IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Sub-Optimal Moving Horizon Estimation in Feedback Control of Linear Constrained Systems

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***Abstract*— Moving horizon estimation (MHE) offers benefits relative to other estimation approaches by its ability to explicitly handle constraints, but suffers increased computation cost. To help enable MHE on platforms with limited computation power, we propose to solve the optimization problem underlying MHE sub-optimally for a fixed number of optimization iterations per time step. The stability of the closed-loop system is analyzed using the small-gain theorem by considering the closed-loop controlled system, the optimization algorithm dynamics, and the estimation error dynamics as three interconnected subsystems. By assuming incremental input/output-to-state stability (δ-IOSS) of the system and imposing standard ISS conditions on the controller, we derive conditions on the iteration number such that the interconnected system is input-to-state stable (ISS) w.r.t. the external disturbances. A simulation with an MHE-MPC estimator-controller pair is used for validation.**

I. INTRODUCTION

MHE is an optimization-based method that considers a fixed window of past measurements and the system's constraints in estimating the current state. Due to the inclusion of the constraints explicitly in the problem formulation, MHE has been shown to produce more accurate state estimates compared to the extended Kalman Filter [1]. Assuming detectability of the system, rather than observability, MHE was shown to posses robust global asymptotic stability w.r.t. bounded disturbances and the estimation error converges in case of bounded and vanishing disturbances [2].

Although MHE offers the benefit of considering constraints, its application is limited by the computational cost, particularly in platforms with limited computational resources. To alleviate this issue, [3] introduced an auxiliary observer to provide pre-estimation for MHE. Despite reduced computation time, the iteration number required to solve the MHE problem with stability guarantees cannot be determined offline. In [4], an auxiliary observer is used to provide warm-start solutions, which are improved for a limited amount of iterations, to obtain sub-optimal state estimates that is robustly stable. The proximity-MHE scheme in [5] performs limited optimization iterations with a proximity regularizing term to improve the prior estimate from an auxiliary observer and guarantees the nominal stability of the MHE.

Other approaches concentrated on modifying the optimization scheme used for solving MHE. For example, [6] proposed to enforce move blocking on the disturbance sequence in MHE to reduce the associated computation burden, which also guarantees the nominal stability of MHE. In [7], a real-time iteration scheme is applied to MHE without inequality constraints. Local convergence is guaranteed when a single optimization iteration is performed per time step. The work [8] combined this scheme with automatic code generation to obtain highly efficient source code of MHE algorithms. For noise-free systems, [9] solves the MHE problem for single or multiple iterations with gradient-based, conjugate gradient-based, and Newton methods and achieves local stability.

Compared to the aforementioned works, we study the stability of the closed-loop with a sub-optimal MHE and a feedback control law. Earlier studies often treated MHE and the feedback controller as separate modules, with MHE providing estimates with bounded error [10], and the controller designed to ensure stability. Instead, we aim to jointly determine conditions that guarantee stability of both MHE and the controlled system. To achieve this, we adopt an stability analysis framework from the sub-optimal model predictive control (MPC) literature [11], [12], which formulated the closed-loop system as an interconnection of a controlled system and an optimization algorithm dynamics.

In this paper, we propose a sub-optimal MHE scheme where, at every time step, the MHE problem is warm-started with the previous solution and then solved by an optimization algorithm with a fixed number of iterations. Then, the resulting sub-optimal estimate is used for feedback control of a linear system with state and input constraints.

Our main contribution lies in the stability analysis. We first characterize the interaction between the closed-loop system controlled by an robustly stabilizing controller, the sub-optimality error dynamics of the optimization process, and the state estimation error dynamics of the sub-optimal MHE as three interconnected subsystems, which we then show are input-to-state stable (ISS). Next, we use the small-gain theorem [13] to derive conditions on the optimization iteration number for guaranteeing the interconnected system is ISS w.r.t to external disturbances.

Notations: Let $\mathbb{S}_{\succ 0}$ be the set of positive definite matrices. Let $\mathbf{I}^n$ be the identity matrix of size n. Let $\mathbf{0}^{m\times n}$ be the zero matrix of size $m \times n$. For a vector $x \in \mathbb{R}^{n_x}$ and a matrix $U \in \mathbb{S}_{\succ 0}^{n_x \times n_x}$, let $\|x\|$ and $\|x\|_U$ denote the l_2-norm and the weighted l_2-norm of x, respectively. Consider square matrices U and V. Let $\|U\|$ denote the spectral norm. Let $\overline{\lambda}_U$ and $\underline{\lambda}_U$ denote the largest and smallest eigenvalues of U, respectively. Let $\Lambda_V^U := \overline{\lambda}(U)/\underline{\lambda}(V)$. Let $\mathbb{I}_{[a,b]}$ denote the set of integers in $[a, b] \in \mathbb{R}$. For $v_t \in \mathbb{R}^{n_v}$ and time steps $a, b \in \mathbb{I}_{[a,b]}$, let $\mathbf{v}_{[a,b]} := \{v_a, \cdots, v_b\}$ and $\|\mathbf{v}_{[a,b]}\| := \sup_{t\in\mathbb{I}_{[a,b]}} \|v_t\|$. Our use of the class $\mathcal{K}$, $\mathcal{L}$, $\mathcal{KL}$ functions,

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follow the convention in [11].

II. Controller and MHE Formulation

A. Dynamic System with State Feedback Controller

Consider a system with linear time-invariant dynamics

$$x_{t+1} = Ax_t + Bu_t + w_t^1, \;\; y_t = Cx_t + w_t^2, \tag{1}$$

with state $x_t \in \mathcal{X} \subset \mathbb{R}^{n_x}$, input $u_t \in \mathcal{U} \subset \mathbb{R}^{n_u}$, output measurement $y_t \in \mathcal{Y} \subset \mathbb{R}^{n_y}$, external disturbance $w_t^1 \in \mathcal{W}_1 \subset \mathbb{R}^{n_x}$, and measurement noise $w_t^2 \in \mathcal{W}_2 \subset \mathbb{R}^{n_y}$. Let $w_t := [w_t^{1\top}, w_t^{2\top}]^\top \in \mathcal{W} \subset \mathbb{R}^{n_{x+y}}$ be the augmented external disturbance. Let $\mathcal{Z} := \mathcal{X} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{W}$ be the Cartesian product of the constraint sets.

