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Stability Analysis of Nonlinear Model Predictive Control with Progressive Tightening of Stage Costs and Constraints

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Abstract— **We consider a stage-varying nonlinear model predictive control (NMPC) formulation and provide a stability result for the corresponding closed-loop system under the assumption that cost and constraints are *progressively tightening*. We illustrate the generality of the stage-varying formulation pointing out various approaches proposed in the literature that can be cast as stage-varying and progressively tightening optimal control problems.**

I. Introduction

Nonlinear model predictive control (NMPC) is an optimization-based approach to nonlinear control and relies on the solution of parametric nonlinear programs (NLP) in order to evaluate an implicit feedback policy [1].

In this paper, we consider a rather general stage-varying optimal control problem formulation which allows for additional auxiliary dynamics as well as stage costs and constraints that vary for each stage. The auxiliary system might be leveraged to pass information – in terms of the auxiliary state – along the horizon, which might in turn affect costs or constraints on the system state. We provide a proof of asymptotic stability of the origin for the associated closed-loop system assuming the stage-varying costs and constraints satisfy a *progressive tightening* condition. A stability proof for the stage-varying formulation has first been given in [2], though many important special cases have been analyzed in the literature since the 1980s [3], [4] and are summarized in the survey by Mayne [5].

The contribution of this work is twofold: (1) We provide a generalization of the proof in [2] to additionally include formulations with an auxiliary dynamical system; (2) We illustrate the generality of the stability result by a survey of control approaches that have been proposed in the literature and that can be cast in terms of a stage-varying and progressively tightening formulation with auxiliary dynamics.

A. Outline

The paper is structured as follows: Section II introduces the stage-varying optimal control problem with auxiliary dynamics. In Section III, we derive the main stability result. In Section IV, we review existing approaches that are covered by our formulation and illustrate how they can be cast as a stage-varying and progressively tightening optimal control problem. Section V summarizes our results.

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B. Notation

We denote the set of positive extended real numbers by $\bar{\mathbb{R}}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{\infty\}$. A continuous function V is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded, $\alpha(s) \to \infty$ as $s \to \infty$. For two functions $V : \mathbb{R}^n \to \bar{\mathbb{R}}_{\geq 0}$, $V' : \mathbb{R}^n \to \bar{\mathbb{R}}_{\geq 0}$, we define the relation: $V \leq V' \Leftrightarrow V(x) \leq V'(x)$ for all $x \in \mathbb{R}^n$, where we allow for a comparison with ∞.

II. Stage-Varying OCP Formulation

We regard a discrete-time nonlinear system $x^+ = f^{\mathrm{x}}(x, u)$ where $x \in \mathbb{R}^{n_x}$ and $x^+ \in \mathbb{R}^{n_x}$ is the current and subsequent state of the system. Our aim is to choose inputs $u \in \mathbb{R}^{n_u}$ that steer the system to the origin in an optimal way while respecting state and input constraints. In order to allow for rather general optimal control problem formulations, that for instance include couplings of costs and constraints between stages, we introduce a virtual auxiliary system of the form

$$z^+ = f^{\mathrm{z}}(x, z, u, w). \tag{1}$$

Based on these two systems, we formulate the following stage-varying optimal control problem (OCP):

$$\begin{aligned}
\min_{X,Z,U,W} \quad & \sum_{i=0}^{N-1} l_i(x_i, z_i, u_i, w_i) + V_N(x_N, z_N) && \text{(2a)}\\
\text{s.t.} \quad & x_0 = \underline{x}, && \text{(2b)}\\
& x_{i+1} = f^{\mathrm{x}}(x_i, u_i), \quad i = 0, \ldots, N-1, && \text{(2c)}\\
& z_{i+1} = f^{\mathrm{z}}(x_i, z_i, u_i, w_i), \; i = 0, \ldots, N-1, && \text{(2d)}\\
& 0 \geq h_i(x_i, z_i, u_i, w_i), \; i = 0, \ldots, N-1, && \text{(2e)}\\
& 0 = g_i(x_i, z_i, u_i, w_i), \; i = 0, \ldots, N-1, && \text{(2f)}\\
& 0 \geq h_N(x_N, z_N), && \text{(2g)}\\
& 0 = g_N(x_N, z_N), && \text{(2h)}
\end{aligned}$$

with system states $X = (x_0, \ldots, x_N)$, $x_i \in \mathbb{R}^{n_x}$, auxiliary states $Z = (z_0, \ldots, z_N)$, $z_i \in \mathbb{R}^{n_z}$, control inputs $U = (u_0, \ldots, u_{N-1})$, $u_i \in \mathbb{R}^{n_u}$, auxiliary control inputs $W = (w_0, \ldots, w_{N-1})$, $w_i \in \mathbb{R}^{n_w}$ and a given initial state $\underline{x} \in \mathbb{R}^{n_x}$.

The functions $l_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}_{\geq 0}$ and $V_N : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ denote the stage and terminal costs. The constraints are defined by

$$h_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_{h_i}}, \tag{3}$$

$$g_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_{g_i}}, \tag{4}$$

and $h_N : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_{h_N}}$, $g_N : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_{g_N}}$.

