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Target Controllability and Target Observability of Structured Network Systems

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***Abstract*— The duality between controllability and observability enables methods developed for full-state control to be applied to full-state estimation, and vice versa. In applications in which control or estimation of all state variables is unfeasible, the generalized notions of output controllability and functional observability establish the minimal conditions for the control and estimation of a target subset of state variables, respectively. Given the seemly unrelated nature of these properties, thus far methods for target control and target estimation have been developed independently in the literature. Here, we characterize the graph-theoretic conditions for target controllability and target observability (which are, respectively, special cases of output controllability and functional observability for structured systems). This allow us to rigorously establish a weak and strong duality between these generalized properties. When both properties are equivalent (strongly dual), we show that efficient algorithms developed for target controllability can be used for target observability, and vice versa, for the optimal placement of sensors and drivers. These results are applicable to large-scale networks, in which control and monitoring are often sought for small subsets of nodes.**

I. Introduction

Controllability and observability are properties that respectively enable full-state control and full-state estimation of a dynamical system. The duality between these properties allow methods developed for feedback controller design to be used for observer design, and vice versa. Beyond classical techniques for pole placement in feedback systems, this duality also finds important applications in optimal control theory [1] and decentralized control of networked systems [2]. In the context of complex networks, the pressing problem of optimally placing actuators and sensors to ascertain full-state control and monitoring can be solved by a single efficient algorithm [3] due to the duality between the graph-theoretic notions of *structural* controllability and *structural* observability [4].

Full-state control and estimation are, however, often unfeasible or unneeded in high-dimensional applications such as large-scale networks [5], [6]. Physical, cost, and energy constraints in the placement and operation of actuators and sensors often limit our ability to fully control or observe a network [7]–[9]. To circumvent these limitations, the generalized notions of output controllability [10] and functional observability [11] establish the minimal conditions under which part of the state vector (e.g., a *target* subset of state variables) can be controlled and estimated, rather than the full-state vector. These properties enable the control and estimation of target nodes in networks while requiring substantially less resources [12], [13].

The output controllability of a system does not imply in general the functional observability of the dual (transposed) system, which is in contrast with the classical duality between controllability and observability. Consequently, these generalized properties have been studied separately up until now, leading to the independent development of methods for target/output control [14]–[16], functional observer design [11], [17], and actuator/sensor placement [12], [13], [18]–[21]. Yet, a rigorous relation has been recently established between these properties, as characterized by the principles of weak and strong duality [22]. In particular, the weak duality establishes that the functional observability of a system implies the output controllability of the dual system, whereas the strong duality establishes that under a particular condition the converse also holds and both properties become equivalent. This opens an opportunity for methods developed for output controllability problems to be mapped to functional observability problems, and vice versa.

In this letter, we establish a graph-theoretic characterization of the weak and strong duality principles between target controllability and target observability, which are special notions of output controllability and functional observability for structured systems (Section III). To this end, we also derive the graph-theoretic conditions for target controllability (Theorem 2), which have been so far restricted to special classes of systems in the literature (Remark 2). As an application of our results, we show that, when strong duality holds, the proposed graph-theoretic characterization enables the use of scalable algorithms to solve both optimal driver and optimal sensor placement in large-scale networks (Section IV). The efficacy of our methods in large networks is numerically demonstrated using the *C. elegans* neural network.

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II. Preliminaries

Consider the linear time-invariant dynamical system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{1}$$

$$\boldsymbol{y} = C\boldsymbol{x}, \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^p$ is the input vector, $\boldsymbol{y} \in \mathbb{R}^q$ is the output vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times p}$ is the input matrix, and $C \in \mathbb{R}^{q\times n}$ is the output matrix. The linear function of the state variables

$$\boldsymbol{z} = F\boldsymbol{x} \tag{3}$$

defines the *target vector* $\boldsymbol{z} \in \mathbb{R}^r$ sought to be controlled or estimated ($r \leq n$), where $F \in \mathbb{R}^{r\times n}$ is the functional matrix.

The system (1)–(3) or, equivalently, the triple $(A, B; F)$ is *output controllable* if, for any initial state $\boldsymbol{x}(0)$ and target state $\boldsymbol{z}^* \in \mathbb{R}^r$, there exists an input $\boldsymbol{u}(t)$ that steers $\boldsymbol{x}(0)$ to some final state $\boldsymbol{x}(t_1)$ satisfying $\boldsymbol{z}(t_1) = F\boldsymbol{x}(t_1) = \boldsymbol{z}^*$ in finite time $t \in [0, t_1]$ [10]. A sufficient and necessary condition for this property is given by [10]

$$\operatorname{rank}(F\mathcal{C}) = \operatorname{rank}(F), \tag{4}$$

where $\mathcal{C} = [B\ AB\ \ldots\ A^{n-1}B]$ is the controllability matrix. Despite the terminology "output" controllability, note that condition (4) is defined for any functional F, which is not necessarily related to the output matrix C; whether the target variables $\boldsymbol{z}_i(t)$ sought to be controlled are monitored (e.g., measured or estimated) or not depends on the feedback/feedforward control application under consideration.