Assumption 1 *$\mathcal{Z}$ is convex and contains the origin.*

Assumption 2 *Consider system* (1)*. There exist $P, Q, R \in \mathbb{S}_{\succ 0}$ and $\eta \in [0, 1)$ that satisfy*

$$\begin{pmatrix} A^\top PA - \eta P - C^\top RC & A^\top P\bar{B} - C^\top R\bar{D} \\ \bar{B}^\top PA - \bar{D}^\top RC & \bar{B}^\top P\bar{B} - Q - \bar{D}^\top R\bar{D} \end{pmatrix} \preceq 0,$$
$$\bar{B} = [\mathbf{I}^{n_x}, \mathbf{0}^{n_x \times n_y}], \quad \bar{D} = [\mathbf{0}^{n_y \times n_x}, \mathbf{I}^{n_y},]. \tag{2}$$

The sufficient condition in (2) was established in Corollary 3 of [14] to guarantee system (1) admits a δ-IOSS Lyapunov function, and is thus detectable. Let (x, u, y, w), $(x', u, y', w') \in \mathcal{Z}$, where $y = Cx + w^2$ and $y' = Cx' + w^{2\prime}$. Then, satisfaction of (2) implies the system (1) admits

$$W_\delta(x, x') := \|x - x'\|_P^2 \tag{3}$$

as a δ-IOSS Lyapunov function, satisfying

$$\begin{aligned} &W_\delta(Ax + Bu + w^1, Ax' + Bu + w^{1\prime}) \\ &\quad \leq \eta W_\delta(x, x') + \|w - w'\|_Q^2 + \|y - y'\|_R^2. \end{aligned} \tag{4}$$

Consider the system (1) with a state feedback controller $u_t := \pi(\hat{x}_t) : \mathcal{X} \to \mathcal{U}$ satisfying Assumptions 3 and 4,

$$x_{t+1} = Ax_t + B\pi(\hat{x}_t) + w_t^1, \tag{5}$$

with state estimate $\hat{x}_t \in \mathcal{X}$ and estimation error $e_t := \hat{x}_t - x_t$.

Assumption 3 *There exists a positive constant L_π such that, for any $x, x' \in \mathcal{X}$, $\pi(\cdot)$ satisfies*

$$\|\pi(x) - \pi(x')\| \leq L_\pi \|x - x'\|. \tag{6}$$

Assumption 4 *The closed-loop controlled system in* (5) *is input-to-state stable (ISS): Given an initial state $x_0 \in \mathcal{X}$, an input sequence $\mathbf{u}_{[0,t-1]}$ generated by $\pi(\cdot)$, an estimation error sequence $\mathbf{e}_{[0,t-1]}$, and a disturbance sequence $\mathbf{w}_{[0,t-1]}$, there exist $\beta_1 \in (0, 1)$, $\alpha_1, \gamma_{1,3} > 0$, and $\gamma_1^w \in \mathcal{K}$ such that, for all $t > 0$, the resulting state $x_t \in \mathcal{X}$ satisfies*

$$\|x_t\| \leq \beta_1^t \alpha_1 \|x_0\| + \gamma_{1,3} \|\mathbf{e}_{[0,t-1]}\| + \gamma_1^w(\|\mathbf{w}_{[0,t-1]}\|). \tag{7}$$

Assumption 3, 4 can be satisfied by, e.g., a MPC controller.

B. Sub-Optimal Moving Horizon Estimation

At time step t, we obtain the state estimate $\hat{x}_t$ by solving a MHE problem based on a prior estimation $x_{t-M_t}^{\text{prior}}$, past inputs $\mathbf{u}_t = \mathbf{u}_{[t-M_t,t-1]}$, and past measurements $\mathbf{y}_t = \mathbf{y}_{[t-M_t,t-1]}$, with estimation horizon $M_t := \min(M, t)$, $M \in \mathbb{I}_{>0}$. The MHE problem $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ is formulated as

$$(\hat{\mathbf{x}}_t^*, \hat{\mathbf{w}}_t^*, \hat{\mathbf{y}}_t^*) = \underset{\hat{\mathbf{x}}_t, \hat{\mathbf{w}}_t, \hat{\mathbf{y}}_t}{\operatorname{argmin}} \; V_{\text{MHE}}(\hat{x}_{t-M_t|t}, \hat{\mathbf{w}}_t, \hat{\mathbf{y}}_t) \tag{8a}$$
$$\text{s.t. } \hat{x}_{i+1|t} = A\hat{x}_{i|t} + Bu_i + \hat{w}_{i|t}^1, \quad i \in \mathbb{I}_{[t-M_t,t-1]}, \tag{8b}$$
$$\hat{y}_{i|t} = C\hat{x}_{i|t} + \hat{w}_{i|t}^2, \quad i \in \mathbb{I}_{[t-M_t,t-1]}, \tag{8c}$$
$$\hat{w}_{i|t} \in \mathcal{W}, \; \hat{y}_{i|t} \in \mathcal{Y}, \quad i \in \mathbb{I}_{[t-M_t,t-1]}, \tag{8d}$$
$$\hat{x}_{i|t} \in \mathcal{X}, \quad i \in \mathbb{I}_{[t-M_t,t]}, \tag{8e}$$

where the cost is defined as

$$\begin{aligned} &V_{\text{MHE}}(\hat{x}_{t-M_t|t}, \hat{\mathbf{w}}_t, \hat{\mathbf{y}}_t) := 2\eta^{M_t} W_\delta(\hat{x}_{t-M_t|t}, x_{t-M_t}^{\text{prior}}) \\ &\quad + \sum_{i=1}^{M_t} \eta^{i-1} \left(2 \left\|\hat{w}_{t-i|t}\right\|_Q^2 + \left\|\hat{y}_{t-i|t} - y_{t-i}\right\|_R^2\right), \end{aligned} \tag{9}$$

with η, P, Q, and R satisfying (2). The decision variables $\hat{\mathbf{x}}_t := \{\hat{x}_{t-M_t|t}, \cdots, \hat{x}_{t|t}\}$, $\hat{\mathbf{w}}_t := \{\hat{w}_{t-M_t|t}, \cdots, \hat{w}_{t-1|t}\}$, and $\hat{\mathbf{y}}_t := \{\hat{y}_{t-M_t|t}, \cdots, \hat{y}_{t-1|t}\}$ denote the estimated states, augmented disturbances, and measurements, respectively. The cost functions (9) can be reformulated as

$$V_{\text{MHE}}(\hat{x}_{t-M_t|t}, \hat{\mathbf{w}}_t, \hat{\mathbf{y}}_t) := \|z_t - \tilde{z}_t\|_{H_t}^2, \tag{10}$$

where

$$\begin{aligned} z_t &:= [\hat{x}_{t-M_t|t}^\top, \hat{w}_{t-M_t|t}^\top, \hat{y}_{t-M_t|t}^\top, \cdots, \hat{w}_{t-1|t}^\top, \hat{y}_{t-1|t}^\top]^\top, \\ \tilde{z}_t &:= [x_{t-M_t}^{\text{prior}\top}, \mathbf{0}^{n_w \top}, y_{t-M_t}^\top, \cdots, \mathbf{0}^{n_w \top}, y_{t-1}^\top]^\top, \\ H_t &:= \text{blkdiag}(2\eta^{M_t} P, 2\eta^{M_t - 1} Q, \eta^{M_t - 1} R, \cdots, 2Q, R). \end{aligned} \tag{11}$$

Given $\mathbf{u}_t$, the state sequence $\hat{\mathbf{x}}_t$ can be constructed from z_t. Let $(\hat{\mathbf{x}}_t^*, \hat{\mathbf{y}}_t^*, \hat{\mathbf{w}}_t^*)$ and z_t^* denote the optimal solution to (8), with the cost in (9) and (10), respectively. We solve $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ with an optimization algorithm denoted by the nonlinear mapping $z_t^K = \Phi_K(z_t^0, \mathbb{P}_t)$, with an initial solution z_t^0 and iteration number $K > 0$. Suppose this optimization algorithm satisfies Assumption 5.