We can formulate an unconstrained OCP that is equivalent to (2) by assigning infinite costs to infeasible points. To this end, we define $\bar{l}_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \bar{\mathbb{R}}_{\geq 0}$ with

$$\bar{l}_i(x,z,u,w) := \begin{cases} l_i(x,z,u,w) & \text{if } h_i(x,z,u,w) \leq 0, g_i(x,z,u,w) = 0, \\ \infty & \text{otherwise,} \end{cases}$$

for $i = 0, \ldots, N-1$, and $\bar{V}_N^{\mathrm{s}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \bar{\mathbb{R}}_{\geq 0}$ with

$$\bar{V}_N(x,z) := \begin{cases} V_N(x,z) & \text{if } h_N(x,z) \leq 0, g_N(x,z) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

To simplify the notation, we introduce

$$s_i := \begin{bmatrix} x_i \\ z_i \end{bmatrix},\ a_i := \begin{bmatrix} u_i \\ w_i \end{bmatrix},\ f^{\mathrm{s}}(s_i, a_i) := \begin{bmatrix} f^{\mathrm{x}}(x_i, u_i) \\ f^{\mathrm{z}}(x_i, z_i, u_i, w_i) \end{bmatrix},$$

as well as

$$\begin{aligned} l_i^{\mathrm{s}}(s_i, a_i) &:= l_i(x_i, z_i, u_i, w_i), \quad V_N^{\mathrm{s}}(s_N) := V_N(x_N, z_N), \\ \bar{l}_i^{\mathrm{s}}(s_i, a_i) &:= \bar{l}_i(x_i, z_i, u_i, w_i), \quad \bar{V}_N^{\mathrm{s}}(s_N) := \bar{V}_N(x_N, z_N). \end{aligned}$$

Assumption 1. *We assume that the origin is a feasible steady state with* $f^{\mathrm{s}}(0,0) = 0$, $\bar{l}_i^{\mathrm{s}}(0,0) = 0$, *and* $\bar{V}_N^{\mathrm{s}}(0) = 0$. *Furthermore, we assume that* f^{s}, l_i^{s}, $i = 0, \ldots, N-1$, *and* V_N^{s} *are continuous.*

Next, we introduce the *progressive tightening* criterion, which is a crucial requirement for the stability result. It is illustrated in Figure 1.

Assumption 2 (Progressive tightening.)**.** *We assume that the modified stage costs satisfy* $\bar{l}_i^{\mathrm{s}}(s,a) \leq \bar{l}_{i+1}^{\mathrm{s}}(s,a)$ *for all states* $s \in \mathbb{R}^{n_s}$ *and all inputs* $a \in \mathbb{R}^{n_a}$ *and* $i = 0, \ldots, N-2$.

Remark 1. *Assumption 2 is equivalent to the assumption on epigraph inclusion, in [2], Assumption 5.1.3., if there are no auxiliary dynamics. The epigraph inclusion is illustrated in Fig. 1, where clearly* epi $\bar{l}_{i+1} \subseteq$ epi $\bar{l}_i$.

III. Asymptotic Stability of Stage-Varying NMPC

In the following, we derive the main result of this paper: asymptotic stability of the origin for the closed-loop control system assuming that cost and constraints are progressively tightening along the horizon. To this end, we define the sets

$$\mathbb{Y}_i := \{(x,z,u,w) : h_i(x,z,u,w) \leq 0, g_i(x,z,u,w) = 0\},$$

$i = 0, \ldots, N-1$, as well as

$$\begin{aligned} \mathbb{U}_0(x) &:= \{u \in \mathbb{R}^{n_u} : (x,z,u,w) \in \mathbb{Y}_0 \text{ for some } z, w\}, \\ \mathbb{Q}_0(x) &:= \{(z,u,w) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} : (x,z,u,w) \in \mathbb{Y}_0\}, \end{aligned}$$

and $\hat{\mathbb{X}}_0 := \{x \in \mathbb{R}^{n_x} : \mathbb{U}_0(x) \neq \emptyset\}$. Note that all of the above sets contain the origin due to Assumption 1.
We now introduce the value functions $\bar{V}_i^{\mathrm{s}}(s)$ and $\bar{V}_0^{\mathrm{x}}(x)$. For stage N, the value function is equivalent to the modified terminal cost $\bar{V}_N^{\mathrm{s}}(s)$. For all other stages, the value functions are recursively defined via the Bellman equation.

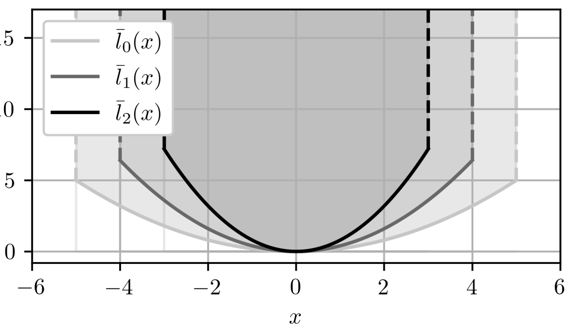

Fig. 1. Progressive tightening condition: As a simple example, we consider a stage cost, which depends only on the state x. The modified stage costs $\bar{l}_i$ summarize both costs and path constraints, which in this example are lower and upper bounds on x. The shaded areas denote the epigraphs epi $\bar{l}_i$.

Assumption 3 (Existence of solutions)**.** *For* $i = N-1, \ldots, 1$, *we define the value function* $\bar{V}_i^{\mathrm{s}} : \mathbb{R}^{n_s} \to \bar{\mathbb{R}}_{\geq 0}$ *associated with stages* i *recursively via*

$$\bar{V}_i^{\mathrm{s}}(s) := \min_a \ \bar{l}_i^{\mathrm{s}}(s,a) + \bar{V}_{i+1}^{\mathrm{s}}(f^{\mathrm{s}}(s,a)), \quad (5)$$

where we assume that for each s *the problem is either infeasible, i.e.* $\bar{V}_i^{\mathrm{s}}(s) = \infty$, *or there is* a^* *minimizing* (5). *The value function* $\bar{V}_0^{\mathrm{x}} : \mathbb{R}^{n_x} \to \bar{\mathbb{R}}_{\geq 0}$ *associated with stage 0 is given by*

$$\bar{V}_0^{\mathrm{x}}(x) := \min_{z,u,w} \bar{l}_0(x,z,u,w) + \bar{V}_1^{\mathrm{x,z}}(f^{\mathrm{x}}(x,u), f^{\mathrm{z}}(x,z,u,w)). \quad (6)$$

again assuming that a minimizer (z^*, u^*, w^*) *exists whenever the problem is feasible.*