Moreover, the system (1)–(3) or the triple $(C, A; F)$ is *functionally observable* if, for any unknown initial state $\boldsymbol{x}(0)$, there exists a finite time $t_1 > 0$ such that knowledge of the output $\boldsymbol{y}(t)$ and input $\boldsymbol{u}(t)$ over $t \in [0, t_1]$ suffices to uniquely determine the target state $\boldsymbol{z}(0) = F\boldsymbol{x}(0)$. A sufficient and necessary condition is given by [23]

$$\operatorname{rank}\left(\begin{bmatrix}\mathcal{O}\\ F\end{bmatrix}\right) = \operatorname{rank}(\mathcal{O}), \tag{5}$$

where $\mathcal{O} = [C^\mathsf{T}\ (CA)^\mathsf{T}\ \ldots\ (CA^{n-1})^\mathsf{T}]^\mathsf{T}$ is the observability matrix. Here, assume that $\operatorname{rank}[C^\mathsf{T}\ F^\mathsf{T}]^\mathsf{T} = \operatorname{rank}(C) + \operatorname{rank}(F)$; otherwise, $\boldsymbol{z}_i = \boldsymbol{\alpha}^\mathsf{T}\boldsymbol{y}$, for some i and $\boldsymbol{\alpha} \in \mathbb{R}^r$, allowing $\boldsymbol{z}_i$ to be trivially estimated without an observer.

In spite of the duality between the (full-state) observability of a system (C, A) and the controllability of the dual system $(C^\mathsf{T}, A^\mathsf{T})$, functional observability and output controllability are not dual properties in general when $\operatorname{rank}(F) < n$ [22]. To see this, consider a pair of dynamical systems $(C, A; F)$ and $(A^\mathsf{T}, C^\mathsf{T}; F)$, where $\mathcal{O}$ is the observability matrix of the former system and $\mathcal{C} = \mathcal{O}^\mathsf{T}$ is the controllability matrix of the latter. Note that condition (4) is equivalent to $\operatorname{rank}(F\mathcal{O}^\mathsf{T}) = \operatorname{rank}(F)$ for a triple $(A^\mathsf{T}, C^\mathsf{T}; F)$. Thus, it follows that any system $(C, A; F)$ that satisfies condition (5) also satisfies condition (4) for the dual $(A^\mathsf{T}, C^\mathsf{T}; F)$. The converse, however, is not always true. As a consequence, $(A^\mathsf{T}, C^\mathsf{T}; F)$ may be output controllable without necessarily implying that $(C, A; F)$ is functionally observable (see Example 1 below).

III. Target Controllability and Observability

We show that the relation and equivalence between output controllability and functional observability are characterized by the notions of weak and strong duality. This duality follows directly from an intuitive graph-theoretic representation of output controllability and functional observability, which allows us to explicitly leverage the structure of the system matrix A and its inputs, outputs, and target variables (defined by matrices B, C, and F, respectively). Before stating our results, we first define graph concepts for structured systems.

A. Structured systems and graph theory

A matrix $M \in \{0, \star\}^{m\times n}$ is a *structured matrix* if M_{ij} is either a fixed zero entry or an independent nonzero entry, denoted by a $\star$. A matrix $\tilde{M}$ is a numerical realization of M if real numbers are assigned to all nonzero entries of M.

The *inference graph* of a system (1)–(3) is denoted by $\mathcal{G}(A, B, C; F) = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}$ is the set of nodes, $\mathcal{E} = \mathcal{E}_\mathcal{X} \cup \mathcal{E}_\mathcal{U} \cup \mathcal{E}_\mathcal{Y}$ is the set of edges, and (A, B, C, F) are structured matrices. Nodes represent state variables $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$, inputs $\mathcal{U} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ (driver nodes), and outputs $\mathcal{Y} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q\}$ (sensor nodes). Let $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathcal{E}_\mathcal{X}$ (directed edge from $\boldsymbol{x}_j$ to $\boldsymbol{x}_i$) if $A_{ij} \neq 0$, $(\boldsymbol{x}_i, \boldsymbol{u}_j) \in \mathcal{E}_\mathcal{U}$ if $B_{ij} \neq 0$, and $(\boldsymbol{y}_i, \boldsymbol{x}_j) \in \mathcal{E}_\mathcal{Y}$ if $C_{ij} \neq 0$. The set of *target nodes* $\mathcal{T} \subseteq \mathcal{X}$ defines a set of state variables sought to be controlled or estimated, where $\boldsymbol{x}_j \in \mathcal{T}$ if $F_{ij} \neq 0$ for some i. The inference graph is denoted simply by $\mathcal{G}(A, B; F)$ and $\mathcal{G}(C, A; F)$ when considering the output controllability and functional observability of a triple, respectively.