Assumption 5 *Given an initial solution z_t^0, the K_{th}-iteration solution z_t^K obtained from solving $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ with $\Phi_K(z_t^0, \mathbb{P}_t)$ satisfies* (8b)-(8e)*, and*

$$e_t := \|z_t^K - z_t^*\| \leq \phi(K) \|z_t^0 - z_t^*\|, \;\; \phi(K) \in \mathcal{L}. \tag{12}$$

Let $(\hat{\mathbf{x}}_t^K, \hat{\mathbf{y}}_t^K, \hat{\mathbf{w}}_t^K)$ and z_t^K denote the sub-optimal solution to (8) and define the sub-optimality error as $\epsilon_t := \|z_t^K - z_t^*\|$. Assumption 5 can be satisfied by a general class of optimization algorithms, for example, the projected gradient algorithm used in [11], where $\phi(K) := \iota^K$, with $0 < \iota < 1$, for strongly convex problems and $1/K$ for convex problems.

III. Sub-Optimal MHE-Based Feedback Control

In this section, we introduce a sub-optimal MHE scheme combined with a given feedback control. We then characterize the resulting closed-loop system as three interconnected subsystems. By showing each subsystem is ISS, we can apply the small-gain theorem [13] to derive conditions on the optimization iteration number that guarantee the interconnected system is ISS w.r.t. external disturbances. We present the proofs of Propositions 1-3 in the Appendix.

A. The Sub-Optimal MHE Scheme

Algorithm 1 (Alg. 1) introduces the proposed sub-optimal

Algorithm 1 Sub-Optimal MHE in Feedback Control

Require: M, K, $\Phi_K(\cdot)$, $\pi(\cdot)$, z_0^0, x_0^{prior}, $\mathbf{u}_0$, $\mathbf{y}_0$;
For $t = 0, 1, 2, \cdots$ **Do**
1. Obtain $\hat{x}_t^K$ by solving $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ for K iterations using optimization algorithm $\Phi_K(z_t^0, \mathbb{P}_t)$;
2. Warm-starting: $z_{t+1}^0 \leftarrow \Sigma_t z_t^K$;
3. Update the parameters: $x_{t-M_t}^{\text{prior}} \leftarrow \hat{x}_{t-M_t|t}^K$, $\mathbf{u}_{t+1}$, $\mathbf{y}_{t+1}$;
4. Apply $\pi(\hat{x}_t^K)$ to the system (5);
End

MHE scheme. In Step 1, the MHE problem is solved to obtain $\hat{x}_t^K$. In Step 2, the solution z_t^K is copied as the next warm-start solution z_{t+1}^0, This step also allows the optimization process to be treated as a dynamic system in the next section. However, for $t < M$, the formulation in (8) grows in size as more information is obtained. As a result, the solution z_t^K has a lower dimension compared to z_{t+1}^0. To alleviate this issue, we introduce a linear mapping matrix

$$\Sigma_t := \begin{cases} \text{blkdiag}(\mathbf{I}^{n_{z_t}-n_x-n_y}, \mathbf{0}^{n_x+n_y}), & t < M, \\ \mathbf{I}^{n_{z_t}}, & t \geq M, \end{cases} \tag{13}$$

to map z_t^K to the same dimension as z_{t+1}^0. In Step 3, the current data $\pi(\hat{x}_t^K)$ and $\mathbf{y}_t$ are appended to the end of $\mathbf{u}_{t+1}$ and $\mathbf{y}_{t+1}$, respectively. In Step 4, the control input is applied.

B. Interconnection of Three Subsystems

The closed-loop system in (5) using Alg. 1 to estimate $\hat{x}_t$ can be reformulated as three interconnected subsystems:

$$\text{Subsys. 1:} \begin{cases} x_{t+1} = Ax_t + B\pi(x_t + e_t) + w_t^1, \\ \quad y_t = Cx_t + w_t^2, \end{cases} \tag{14a}$$

$$\text{Subsys. 2:} \quad \epsilon_{t+1} = \bar{\Phi}_K(\epsilon_t, x_t, y_t, u_t, e_t), \tag{14b}$$

$$\text{Subsys. 3:} \quad e_{t+1} = \mathcal{E}(e_t, x_t, \epsilon_t), \tag{14c}$$

where $\epsilon_t := \|z_t^K - z_t^*\|$ and $e_t := \hat{x}_t - x_t$. Subsys. 1-3 describe the closed-loop controlled system, the sub-optimality error dynamics, and the estimation error dynamics, respectively. Fig. 1 illustrates the interconnections between them.

In subsys. 1, the controller $\pi(x_t)$ attempts to drive x_t to the origin. However, $\pi(x_t)$ is perturbed to $\pi(\hat{x}_t^K)$ by e_t. In subsys. 2, $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ is solved for K iterations to drive the sub-optimal solution z_{t-1}^K to the optimal solution z_t^* (the sub-optimality error ϵ_t to zero.) In subsys. 3, the MHE attempts to drive the estimation error to zero. This process is disturbed by the change in state x_t and the sub-optimality error ϵ_t. The stability of the interconnected system (14) can be analyzed via the small-gain theorem [13], which first requires each subsystem to be ISS. Note that subsys. 1 in (14a) meets this requirement via Assumption 4.

C. ISS of the Sub-Optimality Error Dynamics (Subsys. 2)

To prove the sub-optimality error dynamics is ISS, we first show the difference between two consecutive optimal solutions z_{t-1}^* and z_t^* is bounded as the problem parameters change, which include $x_{t-M_t}^{\text{prior}}, \mathbf{u}_t$ and $\mathbf{y}_t$.

Lemma 1 *Suppose Assumptions 1-2 hold. Then, there exists a Lipschitz constant $L_\Phi > 1$ such that the optimal solutions of $\mathbb{P}_{t-1}(x_{t-1-M_{t-1}}^{\text{prior}}, \mathbf{u}_{t-1}, \mathbf{y}_{t-1})$, $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ satisfy*

$$\|\Sigma_{t-1} z_{t-1}^* - z_t^*\| \leq L_\Phi(\|\tilde{z}_{t-1} - \Sigma_{t-1}^\top \tilde{z}_t\| + \tilde{\mathbf{u}}_t + \sigma_t), \tag{15}$$

where $\tilde{\mathbf{u}}_t = \|u_{t-1}\|$ for $t \leq M$ and $\tilde{\mathbf{u}}_t = \sum_{i=0}^{M-1} \|u_{t-1-i} - u_{t-2-i}\|$ for $t > M$, and $\sigma_t = (\eta^{-1}-1)\|H_t\| + \|A\| + \|B\| + \|C\| + 2$ for $t \leq M$ and $\sigma_t = 0$ for $t > M$.