Note that for stage 0, both the auxiliary state z and the auxiliary control w act as inputs. For stages $i = 1, \ldots, N-1$, only w is free and z is defined via the auxiliary dynamics. We furthermore introduce the shorthand $\bar{V}_i^{\mathrm{x,z}}(x,z) = \bar{V}_i^{\mathrm{s}}(s)$, $i = 1, \ldots, N-1$, as well as the sets $\mathbb{X}_0$ and $\mathbb{S}_i$, $i = 1, \ldots, N$, that contain all states for which the corresponding value function takes finite values,

$$\mathbb{X}_0 := \left\{x \in \mathbb{R}^{n_x} : \bar{V}_0^{\mathrm{x}}(x) < \infty\right\}, \quad (7)$$
$$\mathbb{S}_i := \left\{s \in \mathbb{R}^{n_s} : \bar{V}_i^{\mathrm{s}}(s) < \infty\right\}. \quad (8)$$

Definition 1 (NMPC policy)**.** *We define*

$$\begin{bmatrix} \pi_0^{\mathrm{z}}(x) \\ \pi_0^{\mathrm{u}}(x) \\ \pi_0^{\mathrm{w}}(x) \end{bmatrix} \in \arg\min_{z,u,w} \ l_0(x,z,u,w) + \bar{V}_1^{\mathrm{x,z}}(f^{\mathrm{x}}(x,u), f^{\mathrm{z}}(x,z,u,w)), \quad (9)$$

where we assume that – in case of multiple minimizers – an appropriate minimum norm selection criterion is used to uniquely define $\pi_0^{\mathrm{z}}, \pi_0^{\mathrm{u}}$, *and* π_0^{w}. *The NMPC policy is given by the map* $\pi_0^{\mathrm{u}}(x)$. *We furthermore define* $\pi_0^{\mathrm{s}} : \mathbb{X}_0 \to \mathbb{R}^{n_s}$ *and* $\pi_0^{\mathrm{a}} : \mathbb{X}_0 \to \mathbb{R}^{n_a}$,

$$\pi_0^{\mathrm{s}}(x) = \begin{bmatrix} x \\ \pi_0^{\mathrm{z}}(x) \end{bmatrix}, \quad \pi_0^{\mathrm{a}}(x) = \begin{bmatrix} \pi_0^{\mathrm{u}}(x) \\ \pi_0^{\mathrm{w}}(x) \end{bmatrix}. \quad (10)$$

Assumption 4. *We assume that the policy $\pi_0^{\mathrm{u}} : \mathbb{X}_0 \to \mathbb{R}^{n_u}$ is locally bounded.*

Remark 2. *Note that Assumption 4 is satisfied if the policy $\pi_0^{\mathrm{u}}(x)$ is continuous, which is for instance the case for the LQR. Local boundedness of the policy is also guaranteed if the set $\mathbb{U}_0(x)$ is uniformly bounded for all $x \in \hat{\mathbb{X}}_0$.*

Before proceeding to the stability proof, we state three additional assumptions, which are standard requirements in stability theory for nonlinear model predictive control, compare e.g. [1].

Assumption 5 (Lower bound on l_0)**.** *Assume that there is a $\mathcal{K}_\infty$ function α_l such that $l_0(x, z, u, w) \geq \alpha_l(\|x\|)$, for all $(x, z, u, w) \in \mathbb{Y}_0$.*

Assumption 6 (Weak controllability)**.** *Assume there exists a $\mathcal{K}_\infty$ function α_2 such that $\bar{V}_0^{\mathrm{x}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{X}_0$.*

Assumption 7 (Terminal stability condition)**.** *We assume that the terminal cost satisfies $\bar{V}_{N-1}^{\mathrm{s}} \leq \bar{V}_N^{\mathrm{s}}$.*

Remark 3. *In practice, Assumption 5 can always be satisfied by including a small regularization term, i.e. the cost $l_0(x, z, u, w) = \hat{l}_0(x, z, u, w) + \epsilon\|x\|_Q^2$ with $\hat{l}_0$ nonnegative, $Q \succ 0$ and $\epsilon > 0$ satisfies Assumption 5.*

Next, we show monotonicity of the value functions as well as positive invariance of the set $\mathbb{X}_0$ with respect to the closed-loop system, which are the two main results required for asymptotic stability.

Lemma 1. *Suppose Assumptions 1, 2, 3, and 7 are satisfied. It then holds that $\bar{V}_1^{\mathrm{s}} \leq \ldots \leq \bar{V}_{N-1}^{\mathrm{s}} \leq \bar{V}_N^{\mathrm{s}}$.*

Proof. From Assumption 7, we have $\bar{V}_{N-1}^{\mathrm{s}} \leq \bar{V}_N^{\mathrm{s}}$. Now suppose $\bar{V}_i^{\mathrm{s}} \leq \bar{V}_{i+1}^{\mathrm{s}}$ and consider any $s \in \mathbb{R}^{n_s}$. From Assumption 3, we either have $\bar{V}_i^{\mathrm{s}}(s) = \infty$, which implies that $\bar{V}_{i-1}^{\mathrm{s}}(s) \leq \bar{V}_i^{\mathrm{s}}(s)$ is trivially satisfied, or there exists a^* minimizing

$$\bar{V}_i^{\mathrm{s}}(s) = \min_a \; \bar{l}_i^{\mathrm{s}}(s, a) + \bar{V}_{i+1}^{\mathrm{s}}(f^{\mathrm{s}}(s, a)). \tag{11}$$