A subset of nodes $\mathcal{V}' \subseteq \mathcal{X}$ has a *dilation* in a graph $\mathcal{G}$ if $|P(\mathcal{V}')| < |\mathcal{V}'|$, where $|\cdot|$ denotes the set cardinality and $P(\mathcal{V}')$ is the set of all nodes $v_i \in \mathcal{X} \cup \mathcal{U}$ that have a direct link to $\mathcal{V}'$ (i.e., the set of predecessors of $\mathcal{V}'$). Similarly, $\mathcal{V}' \subseteq \mathcal{X}$ has a *contraction* in $\mathcal{G}$ if $|S(\mathcal{V}')| < |\mathcal{V}'|$, where $S(\mathcal{V}')$ is the set of all nodes $v_i \in \mathcal{X} \cup \mathcal{Y}$ that have a direct link from $\mathcal{V}'$ to v_i (i.e., the set of successors of $\mathcal{V}'$). Let $\mathcal{D}_k$ ($\mathcal{K}_k$) be a *minimal dilation (contraction) set* of $\mathcal{G}$ if $\mathcal{D}_k$ ($\mathcal{K}_k$) has a dilation (contraction) and no subset $\mathcal{D}'_k \subset \mathcal{D}_k$ ($\mathcal{K}'_k \subset \mathcal{K}_k$) has a dilation (contraction).

Remark 1. $\mathcal{G}(A, B)$ has a dilation if there is a set of k rows of $[A\ B]$ that contains nonzero entries in less than k columns of the submatrix formed by these k rows. In fact, $\mathcal{G}(A, B)$ has a dilation if and only if $\operatorname{rank}[A\ B] < n$ [24].

We now revisit a fundamental result on the controllability [4] and, by duality, observability of structured systems.

Definition 1. *The structured system (A, B) [(C, A)] is structurally controllable [observable] if there exists a numerical realization $(\tilde{A}, \tilde{B})$ [$(\tilde{C}, \tilde{A})$] that is controllable [observable].*

Theorem 1. [4] *The system (A, B) [or (C, A)] is structurally controllable [or observable] if and only if $\mathcal{G}(A, B)$ [or $\mathcal{G}(C, A)$] satisfies the following conditions:*

1) *for each state variable $\boldsymbol{x}_i \in \mathcal{X}$, there exists a path from some driver node $\boldsymbol{u}_i \in \mathcal{U}$ to $\boldsymbol{x}_i$ [or every $\boldsymbol{x}_i \in \mathcal{X}$ has a path to some sensor node $\boldsymbol{y}_i \in \mathcal{Y}$];*
2) *$\mathcal{G}$ has no dilations [or contractions].*

B. Target controllability and target observability

We now establish the graph-theoretic conditions for output controllability and functional observability. These conditions are presented for systems in which nodes are independently driven, measured, and targeted, as formalized below.

Assumption 1. We assume that each column of B and each row of C and F have a single nonzero entry. We also assume that one of the following graph-theoretical conditions on the structured matrix A is satisfied: (i) there exists some numerical realization $\tilde{A}$ that is diagonalizable; (ii) $A_{ii} \neq 0$ for every target node $\boldsymbol{x}_i \in \mathcal{T}$. We note that the assumption on A can be relaxed to a weaker algebraic condition based on the Jordan form, which will be presented in future work.

Definition 2. *The structured system* $(A, B; F)$ *is target controllable if there exists some numerical realization* $(\tilde{A}, \tilde{B}; \tilde{F})$ *that is output controllable. Likewise, the structured system* $(C, A; F)$ *is target observable if there exists some numerical realization* $(\tilde{C}, \tilde{A}; \tilde{F})$ *that is functionally observable.*

Given the large adoption of the term "target controllability" by the community [12], [18]–[21], [25], [26] and the duality between output controllability and functional observability [22], it seems appropriate to unify these two structural properties under a common nomenclature—*target controllability* and *target observability*—as in Definition 2.

We present the following theorem on target controllability, which establishes graph-theoretic conditions equivalent to condition (4) for a structured system $(A, B; F)$.

Theorem 2. *The system* $(A, B; F)$ *is target controllable if and only if* $\mathcal{G}(A, B; F)$ *satisfies the following conditions:*

1) *for each target node* $\boldsymbol{x}_i \in \mathcal{T}$*, there exists a path from some driver node* $\boldsymbol{u}_i \in \mathcal{U}$ *to* $\boldsymbol{x}_i$*;*
2) *no subset* $\mathcal{T}_\ell \subseteq \mathcal{T}$ *in* $\mathcal{G}'(A, B; F)$ *has a dilation, where* $\mathcal{G}'$ *is a subgraph of* $\mathcal{G}$ *containing all possible paths from every* $\boldsymbol{u}_i \in \mathcal{U}$ *to any* $\boldsymbol{x}_i \in \mathcal{T}$.

Proof: See Appendix. ■

Remark 2. Theorem 2 generalizes previous results on target controllability [12], [26], [27], as shown next. Assume $\boldsymbol{u}_1$ has a path to all $\boldsymbol{x}_i \in \mathcal{T}$. For directed tree graphs $\mathcal{G}(A)$, a system is target controllable if and only if the path length from a driver node to each target node is unique [12, Th. 2], which is equivalent to $\mathcal{T}_\ell \subseteq \mathcal{T}$ having no dilations in $\mathcal{G}'$ since $\mathcal{G}'$ is also a directed tree and thus has no cycles. For systems with single-input matrix B, target controllability holds if $\mathcal{G}'$ has a perfect matching [26, Th. 2], which is sufficient for $\mathcal{G}'$ to have no dilations, satisfying condition 2 of Theorem 2. Likewise, a multiple-input system is target controllable if $\mathcal{G}$ can be covered by a union of cacti structures [27, Th. 17], which is sufficient for $\mathcal{G}'$ to have no dilations. Note that Refs. [26], [27] established only sufficient conditions.