Proof: *We prove* (15) *by treating $\mathbb{P}_t(\cdot)$ as a parametric optimization problem with a strongly convex cost function (due to Assumption 2), convex inequality constraints, and affine equality constraints (due to Assumption 1). For $t > M$, from Theorem 3.1 in [15], we know the optimal solution of $\mathbb{P}_t(\cdot)$ is Lipschitz continuous w.r.t the parameters, i.e., there exists a Lipschitz constant $L_\Phi > 1$ such that*

$$\|\Sigma_{t-1} z_{t-1}^* - z_t^*\| \leq L_\Phi(\|\tilde{z}_{t-1} - \Sigma_{t-1}^\top \tilde{z}_t\| + \tilde{\mathbf{u}}_t). \tag{16}$$

For $t \leq M$, we consider an equivalent expression of $\mathbb{P}_t(\hat{x}_0, \mathbf{u}_t, \mathbf{y}_t)$, given by $\mathbb{P}'_t(\hat{x}_0, \mathbf{u}_t, \mathbf{y}_t, H_t, A, B, \mathbf{I}^{n_x}, C, \mathbf{I}^{n_y})$, where H_t is from (10) *and the last five matrices are from the constraints* (8b) *and* (8c)*, with $i = t-1$, respectively. Let $\check{\mathbb{P}}_t := \mathbb{P}'_t(\hat{x}_0, \mathbf{u}_t, \mathbf{y}_t, \eta^{-1} H_t, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, with optimal solution $\check{z}_t^*$. From [15], there exists $L_\Phi > 1$ such that*

$$\|\check{z}_t^* - z_t^*\| \leq L_\Phi(\sigma_t + \tilde{\mathbf{u}}_t). \tag{17}$$

By replacing H_t with $\eta^{-1} H_t$ and $A, B, \mathbf{I}^{n_x}, C, \mathbf{I}^{n_y}$ with $\mathbf{0}$ in $\check{\mathbb{P}}_t$, we restrict $\hat{x}_{t|t}^* = 0$, $\hat{y}_{t-1|t}^* = 0$ in $\check{z}_t^*$, and enforce $\check{\mathbb{P}}_t$ and $\mathbb{P}_{t-1}(\hat{x}_0, \mathbf{u}_{t-1}, \mathbf{y}_{t-1})$ to have the same cost. As a result, $\check{z}_t^* = \Sigma_{t-1} z_{t-1}^*$. Then, replacing $\check{z}_t^*$ in (17) *gives*

$$\|\Sigma_{t-1} z_{t-1}^* - z_t^*\| \leq L_\Phi(\sigma_t + \tilde{\mathbf{u}}_t). \tag{18}$$

Lastly, we can combine (16) *and* (18) *to obtain* (15). ■

With the bound in (15), we can show subsys. 2 is ISS:

Proposition 1 *Consider $\mathbb{P}_t(x_{t-M_t}^{\text{prior}}, \mathbf{u}_t, \mathbf{y}_t)$ solved by an optimization algorithm $\Phi_K(z_t^0, \mathbb{P}_t)$, with warm-start solution $z_{t+1}^0 \leftarrow \Sigma_t z_t^K$. Suppose Assumptions 1-5 hold. For $t > 0$, the sub-optimality error ϵ_t satisfies*

$$\begin{aligned} \|\epsilon_t\| \leq & \beta_2(\|\epsilon_0\|, t) + \gamma_{2,1}(\|\mathbf{x}_{[0,t-1]}\|) + \gamma_{2,3}(\|\mathbf{e}_{[0,t-1]}\|) \\ & + \gamma_2^w(\|\mathbf{w}_{[0,t-1]}\|) + \gamma_2^\sigma(\|\boldsymbol{\sigma}_{[0,t-1]}\|), \end{aligned} \tag{19}$$

where $\beta_2(s,t) := \phi(K)^t s$, $\gamma_{2,1}(s) := C_1 \phi(K)/(1-\phi(K))$ s, $\gamma_{2,3}(s) := C_2 \phi(K)/(1-\phi(K)) s$, $\gamma_2^w(s) := C_3$ $\phi(K)/(1-\phi(K)) s$, and $\gamma_3^\sigma(s) := \phi(K) L_\Phi/(1-\phi(K)) s$, with C_1, C_2, C_3 defined in (25)-(26).

D. ISS of the Estimation Error Dynamics (Subsys. 3)

Inspired by [14], we first construct an M-step Lyapunov function for (14c) based on $W_\delta(\cdot)$ defined in (3).

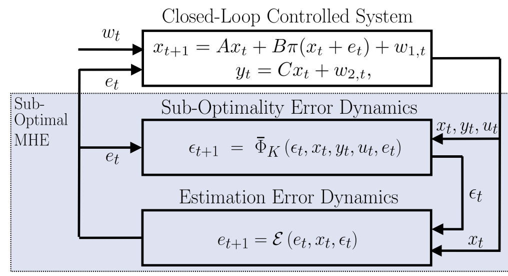

Fig. 1: The interconnection of three subsystems.

Proposition 2 *Suppose Assumptions 1-5 hold. For $t \geq 0$, the state estimate $\hat{x}_t^K$ satisfies*

$$W_\delta(\hat{x}^K_{t|t}, x_t) \leq 6\eta^{M_t} W_\delta(\hat{x}^K_{t-M_t|t-M_t}, x_{t-M_t}) + 2\bar{H}\|\epsilon_t\|^2 + 6\sum_{j=1}^{M_t} \eta^{j-1}\|w_{t-j}\|_Q^2. \quad (20)$$

Based on the M-step Lyapunov function in (20), we show the estimation error dynamics is ISS.

Proposition 3 *Suppose Assumptions 1-5 hold. Then, the estimation error dynamics is ISS and e_t satisfies*

$$\|e_t\| \leq \beta_3(\|e_0\|, t) + \gamma_{3,1}(\|\mathbf{x}_{[0,t-1]}\|) + \gamma_{3,2}(\|\boldsymbol{\epsilon}_{[0,t-1]}\|) + \gamma_3^w(\|\mathbf{w}_{[0,t-1]}\|) + \gamma_3^\sigma(\|\boldsymbol{\sigma}_{[0,t-1]}\|), \quad (21)$$

where $\beta_3(s,t) := C_e(K)\sqrt{\rho}^t s$, $\gamma_{3,1}(s) := \sqrt{2\Lambda_P^{\bar{H}} C_1 \phi(K)s}$, $\gamma_{3,2}(s) := C_\epsilon(K)s$, $\gamma_3^w(s) := C_w(K)s$, and $\gamma_3^\sigma(s) := \sqrt{2\Lambda_P^{\bar{H}}\phi(K)L_\Phi s}$, with ρ satisfying $\rho^M = 6\eta^M$ and $C_e(K)$, $C_w(K)$, and $C_\epsilon(K)$ defined in (27)-(29).

Since K is computed offline, we slightly abuse the notation and drop the K-dependence of γ-functions in (19) and (21).

E. Stability of the Interconnected System

We now apply the small-gain theorem to determine conditions on the optimization iteration number K for guaranteeing (14) in Fig. 1 to be ISS w.r.t external disturbances.