From $\bar{V}_i^{\mathrm{s}} \leq \bar{V}_{i+1}^{\mathrm{s}}$, we conclude that

$$\begin{aligned}
\bar{V}_i^{\mathrm{s}}(s) &= \bar{l}_i^{\mathrm{s}}(s, a^*) + \bar{V}_{i+1}^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*)) && (12)\\
&\geq \bar{l}_i^{\mathrm{s}}(s, a^*) + \bar{V}_i^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*)) && (13)\\
&\geq \bar{l}_{i-1}^{\mathrm{s}}(s, a^*) + \bar{V}_i^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*)) && (14)\\
&\geq \min_a \bar{l}_{i-1}^{\mathrm{s}}(s, a) + \bar{V}_i^{\mathrm{s}}(f^{\mathrm{s}}(s, a)) = \bar{V}_{i-1}^{\mathrm{s}}(s), && (15)
\end{aligned}$$

where $\bar{l}_i^{\mathrm{s}}(s, a^*) \geq \bar{l}_{i-1}^{\mathrm{s}}(s, a^*)$ due to Assumption 2. By induction, we conclude that $\bar{V}_1^{\mathrm{s}} \leq \ldots \leq \bar{V}_N^{\mathrm{s}}$.

□

Lemma 2. *Suppose Assumptions 1, 2, 3, 4, and 7 are satisfied. It then holds that $\bar{V}_0^{\mathrm{x}}(x) \leq \bar{V}_1^{\mathrm{x,z}}(x, z)$ for all $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}$.*

Proof. Consider any $s = (x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. If $\bar{V}_1^{\mathrm{x,z}}(x, z) = \bar{V}_1^{\mathrm{s}}(s) = \infty$, the inequality is trivially satisfied. Otherwise, there is a^* such that

$$\bar{V}_1^{\mathrm{s}}(s) = l_1(s, a^*) + \bar{V}_2^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*)). \tag{16}$$

Together with Assumption 2 and Lemma 1, we obtain

$$\begin{aligned}
\bar{V}_1^{\mathrm{s}}(s) &= l_1^{\mathrm{s}}(s, a^*) + \bar{V}_2^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*))\\
&\geq l_0^{\mathrm{s}}(s, a^*) + \bar{V}_1^{\mathrm{s}}(f^{\mathrm{s}}(s, a^*))\\
&\geq \min_{z,u,w} \; l_0(x, z, u, w) + \bar{V}_1^{\mathrm{x,z}}(f^{\mathrm{x}}(x, u), f^{\mathrm{z}}(x, z, u, w))\\
&= \bar{V}_0^{\mathrm{x}}(x),
\end{aligned}$$

which concludes the proof. □

Lemma 3 (Positive invariance of $\mathbb{X}_0$)**.** *Suppose Assumptions 1, 2, 3, 4, 7 are satisfied. The set $\mathbb{X}_0$ is positive invariant for the closed-loop system $x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))$ with MPC policy $\pi_0^{\mathrm{u}}(x)$ as defined in* (9)*.*

Proof. For any $x \in \mathbb{X}_0$, we have

$$\bar{V}_0^{\mathrm{x}}(x) = l_0^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)) + \bar{V}_1^{\mathrm{s}}\left(f^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)\right) < \infty,$$

which implies that $\bar{V}_1^{\mathrm{s}}\left(f^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)\right)$ must be finite. Together with Lemma 2, we conclude that

$$\infty > \bar{V}_1^{\mathrm{s}}\left(f^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)\right) \geq \bar{V}_0^{\mathrm{x}}(f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))), \tag{17}$$

which implies that $x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x)) \in \mathbb{X}_0$. □

We now derive the main stability result. In particular, we show that $\bar{V}_0^{\mathrm{x}}(x)$ is a Lyapunov function for the closed-loop system obtained by controlling the dynamic system with the optimal policy $\pi_0^{\mathrm{u}}(x)$.

Definition 2 (Lyapunov function)**.** *Suppose that $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ is positive invariant for the system $x^+ = \psi(x)$ with ψ locally bounded. Furthermore assume that the origin is an equilibrium point, $\psi(0) = 0$, and $0 \in \bar{\mathbb{X}}$.*
A function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function in $\mathbb{X}$ for the system $x^+ = \psi(x)$ if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous, positive definite function α_3 such that for any $x \in \mathbb{X}$,

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{18}$$

$$V(\psi(x)) - V(x) \leq -\alpha_3(\|x\|). \tag{19}$$

We make use of the following classical result from Lyapunov stability theory. A proof is given e.g. in [1], Theorem B.13.

Theorem 1. *Suppose that $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ is positive invariant for the system $x^+ = \psi(x)$ with ψ locally bounded. Furthermore assume that the origin is an equilibrium point, $\psi(0) = 0$, and $0 \in \mathbb{X}$. If V is a Lyapunov function in $\mathbb{X}$ for $x^+ = \psi(x)$, then the origin is globally asymptotically stable in $\mathbb{X}$ for the system $x^+ = \psi(x)$.*

Theorem 2 (Asymptotic stability of stage-varying NMPC)**.** *Suppose Assumptions 1, 2, 3, 4, 5, 6 and 7 are satisfied. Let π_0^{u} be the MPC policy as defined in* (9)*.*
Then the following hold.