The related studies [20], [25] on target controllability are complementary to our results, providing conditions for less generic types of structured systems (e.g., symmetric matrices [20]) or for a stronger notion of target controllability in which *all* (rather than *some*, as in Definition 2) numerical realizations $(\tilde{A}, \tilde{B}; \tilde{F})$ are output controllable [25].

The graph-theoretic conditions for target observability have already been established in Ref. [13] (under the nomenclature of "structural functional observability"), being equivalent to condition (5) for a structured system $(C, A; F)$.

Theorem 3. [13] *The system* $(C, A; F)$ *is target observable if and only if* $\mathcal{G}(C, A; F)$ *satisfies the following conditions:*

1) *every target node* $\boldsymbol{x}_i \in \mathcal{T}$ *has a path to some sensor node* $\boldsymbol{y}_i \in \mathcal{Y}$*;*
2) $\mathcal{T} \cap \mathcal{K} = \emptyset$*, where* $\mathcal{K} = \bigcup_k \mathcal{K}_k$ *is the union of all minimal contraction sets in* $\mathcal{G}(C, A; F)$.

C. Duality principle

We now establish the weak and strong duality principles for target controllability and target observability. To this end, consider a pair of structured systems $(C, A; F)$ and $(A^\mathsf{T}, C^\mathsf{T}; F)$ and their inference graphs $\mathcal{G}(C, A; F) = \{\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_\mathcal{X} \cup \mathcal{E}_\mathcal{Y}\}$ and $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F) = \{\mathcal{X} \cup \mathcal{U}, \mathcal{E}_\mathcal{X} \cup \mathcal{E}_\mathcal{U}\}$.

Remark 3. $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ is equivalent to graph $\mathcal{G}(C, A; F)$ with reversed edges and $\mathcal{U} = \mathcal{Y}$. Moreover, a set $\mathcal{V}' \subseteq \mathcal{X}$ has a dilation in $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ if and only if $\mathcal{V}'$ has a contraction in $\mathcal{G}(C, A; F)$. This is later illustrated in Fig. 1.

Theorem 4. (Weak duality) *If* $(C, A; F)$ *is target observable, then* $(A^\mathsf{T}, C^\mathsf{T}; F)$ *is target controllable.*

Proof: Since $(C, A; F)$ is target observable, the conditions of Theorem 3 are satisfied. First, given Remark 3, if condition 1 of Theorem 3 holds, then for each $\boldsymbol{x}_i \in \mathcal{T}$ in the reversed graph $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ there exists a path from some $\boldsymbol{u}_i \in \mathcal{U}$ to $\boldsymbol{x}_i$, satisfying condition 1 of Theorem 2. Second, it follows from Remark 3 that $\mathcal{K}_k = \mathcal{D}_k, \forall k$, where $\mathcal{K}_k$ and $\mathcal{D}_k$ are minimal contraction and dilation sets in $\mathcal{G}(C, A; F)$ and $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$, respectively. By induction, $\mathcal{K} = \mathcal{D} = \bigcup_k \mathcal{D}_k$. Since $\mathcal{T} \cap \mathcal{K} = \emptyset$ holds in $\mathcal{G}(C, A; F)$, it follows that $\mathcal{T} \cap \mathcal{D} = \emptyset$ also holds in $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$. If $\mathcal{T} \cap \mathcal{D} = \emptyset$, then $\mathcal{D}_k \nsubseteq \mathcal{T}_\ell$ for any subset $\mathcal{T}_\ell \subseteq \mathcal{T}$. Thus, no subset $\mathcal{T}_\ell \subseteq \mathcal{T}$ has a dilation in $\mathcal{G}$ and hence in $\mathcal{G}'(A^\mathsf{T}, C^\mathsf{T}; F)$. This satisfies condition 2 of Theorem 2 and thus $(A^\mathsf{T}, B^\mathsf{T}; F)$ is target controllable. ■

Theorem 5. (Strong duality) *The system* $(C, A; F)$ *is target observable if and only if* $(A^\mathsf{T}, C^\mathsf{T}; F)$ *is target controllable and* $\mathcal{T} \cap \mathcal{D} = \emptyset$*, where* $\mathcal{D} = \bigcup_k \mathcal{D}_k$ *is the union of all minimal dilation sets* $\mathcal{D}_k$ *in* $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$.

Proof: We show that Theorems 2 and 3 are equivalent for graphs $\mathcal{G}(C, A; F)$ and $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ under the stated conditions. The equivalence between conditions 1 of Theorems 2 and 3 follows directly from Remark 3. It also follows from Remark 3 that $\mathcal{D}_k = \mathcal{K}_k, \forall k$, and $\mathcal{D} = \mathcal{K}$. Thus, condition 2 of Theorem 3 is equivalent to $\mathcal{T} \cap \mathcal{D} = \emptyset$. ■

Remark 4. When all state variables are targeted ($\mathcal{T} = \mathcal{X}$), Theorems 2 and 3 reduce to Theorem 1. This is evident for condition 1 of both theorems. For condition 2 of Theorem 2, when $\mathcal{T} = \mathcal{X}$, $\mathcal{G}' = \mathcal{G}$ and thus $\mathcal{G}$ must have no dilations. For condition 2 of Theorem 3, when $\mathcal{T} = \mathcal{X}$ it follows that $\mathcal{T} \cap \mathcal{K} = \emptyset$ if and only if $\mathcal{K} = \emptyset$, implying that $\mathcal{G}$ must have no contractions. Thus, the strong duality reduces to the classical duality between (structural) controllability and observability.