Theorem 1 *Suppose Assumptions 1-5 hold. Let T_1, T_2, T_3 be defined in (30)-(32). The interconnected system (14) is ISS w.r.t. the external disturbance w_t, if*

$$K \geq \lceil \phi^{-1}(\min(T_1, T_2, T_3)) \rceil. \quad (22)$$

Proof: Given Assumptions 1-5 hold, Propositions 1, 3, and Assumption 4, show that subsys. 1-3 are ISS, admitting (7), (21), (19), for all $t \geq 0$, respectively. Since $\phi(K) \in \mathcal{L}$ is invertible and $T_1, T_2, T_3 > 0$, choosing a K value satisfying (22) leads to $\phi(K) < \min(T_1, T_2, T_3)$, which guarantees that the small-gain condition holds for all loops in Fig. 1:

$$\gamma_{1,3} \cdot \gamma_{3,1}(s) < s, \;\; \gamma_{2,3} \circ \gamma_{3,2}(s) < s, \qquad \gamma_{1,3} \cdot \gamma_{3,2} \circ \gamma_{2,1}(s) < s, \quad s > 0. \quad (23)$$

Note we treated the last two conditions as quadratic functions and used the quadratic formula to derive T_2 and T_3. Since all subsystems are ISS and the small-gain conditions in (23) hold, we know from Remark 1 of [13] that (14) is ISS w.r.t external disturbance w_t. ■

IV. Case Study with an MHE-MPC

To demonstrate Alg. 1 and the theoretical findings, we consider the discrete-time linear system and the corresponding MPC controller in the case study of [11]. We add an output matrix $C = [0.1, 0.3, 0.8, 0.5]$ to the system such that the system is observable. The state $x \in \mathbb{R}^4$ and measurement $y \in \mathbb{R}$ are unconstrained, and the input $u \in [-1,1] \times [-1,1]$. Each element of the disturbance vector w_t is sampled independently and uniformly from $[-0.1, 0.1]$. We found $\gamma_{1,3}(s) := 28.8s$, through the method used in Proposition 2 of [16], and $L_\pi = 2.65$, through a sample-based method.

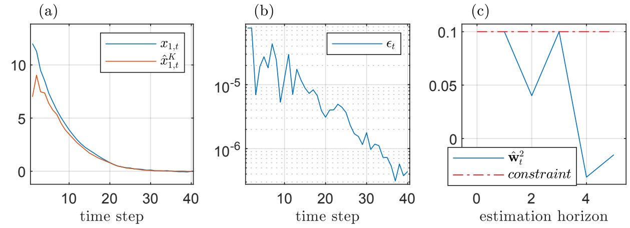

Fig. 2: (a) True state vs. Sub-optimal estimate; (b) Change in sub-optimality error; (c) The estimated measurement noise obtained from solving the MHE problem at $t = 6$.

The parameters of the MHE problem in (8) are $M = 5$, $Q = \mathbf{I}^4$, $R = 1$, and $\eta = 0.8$, with P computed to satisfy (2). Problem (8) is written in a condensed form and solved using the partial gradient method [11] with convergence rate $\|z_t^K - z_t^*\| \leq 0.98^K \|z_t^0 - z_t^*\|$. Accordingly, we define $\phi(K) := 0.98^K$. The Lipschitz constant $L_\Phi = 5.32$ is determined through a sample-based method. Finally, the iteration number $K = 652$ is computed, which satisfies (23) with the previously defined parameters.

Given an initial state $x_0 = [12, -10, 10, -10]^\top$, $z_0^0 = x_0^{\text{prior}} = [7, -7, 3, -5]^\top$, and empty sequences $\mathbf{y}_0$ and $\mathbf{u}_0$, Alg. 1 is applied for 40 time steps. Fig. 2(a) shows the state $x_{1,t}$ converges asymptotically to a neighbourhood of 0 and the sub-optimal estimate $\hat{x}_{1,t}^K$ converges asymptotically to a neighbourhood of $x_{1,t}$. Fig. 2(b) shows that the sub-optimality error ϵ_t converges asymptotically to a neighbourhood of 0. Thus, subsystems 1-3 defined in (14a)-(14c) are ISS. Fig. 2(c) shows the estimated measurement noise sequence $\hat{\mathbf{w}}_t^{2,K}$ obtained from solving (8) at time step $t = 6$, which respects the constraint (in red) by the design of MHE.

V. Conclusion

In this work, we proposed a sub-optimal MHE scheme applied to the control of linear systems with constraints. By characterizing Alg. 1 as three interconnected subsystems, we derived conditions on the optimization iteration number for guaranteeing ISS of the interconnected system w.r.t. to external disturbances. A possible extension is to consider the stability of systems controlled by a sub-optimal MPC-MHE pair in applications with limited computation resources.

VI. Appendix

We define some terms here for clarity:

$$c := 2(3\Lambda_P^P \Lambda_P^{\bar{H}})^{1/2} L_\Phi \Big(2L_\pi \sum_{i=1}^{M-1} \sqrt{\rho}^{-1-i} + (\sqrt{\rho}^{-M} + L_\pi \sqrt{\rho}^{-1}) + (L_\pi + 1)\sqrt{\rho}^{-M-1}\Big), \quad (24)$$

$$C_1 := 2L_\Phi(1 + M(\|C\| + L_\pi)), \quad (25)$$

$$C_2 := 2L_\Phi(1 + ML_\pi), \quad C_3 := 2L_\Phi M, \quad (26)$$

$$C_e(K) := c\phi(K) + (6\Lambda_P^P)^{1/2}, \quad (27)$$

$$C_w(K) := (2\Lambda_P^{\bar{H}})^{1/2} C_3 \phi(K) + (6\Lambda_P^Q)^{1/2}(1 - \sqrt{\rho})^{-1} + 4(3\Lambda_P^{\bar{H}}\Lambda_P^Q)^{1/2}\phi(K) L_\Phi (L_\pi M + 1)(1 - \sqrt{\rho})^{-1}, \quad (28)$$

$$C_\epsilon(K) := (2\Lambda_P^{\bar{H}})^{1/2}\phi(K) + (2\Lambda_P^{\bar{H}})^{1/2}(1 - \sqrt{\rho^M})^{-1}$$

$$+ 4\Lambda_P^{\bar{H}}\phi(K)L_\Phi\left(L_\pi M+1\right)(1-\sqrt{\rho^M})^{-1}, \quad (29)$$

$$T_1 := 1/\gamma_{1,3}(C_1(2\Lambda_P^{\bar{H}})^{1/2})^{-1}, \quad (30)$$

$$T_2 := ((1+6C_2^2\Lambda_P^P-2C_2(6\Lambda_P^P)^{1/2}+4C_2c)^{1/2} +C_2(6\Lambda_P^P)^{1/2}-1)/(2C_2c), \quad (31)$$

$$T_3 := (((1/\gamma_{1,2}^2+4C_1C_2(2\Lambda_P^{\bar{H}})^{1/2}\gamma_{1,2}^{-1}(1))^{1/2} -\gamma_{1,2}^{-1}(1))/(2C_1C_2(2\Lambda_P^{\bar{H}})^{1/2}). \quad (32)$$

