.1 The trajectories of states.

V. Conclusion

Mean-field social control problems for a class of stochastic systems with both ordinary and delayed control inputs have been studied. After defining the stochastic stabilization problem, it has been shown that the problem of minimizing the upper bound of the cost function does not make sense when only delayed control inputs are present. Furthermore, it is importantly pointed out that, when considering delayed control inputs using the Lyapunov-Krasovsky function method, delayed control inputs based on individual SOF strategies do not have a solution set and the input matrix must have the same dimension as the state matrix. As a result, a state feedback strategy class is introduced. Note that each strategy set can be computed by solving the SCCME derived from the KKT condition. Finally, the reliability and usefulness of the proposed strategies are examined using a order-reduced scheme based on the direct method. The results show that each state feedback gain can be obtained under low-dimensional computation, even when there are many decision makers.

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