Remark 5. For many sparse directed networks, the strong duality condition $\mathcal{T} \cap \mathcal{D} = \emptyset$ can be computationally tested in $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ as follows. For every target node $\boldsymbol{x}_i \in \mathcal{T}$, a

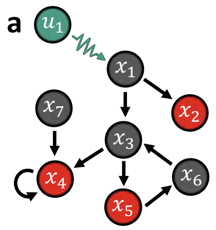

Fig. 1. Inference graphs of a dual pair of dynamical systems. (a) Target controllability of a structured system $(A, B; F)$. (b) Target observability of the dual structured system $(B^\mathsf{T}, A^\mathsf{T}; F)$. The driver, sensor, and target nodes are indicated in green, blue, and red, respectively.

breadth-first search algorithm can be used to build the set of nodes $\mathcal{S} \subseteq \mathcal{X} \cup \mathcal{U}$ composed of the union of sets $S(P(\boldsymbol{x}_i))$, $S(P(S(P(\boldsymbol{x}_i))))$, and so forth, incurring in a computational complexity of order $O(n + |\mathcal{E}|)$. The existence of a minimal dilation set $\mathcal{D}_k \supseteq \{\boldsymbol{x}_i\}$ can then be readily verified by testing the condition $|P(\mathcal{S}')| < |\mathcal{S}'|$ for all possible subsets $\mathcal{S}' \subseteq \mathcal{S}$. This procedure is feasible if $\mathcal{S}$ is sufficiently small, which holds in general for high-dimensional networks when $\mathcal{G}$ has few cycles, small node degrees ($\sum_j A_{ij} \ll n, \forall i$), and few targets ($r \ll n$). For undirected networks, however, $\mathcal{S} = \mathcal{X}$, making this test computationally expensive for large n.

A sufficient condition based on the structure of the inference graph $\mathcal{G}(A)$ is provided below for strong duality.

Corollary 1. *$(C, A; F)$ is target observable if $(A^\mathsf{T}, C^\mathsf{T}; F)$ is target controllable and every $\boldsymbol{x}_i \in \mathcal{T}$ has a self-edge.*

Proof: If a target node has a self-edge, then it does not belong to a minimal contraction (dilation) set. Since this holds for all $\boldsymbol{x}_i \in \mathcal{T}$, then conditions 2 of Theorems 2 and 3 are satisfied for graphs $\mathcal{G}(A^\mathsf{T}, C^\mathsf{T}; F)$ and $\mathcal{G}(C, A; F)$, respectively. ■

Example 1. Consider the dual pair of systems illustrated in Fig. 1. System $(A, B; F)$ is target controllable since there is a path from u_1 to every target node and no subset $\mathcal{T}_\ell \subseteq \mathcal{T}$ has a dilation in $\mathcal{G}' = \mathcal{G}\backslash\{\boldsymbol{x}_7\}$ (e.g., for $\mathcal{T}_\ell = \{\boldsymbol{x}_4, \boldsymbol{x}_5\}$, we have that $P(\mathcal{T}_\ell) = \{\boldsymbol{x}_3, \boldsymbol{x}_4\}$ and thus $|P(\mathcal{T}_\ell)| = |\mathcal{T}_\ell|$). However, the dual system $(B^\mathsf{T}, A^\mathsf{T}; F)$ is not target observable since $\mathcal{D}_k = \{\boldsymbol{x}_2, \boldsymbol{x}_3\}$ is a minimal dilation set and $\mathcal{T} \cap \mathcal{D} = \{\boldsymbol{x}_2\}$, hence strong duality does not hold. There are several ways to enforce strong duality and make $(B^\mathsf{T}, A^\mathsf{T}; F)$ target observable; for example, by adding a self-edge to $\boldsymbol{x}_2$, connecting a second driver node $\boldsymbol{u}_2$ to $\boldsymbol{x}_2$, or removing $\boldsymbol{x}_2$ from the set of target nodes (corresponding to changes in the structure of matrices A, B, and F, respectively). Following Definition 2, the conditions for target controllability and target observability are generic and hold for all numerical matrices $(\tilde{A}, \tilde{B}; \tilde{F})$ sharing the structure of $(A, B; F)$ except for a set of matrices of Lesbegue measure zero.

IV. Optimal Driver and Sensor Placement

A. Duality and algorithms

The weak duality principle shows that methods developed for target observability problems can be directly applied to target controllability problems (by using the dual graph), as well as the converse when strong duality holds. Such methods include algorithms designed to test the conditions of Th. 2 and 3 for high-dimensional systems, or—as we consider next—to find a minimum set of driver nodes $\mathcal{U}$ for target controllability or sensor nodes $\mathcal{Y}$ for target observability. The latter are respectively addressed as the problems of minimum driver placement for target controllability (MDPt) and minimum sensor placement for target observability (MSPt).