Proof of Proposition 1: We break the proof into two cases. **Case 1**: For $t \le M$, we have

$$\|z_t^0 - z_t^*\| \overset{\text{Step 2}}{=} \|\Sigma_{t-1}z_{t-1}^K - z_t^*\| \quad (33)$$
$$\le \|\Sigma_{t-1}z_{t-1}^K - \Sigma_{t-1}z_{t-1}^*\| + \|\Sigma_{t-1}z_{t-1}^* - z_t^*\| \quad (34)$$
$$\overset{(15)}{\le} \|z_{t-1}^K - z_{t-1}^*\| + L_\Phi(\sigma_t + \tilde{\mathbf{u}}_t), \quad (35)$$

where $\|\Sigma_{t-1}\| = 1$ was used in (35). By multiplying $\phi(K)$ on both sides of the above inequality and using (12), we have

$$\|\epsilon_t\| \le \phi(K)(\|\epsilon_{t-1}\| + L_\Phi(\|\boldsymbol{\sigma}_{[0,t-1]}\| + \|u_{t-1}\|)), \quad (36)$$

where $\tilde{\mathbf{u}}_t = \|u_{t-1}\|$. Furthermore, $||\sigma_t|| \le ||\boldsymbol{\sigma}_{[0,t-1]}||$ since $\|H_t\| = \max(2\|Q\|, \|R\|, 2\eta^{M_t}\|P\|) \le \max(2\|Q\|, \|R\|, 2\eta^{M_{t-1}}\|P\|) = \|H_{t-1}\|$ for $t \le M$.
Case 2: For $t > M$, we have

$$\|z_t^0 - z_t^*\| \overset{\text{Step 2}}{\le} \|z_{t-1}^K - z_{t-1}^*\| + \|z_{t-1}^* - z_t^*\| \quad (37)$$
$$\overset{(15)}{\le} \|z_{t-1}^K - z_{t-1}^*\| + L_\Phi(\|\tilde{z}_{t-1} - \tilde{z}_t\| + \tilde{\mathbf{u}}_t) \quad (38)$$
$$\le L_\Phi \sum_{i=0}^{M_{t-1}} (\|u_{t-1-i} - u_{t-2-i}\| + \|y_{t-1-i} - y_{t-2-i}\|)$$
$$+ L_\Phi\|\hat{x}_{t-M|t-M}^K - \hat{x}_{t-M-1|t-M-1}^K\| + \|\epsilon_{t-1}\|, \quad (39)$$

where we used $x_{t-M}^{\text{prior}} = \hat{x}_{t-M|t-M}^K$ and $x_{t-M-1}^{\text{prior}} = \hat{x}_{t-M-1|t-M-1}^K$ in (39). Given the above inequality, we can bound $\|\hat{x}_{t-M|t-M}^K - \hat{x}_{t-M-1|t-M-1}^K\|$ with

$$\|\hat{x}_{t-M|t-M}^K - \hat{x}_{t-M-1|t-M-1}^K\|$$
$$= \|(x_{t-M} + e_{t-M}) - (x_{t-M-1} + e_{t-M-1})\| \quad (40)$$
$$\le \|x_{t-M_t}\| + \|x_{t-M_t-1}\| + \|e_{t-M_t}\| + \|e_{t-M_t-1}\|, \quad (41)$$

bound $\|u_{t-1-i} - u_{t-2-i}\|$ with

$$\|u_{t-1-i} - u_{t-2-i}\| \overset{(6)}{\le} L_\pi\|\hat{x}_{t-1-i}^K - \hat{x}_{t-2-i}^K\|$$
$$\le L_\pi(\|x_{t-1-i}\| + \|x_{t-2-i}\| + \|e_{t-1-i}\| + \|e_{t-2-i}\|), \quad (42)$$

and bound $\|y_{t-1-i} - y_{t-2-i}\|$ with

$$\|y_{t-1-i} - y_{t-2-i}\| \le \|w_{t-1-i}\| + \|w_{t-2-i}\| + \|C\|\|x_{t-1-i}\| + \|C\|\|x_{t-2-i}\|. \quad (43)$$

Using the resulting bound to replace the term $\|z_t^0 - z_t^*\|$ on the r.h.s. of (12), we have that

$$\|\epsilon_t\| \le \phi(K)\|\epsilon_{t-1}\| + C_1\phi(K)\|\mathbf{x}_{[0,t-1]}\|$$
$$+ \phi(K)L_\Phi\left(\|e_{t-M}\| + \|e_{t-M-1}\|\right) + C_3\phi(K)\|\mathbf{w}_{[0,t-1]}\|$$
$$+ \phi(K)L_\Phi \sum_{i=0}^{M-1} \left(L_\pi(\|e_{t-1-i}\| + \|e_{t-2-i}\|)\right). \quad (44)$$

where $\|x_j\|$ and $\|w_j\|$, $j < t$, are bounded with $\|\mathbf{x}_{[0,t-1]}\|$ and $\|\mathbf{w}_{[0,t-1]}\|$, respectively. Next, bounding $\|e_j\|, j < t$ in (44) with $\|\mathbf{e}_{[0,t-1]}\|$ gives

$$\|\epsilon_t\| \le \phi(K)\|\epsilon_{t-1}\| + C_1\phi(K)\|\mathbf{x}_{[0,t-1]}\|$$
$$+ C_2\phi(K)\|\mathbf{e}_{[0,t-1]}\| + C_3\phi(K)\|\mathbf{w}_{[0,t-1]}\|. \quad (45)$$

Like (42), we can apply $\|u_{t-1}\| \le L_\pi(\|x_{t-1}\| + \|e_{t-1}\|)$ in (36). Then, combining the r.h.s. of (36) and (45) gives

$$\|\epsilon_t\| \le \phi(K)\|\epsilon_{t-1}\| + C_1\phi(K)\|\mathbf{x}_{[0,t-1]}\| + C_2\phi(K)\|\mathbf{e}_{[0,t-1]}\|$$
$$+ C_3\phi(K)\|\mathbf{w}_{[0,t-1]}\| + \phi(K)L_\Phi\|\boldsymbol{\sigma}_{[0,t-1]}\|, \quad (46)$$

which holds for all $t > 0$. Finally, applying (46) for t times and applying the bound $\sum_{i=0}^{t-1}\phi(K)^{(t-1-i)} < \sum_{i=0}^{\infty}\phi(K)^{(t-1-i)} = 1/(1-\phi(K))$ give (19). ■

Proof of Proposition 2: We first derive an intermediate bound on $W_\delta(\hat{x}_{t|t}^K, x_t)$. Due to Assumption 5, the sub-optimal solution $(\hat{\mathbf{x}}_t^K, \hat{\mathbf{y}}_t^K, \hat{\mathbf{w}}_t^K)$ is feasible for (8) and forms a feasible trajectory of the system in (1). Given the actual trajectory $(\mathbf{x}_{[t-M,t]}, \mathbf{y}_{[t-M,t-1]}, \mathbf{w}_{[t-M,t-1]})$, we can apply the bound in (4) for M_t times to obtain