1) *The value function $\bar{V}_0^{\mathrm{x}}$ is a Lyapunov function in $\mathbb{X}_0$ for the closed-loop system $x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))$.*
2) *The origin $x = 0$ is an asymptotically stable equilibrium with region of attraction $\mathbb{X}_0$ for the closed-loop system*

$$x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x)). \tag{20}$$

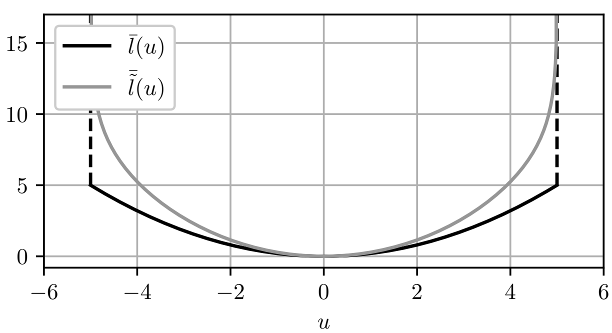

Fig. 2. Partial Tightening: As a simple example, we consider here stage costs on controls only. The stage cost $\bar{l}(u)$ in the first part of the horizon, $i = 0, \ldots, M-1$, is quadratic and there is a lower and an upper bound on the control, $u_{\min} \leq u \leq u_{\max}$ with $u_{\max} = -u_{\min} = 5$. In the tightened part of the horizon, $i = M, \ldots, N-1$, the stage cost $\bar{\bar{l}}(u)$ includes the additional barrier terms. Note that the progressive tightening condition, $\bar{l} \leq \bar{\bar{l}}$, $i = 0, \ldots, N-2$, is satisfied.

Proof. First, note that $f^{\mathrm{x}}(\cdot, \pi_0^{\mathrm{u}}(\cdot))$ is locally bounded due to continuity of f^{x} and the assumption that $\pi_0^{\mathrm{u}}(\cdot)$ is locally bounded, cf. Assumption 1 and 4. Furthermore, $\mathbb{X}_0$ is positive invariant for $x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))$ due to Lemma 3 and contains the origin due to Assumption 4.

We first show that the value function $\bar{V}_0^{\mathrm{x}}$ is a Lyapunov function in $\mathbb{X}_0$ for $x^+ = f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))$. Due to Assumption 6, there exists a $\mathcal{K}_\infty$ function α_2 such that $\bar{V}_0^{\mathrm{x}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{X}_0$. From Assumption 5 and 6, we have

$$\alpha_l(\|x\|) \leq l_0(x, z, u, w) \leq \bar{V}_0^{\mathrm{x}}(x) \leq \alpha_2(\|x\|) \tag{21}$$

for any $x \in \mathbb{X}_0$ and corresponding $(z, u, w) \in \mathbb{Q}_0(x)$ and with $\mathcal{K}_\infty$ functions α_l, α_2, which implies that (18) is satisfied.

For any $x \in \mathbb{X}_0$, we furthermore have

$$\begin{aligned}\bar{V}_0^{\mathrm{x}}(x) &= l_0^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)) + \bar{V}_1^{\mathrm{s}}(f^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x))) \\ &\geq l_0^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)) + \bar{V}_0^{\mathrm{x}}(f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x)))\end{aligned} \tag{22}$$

where the inequality follows from Lemma 2. Together with Assumption 5, we obtain

$$\begin{aligned}\bar{V}_0^{\mathrm{x}}(f^{\mathrm{x}}(x, \pi_0^{\mathrm{u}}(x))) - \bar{V}_0^{\mathrm{x}}(x) &\leq -l_0^{\mathrm{s}}(\pi_0^{\mathrm{s}}(x), \pi_0^{\mathrm{a}}(x)) \\ &\leq -\alpha_l(\|x\|),\end{aligned} \tag{23}$$

which shows that the decrease property (19) is satisfied with $\alpha_3(\cdot) = \alpha_l(\cdot)$.

The second part – asymptotic stability of the origin for the closed-loop system – follows directly from Theorem 1 with the value function $\bar{V}_0^{\mathrm{x}}$ as Lyapunov function in the set $\mathbb{X}_0$. □

Remark 4 (Inexact NMPC)**.** *Within the real-time iteration (RTI) framework [6], [7], the optimal control problem* (2) *is only solved approximately resulting in an inexact MPC policy. The main stability results, Theorem 2 together with the analysis in [8] directly implies asymptotic stability of the closed-loop system-optimizer dynamics arising from the inexact RTI approach under the assumption that the sampling time is sufficiently small.*

IV. Stage-Varying Approaches in the Literature

In the following, we give a survey of NMPC approaches that have been proposed in the literature and that can be cast in terms of a progressively tightening optimal control problem.

A. Partial tightening

With partially tightened NMPC formulations as introduced in [9] and applied in [10], the inequality constraints in the later part of the optimization horizon are replaced by corresponding logarithmic barriers in order to reduce the computational complexity. This yields an OCP of the form:

$$\min_{X,U} \quad \sum_{i=0}^{M-1} l(x_i, u_i) + \sum_{i=M}^{N-1} \tilde{l}(x_i, u_i) + \tilde{V}_N(x_N) \tag{24a}$$

$$\text{s.t.} \quad x_0 = \underline{x}, \tag{24b}$$

$$x_{i+1} = f^{\mathrm{x}}(x_i, u_i),\ i = 0, \ldots, N-1, \tag{24c}$$

$$0 \geq h(x_i, u_i), \quad i = 0, \ldots, M-1, \tag{24d}$$

where we introduced $\tilde{V}_N(x) = V_N(x) + B_N(h_N(x))$ and $\tilde{l}(x, u) = l(x, u) + B(h(x, u))$ with barrier functions B and B_N. For the stability result to apply, the modified stage costs $\tilde{l}_i(x, u)$, as well as the terminal cost $\tilde{V}_N(x)$ need to be continuous and positive definite, which typically requires recentered barriers [11], [12], [13].

If the barriers B_i are positive definite, then $l(x, u) \leq \tilde{l}(x, u)$ and Assumption 2 is satisfied, which is illustrated in Figure 2.

B. Progressive barrier tightening

Similarly to partial tightening, progressive barrier tightening formulations replace inequality constraints with corresponding barrier terms. Besides, progressive barrier tightening approaches employ increasing barrier parameters along the horizon, i.e. $\tilde{l}_i(x, u) = l(x, u) + \tau_i B(h(x, u))$, $i = M, \ldots, N-1$, with $0 < \tau_i < \tau_{i+1}$ for all i. Progressive barrier tightening has been used in [14], [2], [15].