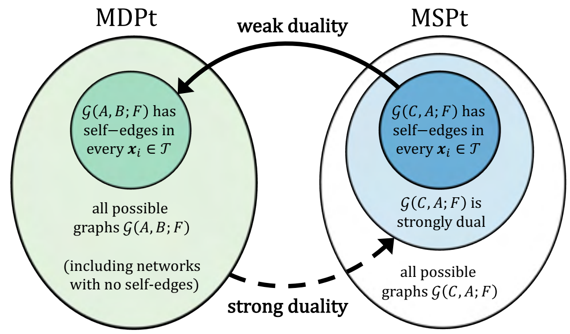

Fig. 2. Sets of MDPt and MSPt problems. Depending on the structure of the inference graph $\mathcal{G}(A)$, algorithms developed to solve a MDPt problem can be employed to solve a dual MSPt problem, and vice versa. The origin of the dashed and solid arrows are the set of problems that were originally solved by the algorithms presented in Refs. [12] and [13], respectively. The endpoints of these arrows indicate the new sets of problems that these algorithms can solve due to the weak and strong duality principles. The light (dark) shades represent weakly (strongly) dual sets of problems.

Thus far, no algorithm has been developed to solve the MSPt for *general* inference graphs $\mathcal{G}(A)$ due to the computational challenges in verifying condition 2 of Theorem 3 for generic (possibly undirected) graphs (Remark 5). For a broad class of applications where every target node has a self-edge in $\mathcal{G}(A)$ (Corollary 1), the MSPt can be formulated as a set cover problem, which can be approximately solved by combining a greedy algorithm and breadth-first searches, as presented in [13, Alg. 1]. Owing to the weak duality principle, it follows that the MDPt can also be formulated as a set cover problem for the dual graph $\mathcal{G}(A^\mathsf{T})$ and solved by the same algorithm when every target node has a self-edge.

When strong duality holds, we show that a new class of problems can be solved. Unlike the MSPt, the MDPt can be solved efficiently (though approximately) for *any* inference graph $\mathcal{G}(A)$. This is enabled by the fact that condition 2 of Theorem 2 is weaker than condition 2 of Theorem 3, which allows it to be enforced using a greedy algorithm that recursively solves a maximum matching problem in an induced bipartite graph, as proposed in [12, Alg. 3]. The strong duality principle thus enables this MDPt algorithm to be employed for MSPt problems, providing an efficient (approximate) solution for the set of all $\mathcal{G}(C, A; F)$ that satisfy the strong duality condition in Theorem 5.

Fig. IV-A summarizes the relation between the MDPt and MSPt problems, illustrating how the weak and strong duality principles can be applied for the conversion of algorithms from one problem to the other. The light green and dark blue sets are those containing the problems originally solved in Refs. [12] and [13], respectively. It is now evident that the strong duality principle enables the translation of algorithms to solve a new class of problems (contained in the light blue region) that did not have a solution available in the literature yet. The white region remains as the most general set of MSPt problems with no available solvers.

B. Numerical results

Fig. IV-B illustrates the MDPt and MSPt problems applied to a high-dimensional system, the *C. elegans* neural network. The network is modeled as a linear system (1) where each variable $\boldsymbol{x}_i$ represents a neuron (node) and A is the adjacency matrix. Given the highly directed and sparse nature of the inference graph $\mathcal{G}(A)$ and the small number of selected target nodes ($r = 0.05n$), the presence of minimal dilation sets containing $\mathcal{T}$ can be efficiently tested following Remark 5. For the set of target nodes $\mathcal{T}$ shown in Fig. IV-Ba, it holds that $(C, A; F)$ satisfies $\mathcal{T} \cap \mathcal{K} = \emptyset$ for any choice of C and, therefore, the system is strongly dual. The network has no self-edges, implying that this MSPt problem falls into the class of problems that can be solved by the MDPt algorithm (light blue set in Fig. IV-A). Fig. IV-Ba shows the minimum set of drivers and sensors selected with [12, Alg. 3] by considering the original graph $\mathcal{G}(A)$ and the dual graph $\mathcal{G}(A^\mathsf{T})$, respectively. The algorithm provides an efficient approximation, in which the minimum number of sensors and drivers correspond to only 1% and 1.5% of the network size, respectively. As the number of targets increases, Fig. IV-Bb shows that the number of drivers and sensors remain relatively small compared to the network size, as also observed in other complex networks without and with self-edges (cf. [12, Fig. 6] and [13, Fig. 2]).

Algorithmic implementations to solve the MDPt [12, Alg. 3] and MSPt [13, Alg. 1] problems for arbitrary inference graphs $\mathcal{G}(A)$ and target sets $\mathcal{T}$ are available at https://github.com/montanariarthur/TargetCtrb. Beyond the placement of drivers and sensors, our GitHub repository also provides code on how to effectively design feedback controllers [22] and functional observers [11], [13] for the stable control and estimation of target variables, respectively.