$$W_\delta(\hat{x}_{t|t}^K, x_t) \le \eta^{M_t} W_\delta(\hat{x}_{t-M_t|t}^K, x_{t-M_t})$$
$$+ \sum_{j=1}^{M_t} \eta^{j-1}(\|\hat{w}_{t-j|t}^K - w_{t-j}\|_Q^2 + \|\hat{y}_{t-j|t}^K - y_{t-j}\|_R^2) \quad (47)$$
$$\le 2\eta^{M_t}\|\hat{x}_{t-M_t|t}^K - \hat{x}_{t-M_t|t-M_t}^K\|_P^2$$
$$+ 2\eta^{M_t}\|\hat{x}_{t-M_t|t-M_t}^K - x_{t-M_t}\|_P^2 + \sum_{j=1}^{M_t}\eta^{j-1}2\|w_{t-j}\|_Q^2$$
$$+ \sum_{j=1}^{M_t}\eta^{j-1}(\|\hat{y}_{t-j|t}^K - y_{t-j}\|_R^2 + 2\|\hat{w}_{t-j|t}^K\|_Q^2) \quad (48)$$
$$\le 2\eta^{M_t}W_\delta(\hat{x}_{t-M_t|t-M_t}^K, x_{t-M_t}) + \sum_{j=1}^{M_t}\eta^{j-1}2\|w_{t-j}\|_Q^2$$
$$+ V_{\text{MHE}}(\hat{x}_{t-M_t|t}^K, \hat{\mathbf{w}}_t^K, \hat{\mathbf{y}}_t^K) \quad (49)$$

where (48) is obtained by applying $\|a+b\|^2 \le 2(\|a\|^2 + \|b\|^2)$ to $W_\delta(\hat{x}_{t-M_t|t}^K, x_{t-M_t})$ and $\|\hat{w}_{t-j|t}^K - w_{t-j}\|_Q^2$. Next, we derive a bound on $V_{\text{MHE}}(\hat{x}_{t-M_t|t}^K, \hat{\mathbf{w}}_t^K, \hat{\mathbf{y}}_t^K)$. We know

$$V_{\text{MHE}}(\hat{x}_{t-M_t|t}^K, \hat{\mathbf{w}}_t^K, \hat{\mathbf{y}}_t^K) = \|z_t^K - \tilde{z}_t\|_{H_t}^2 \quad (50)$$
$$\le 2\|z_t^K - z_t^*\|_{H_t}^2 + 2\|z_t^* - \tilde{z}_t\|_{H_t}^2 \quad (51)$$
$$\le 2\|\epsilon_t\|_{H_t}^2 + 2V_{\text{MHE}}(\hat{x}_{t-M_t|t}^*, \hat{\mathbf{w}}_t^*, \hat{\mathbf{y}}_t^*) \quad (52)$$
$$\le 2\|\epsilon_t\|_{H_t}^2 + 2V_{\text{MHE}}(x_{t-M_t}, \mathbf{w}_{[t-M_t,t-1]}, \mathbf{y}_{[t-M_t,t-1]})$$

where the last inequality holds since $(\mathbf{x}_{[t-M_t,t]}, \mathbf{w}_{[t-M_t,t-1]}, \mathbf{y}_{[t-M_t,t-1]})$ forms a sub-optimal solution to (8). Using the above bound with (49) and then using (9) give

$$W_\delta(\hat{x}_{t|t}^K, x_t) \le 2\eta^{M_t}W_\delta(\hat{x}_{t-M_t|t-M_t}^K, x_{t-M_t})$$
$$+ \sum_{j=1}^{M_t}\eta^{j-1}2\|w_{t-j}\|_Q^2 + 2\|\epsilon_t\|_{H_t}^2$$
$$+ 2V_{\text{MHE}}(x_{t-M_t}, \mathbf{w}_{[t-M_t,t-1]}, \mathbf{y}_{[t-M_t,t-1]}) \quad (53)$$
$$= \sum_{j=1}^{M_t}\eta^{j-1}(6\|w_{t-j}\|_Q^2 + 2\left\|y_{t-j} - y_{t-j}\right\|_R^2)$$

$$+6\eta^{M_t} W_\delta(\hat{x}^K_{t-M_t|t-M_t}, x_{t-M_t}) + 2\|\epsilon_t\|^2_{H_t}. \quad (54)$$

Using $\|\epsilon_t\|^2_{H_t} \leq \overline{\lambda}(H_t)\|\epsilon_t\|^2 \leq \bar{H}\|\epsilon_t\|^2$ in (55) gives (20). ■

Proof of Proposition 3: Let $t = cM + l$, with $l \in \mathbb{I}_{[0,M-1]}$ and $c \in \mathbb{I}_{\geq 0}$. At time step l, plugging $t = l$ into (20) gives

$$W_\delta(\hat{x}^K_{l|l}, x_l) \leq 6\eta^l W_\delta(\hat{x}^K_{0|0}, x_0) + 2\bar{H}\|\epsilon_l\|^2 + 6\sum_{j=1}^{l} \eta^{j-1}\|w_{l-j}\|^2_Q. \quad (55)$$

At t, using (20) for c times and ρ^M to replace $6\eta^M$ give

$$W_\delta(\hat{x}^K_{t|t}, x_t) \leq \rho^{cM} W_\delta(\hat{x}^K_{l|l}, x_l) + 2\bar{H}\sum_{i=0}^{c-1} \rho^{iM}\|\epsilon_{t-iM}\|^2 + 6\sum_{i=0}^{c-1}\rho^{iM}\sum_{j=1}^{M}\eta^{j-1}\|w_{t-iM-j}\|^2_Q \quad (56)$$
$$\overset{(55)}{\leq} \rho^{cM} 6\eta^l W_\delta(\hat{x}^K_{0|0}, x_0) + 6\rho^{cM}\sum_{j=1}^{l}\eta^{j-1}\|w_{t-cM-j}\|^2_Q + 6\sum_{i=0}^{c-1}\rho^{iM}\sum_{j=1}^{M}\eta^{j-1}\|w_{t-iM-j}\|^2_Q + 2\bar{H}\sum_{i=0}^{c}\rho^{iM}\|\epsilon_{t-iM}\|^2.$$

Then, using $\rho = 6^{\frac{1}{M}}\eta > \eta$ and $\rho^{cM}\rho^l = \rho^t$ yields

$$W_\delta(\hat{x}^K_{t|t}, x_t) \leq 6\rho^t W_\delta(\hat{x}^K_{0|0}, x_0) + 2\bar{H}\sum_{i=0}^{c}\rho^{iM}\|\epsilon_{t-iM}\|^2 + 6\sum_{j=0}^{t-1}\rho^j\|w_{t-j-1}\|^2_Q. \quad (57)$$

Next, using $\underline{\lambda}(P)\|e_t\|^2 \leq W_\delta(\hat{x}^K_{t|t}, x_t)$ on the l.h.s. of (57), and $W_\delta(\hat{x}^K_{0|0}, x_0) \leq \overline{\lambda}(P)\|e_0\|^2$ and $\|w_t\|^2_Q \leq \overline{\lambda}(Q)\|w_t\|^2$ on the r.h.s. of (57), and dividing both sides by $\underline{\lambda}(P)$ give

$$\|e_t\|^2 \leq 6\Lambda^Q_P\sum_{j=0}^{t-1}\rho^j\|w_{t-j-1}\|^2 + 2\Lambda^{\bar{H}}_P\sum_{i=1}^{c}\rho^{iM}\|\epsilon_{t-iM}\|^2 + 2\Lambda^{\bar{H}}_P\|\epsilon_t\|^2 + 6\rho^t\Lambda^P_P\|e_0\|^2. \quad (58)$$