C. Approximate infinite horizon closed-loop costing

With approximate infinite horizon closed-loop costing [16], [17], the horizon is split into two parts. In the second part, the inputs are given by a fixed policy $\pi(x)$, which is typically chosen as the LQR policy obtained by linearization at a steady state. This yields an OCP of the form:

$$\min_{X,U} \quad \sum_{i=0}^{N-1} l(x_i, u_i) + l_N(x_N) \tag{25a}$$

$$\text{s.t.} \quad x_0 = \underline{x}, \tag{25b}$$

$$x_{i+1} = f^{\mathrm{x}}(x_i, u_i),\ i = 0, \ldots, N-1, \tag{25c}$$

$$0 \geq h(x_i, u_i), \quad i = 0, \ldots, N-1, \tag{25d}$$

$$0 \geq h_N(x_N), \tag{25e}$$

$$u_i = \pi(x_i), \qquad i = M, \ldots, N-1. \tag{25f}$$

Obviously, the additional constraint $u_i = \pi(x_i)$ is a form of tightening and Assumption 2 is satisfied. If $\pi(x)$ satisfies

$$\bar{V}_{N-1}^{\mathrm{x}}(x) \leq \bar{l}_{N-1}(x, \pi(x)) + \bar{V}_N^{\mathrm{x}}(f^{\mathrm{x}}(x, \pi(x))) \tag{26}$$

for all $x \in \mathbb{R}^{n_x}$, the terminal stability condition, Assumption 7, is satisfied.

D. Stochastic linear MPC with chance constraints

For linear systems subject to additive disturbances, the stochastic MPC formulation minimizing expected cost and employing chance constraints yields a stage-varying OCP formulation satisfying the progressive tightening condition as the back-off terms, which in the linear case can be precomputed, are typically increasing along the horizon [18].

E. Time-optimal point-to-point motion

The NMPC formulation for time-optimal point-to-point motions which has been introduced in [19] fits perfectly into the class of progressively tightening NMPC formulations. A similar formulation is used in [20].

F. Move blocking

An approach for reducing the computational complexity of NMPC approaches is to introduce move blocking, which corresponds to constraining the inputs to be constant over a certain number of time steps [21], [22], [23].

In the following, we consider a very particular variant of move blocking where the horizon is split into two parts: a control horizon, where the control input is allowed to change in every time step, and a simulation horizon, where the control input is fixed, i.e. $u_i = u_{M-1}$ for $i = M, \dots, N-1$. Introducing the auxiliary dynamics $f^{\mathrm{z}}(u) = u$, we obtain the move blocking formulation:

$$
\begin{aligned}
\min_{X,Z,U} \quad & \sum_{i=0}^{N-1} l_i(x_i, u_i) + V_N(x_N) && \text{(27a)}\\
\text{s.t.} \quad & x_0 = \underline{x}, && \text{(27b)}\\
& x_{i+1} = f^{\mathrm{x}}(x_i, u_i),\ i = 0, \dots, N-1, && \text{(27c)}\\
& z_{i+1} = f^{\mathrm{z}}(u_i), \qquad i = 0, \dots, N-1, && \text{(27d)}\\
& u_i - z_i = 0, \qquad\quad i = M, \dots, N, && \text{(27e)}\\
& z_N \in \tilde{\mathbb{U}}, && \text{(27f)}\\
& 0 \geq h_N(x_N), && \text{(27g)}\\
& 0 = g_N(x_N), && \text{(27h)}
\end{aligned}
$$

where $\tilde{\mathbb{U}}$ is the set of controls that might be applied in the second part of the horizon. The set $\tilde{\mathbb{U}}$ needs to be chosen such that the following assumption is satisfied:

Assumption 8. *Suppose that*

$$l_{N-1}(x, u) + V_N(f^{\mathrm{x}}(x, u)) \leq V_N(x) \tag{28}$$

for all $u \in \tilde{\mathbb{U}}$ and all x satisfying $h_N(x) \leq 0$, $g_N(x) = 0$.

Proposition 1. *Suppose Assumptions 1, 3, 8 are satisfied for some set $\tilde{\mathbb{U}}$. Then, the terminal stability condition, Assumption 7, is satisfied.*

Proof. If $\bar{V}_N^{\mathrm{x,z}}(x, z) = \infty$, the inequality is trivially satisfied. If $\bar{V}_N^{\mathrm{x,z}}(x, z) < \infty$, we conclude that $z \in \tilde{\mathbb{U}}$ and obtain

$$
\begin{aligned}
\bar{V}_{N-1}^{\mathrm{x,z}}(x, z) &= \min_u \ \bar{l}_{N-1}(x, z, u) + \bar{V}_N^{\mathrm{x,z}}(f^{\mathrm{x}}(x, u), f^{\mathrm{z}}(u))\\
&= l_{N-1}(x, z) + V_N(f^{\mathrm{x}}(x, z))\\
&\leq V_N(x),
\end{aligned} \tag{29}
$$

where the last inequality follows from Assumption 8. □

G. Low-dimensional parameterized NMPC

Similar to move blocking, the computational complexity might be reduced by choosing a non-constant, but still low-dimensional parametrization of the control trajectory. An example is the exponential parametrization as introduced in [24]. Introducing the auxiliary dynamics $f^{\mathrm{z}}(z) = Az$, where $A \in \mathbb{R}^{n_z \times n_z}$ is a constant matrix with spectral radius $\rho(A) < 1$, an OCP with an exponential control parametrization is:

$$
\begin{aligned}
\min_{X,Z,U} \quad & \sum_{i=0}^{N-1} l_i(x_i, u_i) + V_N(x_N) && \text{(30a)}\\
\text{s.t.} \quad & x_0 = \underline{x}, && \text{(30b)}\\
& x_{i+1} = f^{\mathrm{x}}(x_i, u_i),\ i = 0, \dots, N-1, && \text{(30c)}\\
& z_{i+1} = f^{\mathrm{z}}(z_i), \qquad i = 0, \dots, N-1, && \text{(30d)}\\
& u_i - Cz_i = 0, \qquad\ i = 0, \dots, N, && \text{(30e)}\\
& z_N \in \tilde{\mathbb{U}}, && \text{(30f)}
\end{aligned}
$$

where $C \in \mathbb{R}^{n_u \times n_z}$. This parametrization limits the degrees of freedom to z_0 whereas the controls $u_0, \dots, u_{N-1}$ are determined by the state of the auxiliary dynamic system. It is easy to check that the progressive tightening criterion, Assumption 2, is fulfilled. For the terminal stability condition, Assumption 7, to be satisfied we need to assume that the set $\tilde{\mathbb{U}}$ is chosen such that Assumption 8 holds.

H. Rate-regularized NMPC

Rate-regularized NMPC formulations, where the initial control u_0 is free but there are additional constraints and/or costs on the difference $u_{i+1} - u_i$, can be expressed in terms of a progressively tightening OCP by making use of the auxiliary dynamics. With $f^{\mathrm{z}}(u) = u$, a rate-regularized formulation is:

$$
\begin{aligned}
\min_{X,Z,U} \quad & l_0(x_0, u_0) + \sum_{i=1}^{N-1} l(x_i, z_i, u_i) + V_N(x_N) && \text{(31a)}\\
\text{s.t.} \quad & x_0 = \underline{x}, && \text{(31b)}\\
& x_{i+1} = f^{\mathrm{x}}(x_i, u_i), \quad i = 0, \dots, N-1, && \text{(31c)}\\
& z_{i+1} = f^{\mathrm{z}}(u_i), \qquad i = 0, \dots, N-1, && \text{(31d)}\\
& -c \leq u_i - z_i \leq c,\ i = 1, \dots, N-1. && \text{(31e)}
\end{aligned}
$$

In addition to rate constraints, we might introduce a penalty $p : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ on the difference between subsequent inputs,

$$l(x, z, u) = l_0(x, u) + p(\|u - z\|) \tag{32}$$

in (31a). Given that Assumptions 1, 3, 4, 5, 6 and 7 are satisfied, it remains to check whether the costs satisfy $l_0(x, u) \leq l(x, z, u)$ for all x, u, z satisfying $|u - z| \leq c$, which is the case if $l(x, z, u)$ is of the form given in (32).

I. Tunnel-following NMPC

With tunnel-following NMPC, a robot or agent is controlled to follow a tunnel or tube around a prescribed path [25], [26]. This is achieved by an NMPC formulation that aims at maximizing progress along the path while keeping the distance of the robot to the reference point on the path below some prescribed value. While the robot dynamics are given by $f^{\mathrm{x}}(x,u)$, the progress along the reference path is described by the auxiliary system $f^{\mathrm{z}}(z,w)=z-w$ where $z\in[0,1]$ denotes the normalized distance along the path to the goal position $z^{\mathrm{target}}=0$. Let $p(x)$ denote the position associated with the robot state x and $\rho(z)$ denote the position associated with the normalized path distance z and assume $p(0)=\rho(0)$. The tunnel-following formulation is:

$$\begin{aligned}
\min_{X,Z,U,W} \quad & \sum_{i=0}^{N-1} l(x,z) + V_N^{\mathrm{z}}(z) && \text{(33a)}\\
\text{s.t.} \quad & x_0 = \underline{x}, && \text{(33b)}\\
& x_{i+1} = f^{\mathrm{x}}(x_i,u_i), \quad i=0,\dots,N-1, && \text{(33c)}\\
& z_{i+1} = f^{\mathrm{z}}(z_i,w_i), \quad i=0,\dots,N-1, && \text{(33d)}\\
& u_i \in \tilde{\mathbb{U}}, x_i \in \tilde{\mathbb{X}}, \quad i=0,\dots,N-1, && \text{(33e)}\\
& z_i \in [0,1], \quad i=0,\dots,N, && \text{(33f)}\\
& 0 \le w_i \le w_{\max}, \quad i=0,\dots,N-1, && \text{(33g)}\\
& r \ge \|p(x_i)-\rho(z_i)\|, i=0,\dots,N-1, && \text{(33h)}\\
& p(x_N) = \rho(z_N), && \text{(33i)}\\
& x_N \in \tilde{\mathbb{X}}_N(z), && \text{(33j)}
\end{aligned}$$

with $l(x,z)=l_{\mathrm{p}}(p(x)-\rho(z))+l_{\mathrm{z}}(z)$ penalizing the deviation of the robot position to the corresponding reference position on the path, as well as the normalized distance to the target position. The terminal stability condition is fulfilled if $V_N^{\mathrm{z}}(z)$ and $\tilde{\mathbb{X}}_N(z)$ satisfy the following assumption:

Assumption 9. *Let $\mathbb{X}_*(z)=\{x\in\tilde{\mathbb{X}}_N(z):\ p(x)=\rho(z)\}$. Suppose that for all $z\in[0,1]$ and all $x\in\mathbb{X}_*(z)$ there is $u\in\tilde{\mathbb{U}}$ and $w\in[0,w_{\max}]$ such that $x^+=f^{\mathrm{x}}(x,u)\in\mathbb{X}_*(z^+)$ and $l_{\mathrm{z}}(z)+V_N^{\mathrm{z}}(z^+)\le V_N^{\mathrm{z}}(z)$ where $z^+=f^{\mathrm{z}}(z,w)$.*

V. CONCLUSIONS

We considered a stage-varying optimal control problem formulation including an auxiliary dynamic system and provided a stability proof for the corresponding closed-loop system controlled with the NMPC policy. In addition to standard assumptions, we require the stage-varying costs and constraints to satisfy a *progressive tightening* condition.