V. Conclusion

Examining the rank-based conditions for output controllability and functional observability, it is not immediately clear for which systems the output controllability of $(A, B; F)$ implies the functional observability of $(B^\mathsf{T}, A^\mathsf{T}; F)$. For network applications where target variables are independently sought to be controlled or estimated (Assumption 1), our results provide a graph-theoretic characterization of target controllability and target observability. Unlike output controllability and functional observability, each characterized by a single rank-based condition, target controllability and target observability are individually depicted by two graph-based conditions that highlight the weak and strong dualities between these properties. The first condition—related to the existence of paths from/to target nodes to/from sensor/driver nodes—is equivalent for any dual pair of inference graphs. However, the second condition—related to dilations and contractions in a graph—is inherently stronger for target observability than for target controllability. In particular, it follows from Theorems 1–5 that the set of structurally

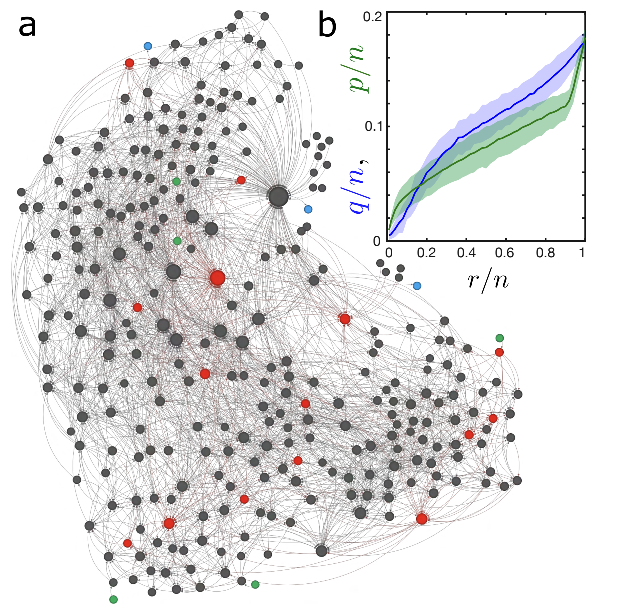

Fig. 3. (a) Optimal driver and sensor placement for the target controllability and target observability of the *C. elegans* neural network. The driver, sensor, and target nodes are indicated in green, blue, and red, respectively. (b) Minimum number of drivers p (green) and sensors q (blue) as a function of the number of target nodes r (normalized by the network size n). Each data point is an average over 100 realizations of randomly selected target nodes, where shaded areas indicate three standard deviations.

observable ((and, equivalently, the dual set of structurally controllable) systems are contained inside the set of target observable systems, which in turn are contained inside the dual set of target controllable systems.

Our application of an MDPt algorithm for a class of MSPt problems is one of many possible uses of the established duality principle. Here, we focused on algorithms proposed in Refs. [12], [13] due to their intrisic connection to the graph-theoretic conditions in Theorems 2 and 3. Nonetheless, we expect that many other methods developed for the broadly explored problem of target controllability (based on graph theory [19], [21], linear programming [25], or structural rank conditions [18], [26]) may also find new applications in functional observability problems, as well as the converse.

Appendix

Proof of Theorem 2. Let $\mathcal{X}_1 \subseteq \mathcal{X}$ be the set of all state variables belonging to a path in $\mathcal{G}$ from some driver node $\boldsymbol{u}_i \in \mathcal{U}$ to some target node $\boldsymbol{x}_i \in \mathcal{T}$, and $\mathcal{X}_2 = \mathcal{X} \backslash \mathcal{X}_1$ be the complement set. Define $|\mathcal{X}_1| = k$ and $|\mathcal{X}_2| = n - k$.

Sufficiency. Suppose that condition 1 is satisfied, i.e., $\mathcal{T} \subseteq \mathcal{X}_1$ and $k \geq r$. After applying a permutation of coordinates such that the nodes in $\mathcal{X}_1$ appear first, we have the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \; F = \begin{bmatrix} F_1 & 0 \end{bmatrix}, \quad (6)$$

where $A_{11} \in \mathbb{R}^{k \times k}$, $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, $B_1 \in \mathbb{R}^{k \times p_1}$, $B_2 \in \mathbb{R}^{(n-k)\times(p-p_1)}$, and $F_1 \in \mathbb{R}^{r \times k}$. Matrices A_{12} and A_{22} correspond to paths between sets $\mathcal{X}_1$ and $\mathcal{X}_2$, B_1 corresponds to all p_1 driver nodes $\boldsymbol{u}_i \in \mathcal{U}_1 \subseteq \mathcal{U}$ that have some path to a target node, and B_2 to $\boldsymbol{u}_i \in \mathcal{U}_2 = \mathcal{U} \backslash \mathcal{U}_1$. This yields the subgraph $\mathcal{G}' = \{\mathcal{X}_1, \mathcal{E}_1\}$, where $\mathcal{E}_1 = \mathcal{E}_{\mathcal{X}_1} \cup \mathcal{E}_{\mathcal{U}_1}$, $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathcal{E}_{\mathcal{X}_1}$ if $[A_{11}]_{ij} \neq 0$, and $(\boldsymbol{x}_i, \boldsymbol{u}_j) \in \mathcal{E}_{\mathcal{U}_1}$ if $[B_1]_{ij} \neq 0$.