Using $\|w_{t-j-1}\| \leq \|\mathbf{w}_{[0,t-1]}\|$, $\|\epsilon_{t-iM}\| \leq \|\boldsymbol{\epsilon}_{[0,t-1]}\|$, and opening square on both sides using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ gives

$$\|e_t\| \leq \sqrt{6\Lambda^P_P}\sqrt{\rho}^t\|e_0\| + \sqrt{6\Lambda^Q_P}(1-\sqrt{\rho})^{-1}\|\mathbf{w}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}(1-\sqrt{\rho^M})^{-1}\|\boldsymbol{\epsilon}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}\|\epsilon_t\|, \quad (59)$$

where $\sum_{j=0}^{t-1}\sqrt{\rho}^j < \sum_{j=0}^{\infty}\sqrt{\rho}^j = (1-\sqrt{\rho})^{-1}$ and $\sum_{i=1}^{c}\sqrt{\rho^M}^i < \sum_{i=1}^{\infty}\sqrt{\rho^M}^i = (1-\sqrt{\rho^M})^{-1}$ were used. To eliminate $\|\epsilon_t\|$ in (59), we consider two cases:

Case 1: For $t \leq M$, $\|\epsilon_t\|$ can be bounded by (36) to obtain

$$\|e_t\| \leq \sqrt{6\Lambda^P_P}\sqrt{\rho}^t\|e_0\| + \sqrt{6\Lambda^Q_P}(1-\sqrt{\rho})^{-1}\|\mathbf{w}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}((1-\sqrt{\rho^M})^{-1} + \phi(K))\|\boldsymbol{\epsilon}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}\phi(K)L_\Phi\|\boldsymbol{\sigma}_{[0,t-1]}\|. \quad (60)$$

where the resulting $\|\epsilon_0\|$ is bounded by $\|\boldsymbol{\epsilon}_{[0,t-1]]}\|$.

Case 2: For $t > M$, $\|\epsilon_t\|$ can be bounded by (44) to obtain

$$\|e_t\| \leq \sqrt{6\Lambda^P_P}\sqrt{\rho}^t\|e_0\| + \sqrt{2\Lambda^{\bar{H}}_P}C_1\phi(K)\|\mathbf{x}_{[0,t-1]}\| + (\sqrt{6\Lambda^Q_P}(1-\sqrt{\rho})^{-1} + \sqrt{2\Lambda^{\bar{H}}_P}C_3\phi(K))\|\mathbf{w}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}((1-\sqrt{\rho^M})^{-1} + \phi(K))\|\boldsymbol{\epsilon}_{[0,t-1]}\| + \sqrt{2\Lambda^{\bar{H}}_P}\phi(K)L_\Phi\sum_{i=0}^{M-1}(L_\pi(\|e_{t-1-i}\| + \|e_{t-2-i}\|)) + \sqrt{2\Lambda^{\bar{H}}_P}\phi(K)L_\Phi(\|e_{t-M}\| + \|e_{t-M-1}\|). \quad (61)$$

Bounding $\|e_{t-M}\|$, $\|e_{t-M-1}\|$, $\|e_{t-1-i}\|$, $\|e_{t-2-i}\|$, $i \in [0, M-1]$ with (59) and simplifying expression give

$$\|e_t\| \leq C_e(K)\sqrt{\rho}^t\|e_0\| + \sqrt{2\Lambda^{\bar{H}}_P}C_1\phi(K)\|\mathbf{x}_{[0,t-1]}\| + C_\epsilon(K)\|\boldsymbol{\epsilon}_{[0,t-1]}\| + C_w(K)\|\mathbf{w}_{[0,t-1]}\|. \quad (62)$$

Since $C_e(K) \geq \sqrt{6\Lambda^P_P}$, $C_\epsilon(K) \geq \sqrt{2\Lambda^{\bar{H}}_P}((1-\sqrt{\rho^M})^{-1} + \phi(K))$, and $C_w(K) \geq \sqrt{6\Lambda^Q_P}(1-\sqrt{\rho})^{-1}$, we can combine (60) and (62) to obtain (21), which holds for $t \geq 0$. ■

References

[1] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model predictive control : theory, computation, and design (2. ed.)*. Nob Hill, 2017.

[2] M. A. Müller, "Nonlinear moving horizon estimation in the presence of bounded disturbances," *Automatica*, vol. 79, pp. 306–314, 2017.

[3] R. Suwantong, S. Bertrand, D. Dumur, and D. Beauvois, "Stability of a nonlinear moving horizon estimator with pre-estimation," in *2014 ACC*, 2014, pp. 5688–5693.

[4] J. D. Schiller and M. A. Müller, "Suboptimal nonlinear moving horizon estimation," *IEEE Trans. Automat.*, vol. 68, no. 4, pp. 2199–2214, 2023.

[5] M. Gharbi, B. Gharesifard, and C. Ebenbauer, "Anytime proximity moving horizon estimation: Stability and regret for nonlinear systems," in *2021 60th IEEE CDC*, 2021, pp. 728–735.

[6] H. Kong and S. Sukkarieh, "Suboptimal receding horizon estimation via noise blocking," *Automatica*, vol. 98, pp. 66–75, 2018.

[7] A. Wynn, M. Vukov, and M. Diehl, "Convergence guarantees for moving horizon estimation based on the real-time iteration scheme," *IEEE Trans. Automat.*, vol. 59, no. 8, pp. 2215–2221, 2014.

[8] H. J. Ferreau, T. Kraus, M. Vukov, W. Saeys, and M. Diehl, "High-speed moving horizon estimation based on automatic code generation," in *2012 IEEE 51st IEEE CDC*, 2012, pp. 687–692.

[9] A. Alessandri and M. Gaggero, "Fast moving horizon state estimation for discrete-time systems using single and multi iteration descent methods," *IEEE Trans. Automat.*, vol. 62, no. 9, pp. 4499–4511, 2017.

[10] M. Ellis, J. Zhang, J. Liu, and P. D. Christofides, "Robust moving horizon estimation based output feedback economic model predictive control," *Systems and Control Letters*, vol. 68, pp. 101–109, 2014.

[11] D. Liao-McPherson, T. Skibik, J. Leung, I. Kolmanovsky, and M. M. Nicotra, "An analysis of closed-loop stability for linear model predictive control based on time-distributed optimization," *IEEE Trans. Automat.*, vol. 67, no. 5, pp. 2618–2625, 2022.

[12] Y. Yang, Y. Wang, C. Manzie, and Y. Pu, "Sub-optimal MPC with dynamic constraint tightening," *IEEE Control Systems Letters*, vol. 7, pp. 1111–1116, 2023.

[13] Y. L. Z-P. Jiang and Y. Wang, "Nonlinear small-gain theorems for discrete-time large-scale systems," in *IEEE CCC*, 2008, pp. 704–708.

[14] J. D. Schiller, S. Muntwiler, J. Köhler, M. N. Zeilinger, and M. A. Müller, "A lyapunov function for robust stability of moving horizon estimation," 2022. [Online]. Available: https://arxiv.org/abs/2202.12744

[15] W. W. Hager, "Lipschitz continuity for constrained processes," *SIAM J Control Optim.*, vol. 17, no. 3, pp. 321–338, 1979.

[16] Y. Yang, Y. Wang, C. Manzie, and Y. Pu, "Real-time distributed model predictive control with limited communication data rates," 2022. [Online]. Available: https://arxiv.org/abs/2208.12531