In the second part, we showed that progressively tightening problem formulations arise in various contexts illustrating the generality of the problem formulation and the corresponding stability result.

Our analysis does not cover formulations with discounted cost, as is commonly used in reinforcement learning, as well as formulations with zero terminal costs as analyzed in [27]. Future work might aim at extending the progressive tightening concept to continuous-time control systems.

REFERENCES

[1] J. B. Rawlings, D. Q. Mayne, and M. M. Diehl, *Model Predictive Control: Theory, Computation, and Design*, 2nd ed. Nob Hill, 2017.

[2] A. Zanelli, "Inexact methods for nonlinear model predictive control: stability, applications, and software," Ph.D. dissertation, University of Freiburg, 2021.

[3] S. Keerthi and E. Gilbert, "Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations," *Journal of Optimization Theory and Applications*, vol. 57, no. 2, pp. 265–293, 1988.

[4] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, 1990.

[5] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[6] M. Diehl, *Real-Time Optimization for Large Scale Nonlinear Processes*, ser. Fortschritt-Berichte VDI Reihe 8, Meß-, Steuerungs- und Regelungstechnik. Düsseldorf: VDI Verlag, 2002, PhD Thesis.

[7] H. G. Bock, M. Diehl, E. A. Kostina, and J. P. Schlöder, "Constrained optimal feedback control of systems governed by large differential algebraic equations," in *Real-Time and Online PDE-Constrained Optimization*. SIAM, 2007, pp. 3–22.

[8] A. Zanelli, Q. Tran-Dinh, and M. Diehl, "A lyapunov function for the combined system-optimizer dynamics in inexact model predictive control," *Automatica*, vol. 134, p. 109901, 2021.

[9] A. Zanelli, R. Quirynen, G. Frison, and M. Diehl, "A partially tightened real-time iteration scheme for nonlinear model predictive control," in *Proceedings of 56th IEEE Conference on Decision and Control*, Melbourne, Australia, December 2017.

[10] A. Zanelli, G. Horn, G. Frison, and M. Diehl, "Nonlinear model predictive control of a human-sized quadrotor," in *Proceedings of the European Control Conference (ECC)*, 2018, pp. 1542–1547.

[11] A. G. Wills and W. P. Heath, "Barrier function based model predictive control," *Automatica*, vol. 40, no. 8, pp. 1415–1422, August 2004.

[12] C. Feller and C. Ebenbauer, "Weight recentered barrier functions and smooth polytopic terminal set formulations for linear model predictive control," in *2015 American Control Conference (ACC)*. IEEE, 2015.

[13] ——, "A stabilizing iteration scheme for model predictive control based on relaxed barrier functions," *Automatica*, vol. 80, June 2017.

[14] G. Valverde and T. Van Cutsem, "Model predictive control of voltages in active distribution networks," *IEEE Transactions on Smart Grid*, vol. 4, no. 4, pp. 2152–2161, 2013.

[15] K. Baumgärtner, Y. Wang, A. Zanelli, and M. Diehl, "Fast nonlinear model predictive control using barrier formulations and squashing with a generalized Gauss-Newton hessian," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*. IEEE, 2022.

[16] M. Diehl, L. Magni, and G. D. Nicolao, "Online NMPC of a looping kite using approximate infinite horizon closed loop costing," in *Proceedings of the IFAC Conference on Control Systems Design*. Bratislava, Slovak Republic: IFAC, September 7-10 2003.

[17] ——, "Efficient NMPC of unstable periodic systems using approximate infinite horizon closed loop costing," *Annual Reviews in Control*, vol. 28, no. 1, pp. 37–45, 2004.

[18] M. Cannon, B. Kouvaritakis, S. V. Raković, and Q. Cheng, "Stochastic tubes in model predictive control with probabilistic constraints," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 194–200, 2010.

[19] R. Verschueren, H. J. Ferreau, A. Zanarini, M. Mercangöz, and M. Diehl, "A stabilizing nonlinear model predictive control scheme for time-optimal point-to-point motions," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, 2017.

[20] T. Schoels, P. Rutquist, L. Palmieri, A. Zanelli, K. O. Arras, and M. Diehl, "CIAO∗: MPC-based safe motion planning in predictable dynamic environments," *IFAC-PapersOnLine*, vol. 53, no. 2, 2020.

[21] P. Tøndel and T. A. Johansen, "Complexity reduction in explicit linear model predictive control," *IFAC Proceedings Volumes*, vol. 35, 2002.

[22] S. J. Qin and T. A. Badgwell, "A survey of industrial model predictive control technology," *Control Engineering Practice*, vol. 11, 2003.

[23] R. Cagienard, P. Grieder, E. C. Kerrigan, and M. Morari, "Move blocking strategies in receding horizon control," *Journal of Process Control*, vol. 17, no. 6, pp. 563–570, 2007.

[24] M. Alamir, *Stabilization of nonlinear systems using receding-horizon control schemes: a parametrized approach for fast systems*. Springer, 2006, vol. 339.

[25] N. van Duijkeren, "Online motion control in virtual corridors – for fast robotic systems," Ph.D. dissertation, KU Leuven, 2019.

[26] A. Astudillo, G. Pipeleers, J. Gillis, W. Decré, and J. Swevers, "Varying-radius tunnel-following NMPC for robot manipulators using sequential convex quadratic programming," *IFAC-PapersOnLine*, vol. 55, no. 37, pp. 345–352, 2022.

[27] L. Grüne, "NMPC without terminal constraints," *IFAC Proceedings Volumes*, vol. 45, no. 17, pp. 1–13, 2012.