The controllability matrix $\mathcal{C}$ of system (A, B) has the form

$$\mathcal{C} = \begin{bmatrix} B_1 & 0 & A_{11}B_1 & 0 & A_{11}^2 B_1 & 0 & \dots \\ 0 & B_2 & A_{21}B_1 & A_{22}B_2 & XB_1 & A_{22}^2 B_2 & \dots \end{bmatrix}, \quad (7)$$

where $X = A_{21}A_{11} + A_{22}A_{21}$. In Eq. (7), we have used the fact that $A_{12}A_{22}^k B_2 = 0, \forall k \in \{0, 1, \dots\}$, since by definition no driver node $\boldsymbol{u}_i \in \mathcal{U}_2$ has a path to some node $\boldsymbol{x}_i \in \mathcal{X}_1$. Likewise, matrices of form $A_{12}A_{21}B_1$ and $A_{12}A_{21}A_{11}B_1$ are zero; otherwise, there would exist a path from $\boldsymbol{u}_i \in \mathcal{U}_1$ to $\boldsymbol{x}_i \in \mathcal{X}_1$ passing by a node $\boldsymbol{x}_j \in \mathcal{X}_2$, which contradicts the assumption that all such paths are already covered in $\mathcal{G}' = \{\mathcal{X}_1, \mathcal{E}_1\}$. Therefore, it follows by construction that $F\mathcal{C} = F_1\mathcal{C}_1$, where $\mathcal{C}_1$ is the controllability matrix of pair (A_{11}, B_1). It remains to show that if no subset $\mathcal{T}_\ell \subseteq \mathcal{T}$ has a dilation in $\mathcal{G}'$ then $\mathrm{rank}(F_1\mathcal{C}_1) = r$.

Assume without loss of generality that the first r nodes in $\mathcal{X}_1$ belong to $\mathcal{T}$. First, suppose no driver node is directly connected to a target node in $\mathcal{G}'$ (first r rows of B_1 are zero). Following Remark 1, if no subset $\mathcal{T}_\ell \subseteq \mathcal{T}$ has a dilation in $\mathcal{G}'$, it follows that the first r rows of A_{11} have nonzero entries in at least r columns of the submatrix formed by these r rows. Since A satisfies Assumption 1 and there exists a path from some driver node in $\mathcal{U}_1$ to every target node in $\mathcal{T}$, it follows that the first r rows of $\mathcal{C}_1$ also have nonzero entries in at least r columns. Therefore, $\mathrm{rank}(F_1\mathcal{C}_1) = r$. Second, suppose a driver node is directly connected to target node $\boldsymbol{x}_1$. Given that B_1 has a single nonzero entry per column (Assumption 1), $\boldsymbol{x}_1$ does not belong to a minimal dilation set and the first row of matrix $F_1\mathcal{C}_1$ is always linearly independent from the other rows. Therefore, $\mathrm{rank}(F_1\mathcal{C}_1) = \mathrm{rank}(F_1'\mathcal{C}_1') + 1$, where $F_1'\mathcal{C}_1'$ is a submatrix of $F_1\mathcal{C}_1$ without the first row. The rest of the proof follows as above for the submatrix $F_1'\mathcal{C}_1'$.

Necessity. The necessity of condition 2 follows from the fact that that if some subset $\mathcal{T}_\ell \subseteq \mathcal{T}$ has a dilation, then the first r rows of A_{11} have nonzero entries in less than r columns, and so does $\mathcal{C}_1$. This implies that $\mathrm{rank}(F_1\mathcal{C}_1) < r$.

For the necessity of condition 1, suppose there are no paths from driver nodes $\mathcal{U}$ to some nodes $\mathcal{X}_1 \subseteq \mathcal{X}$. Let $\mathcal{X}_2 = \mathcal{X} \backslash \mathcal{X}_2$, $|\mathcal{X}_1| = k$, and $|\mathcal{X}_2'| = n - k$. After applying a coordinate permutation such that nodes in $\mathcal{X}_1$ appear first, we have that

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \; F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \quad (8)$$

where $A_{11} \in \mathbb{R}^{k \times k}$, $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, $B_2 \in \mathbb{R}^{(n-k)\times p}$, and other matrices have consistent dimensions. Given Assumption 1, $F_1 \in \mathbb{R}^{r_1 \times k}$ correspond to the subset $\mathcal{T}_1 \subseteq \mathcal{X}_1$, and $F_2 \in \mathbb{R}^{(r-r_1)\times(n-k)}$ to $\mathcal{T}_2 = \mathcal{T} \backslash \mathcal{T}_1$. It follows that

$$\mathrm{rank}(F\mathcal{C}) = \mathrm{rank} \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} 0 \\ \mathcal{C}_2 \end{bmatrix} = \mathrm{rank}(F_2\mathcal{C}_2) \leq r - r_1, \quad (9)$$

where $\mathcal{C}_2$ is the controllability matrix of (A_{22}, B_2). Thus, if there exists a target node with no path coming from a driver node, then $\mathcal{T}_1 \neq \emptyset$, $r_1 > 0$, and condition (4) is violated. ■

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