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Stabilization for a Class of Bilinear Systems: A Unified Approach

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***Abstract*— This letter studies nonlinear dynamic control design for a class of bilinear systems to asymptotically stabilize a given equilibrium point while fulfilling constraints on the control input and state. We design a controller based on integral actions on the system input and output. As special cases, the proposed controller contains a dynamic controller with an integral action of either input or output only and a static controller. Stability analysis of the closed-loop system is performed based on a Lyapunov function. Level sets of the Lyapunov function are utilized to estimate a set of initial states and inputs such that the corresponding state and input trajectories are within specified compact sets. Finally, the proposed control technique is applied to a heat exchanger under constraints on the temperature of each cell (state) and the mass flow rate (input), and simulations show the effectiveness of the proposed approach.**

I. Introduction

Bilinear systems belong to an important subclass of control systems which have been widely used to accurately model various real-life systems and phenomena [1], such as power electronics, chemical reactions, and socio-economics [2], [3]. However, control design for bilinear systems is a challenging task, as evidenced by the numerous publications on the topic [1], [4]–[11]. In this letter, we aim to study the problem of robustly stabilizing a bilinear system while characterizing the trajectories evolution for physical applications.

Static and dynamic switching state feedback controllers have been respectively proposed in [8] and [9] to stabilize classes of bilinear systems. In these works, the control input acts only multiplicatively. In particular, the controller proposed in [8] guarantees the satisfaction of input constraints. However, none of these works addresses the problem of constraining the state variable. Linear state feedback controllers are designed in [7] and [1], where the resulting closed-loop system is shown to admit a locally asymptotically stable equilibrium point. Subject to a suitable tuning of the controller gains, the satisfaction of state constraints, but not input constraints, are guaranteed in [7]. In [1], a method to fulfill the constraints on both state and input is provided. However, this requires solving a set of nonlinear inequalities.

The problem of output regulation of bilinear systems is considered in [6], where the theoretical developments are applied to the problem of temperature control of a heat exchanger. While the asymptotic stability of the system is preserved, restrictions on the control input are met by applying a saturation mechanism on the input. However, input saturation is known to potentially induce deleterious effects on systems stability and performance [12]. For the heat exchanger model, the saturated controller proposed in [6]—in combination with some of the model intrinsic properties—also guarantees the fulfillment of state constraints. To prevent potential undesired effects stemming from the use of input saturation, the authors of [4] propose an anti-windup mechanism to solve the output regulation problem for an electrical system.

In this work, we propose a new unifying dynamic controller approach that simultaneously considers input and output integral actions. The input integral action takes inspiration from the control design approach for Krasovskii passive systems [13], [14]. In addition to satisfying constraints on the control input, the input integral action guarantees convergence of the input to its equilibrium value: for systems in which input convergence implies state convergence, it is possible to ensure stabilization only based on the knowledge of the input equilibrium value [14], [15]. Additionally, the integral action on the output guarantees robust regulation towards a desired output set point. By unifying approach, we mean that the proposed controller strategy can recover two classes of dynamic controllers, including the one proposed in [6] and the one proposed in this letter as a special case, which makes the unconstrained system exponentially stable. Also, a static stabilizing controller can be recovered without integral actions. Furthermore, the proposed controller is not switching. The analysis is based on a Lyapunov function ensuring feasible operative conditions, i.e., we find initial conditions such that the states and the control inputs satisfy defined constraints all the time. Also, the control input satisfies predefined constraints without employing saturation or an anti-windup mechanism. Finally, the proposed control

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technique is applied to the problem of temperature stabilization of a counter-current heat exchanger.

Notation. $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_{>0}$ is the set of positive real numbers. An identity matrix with dimension $n \times n$ is denoted by $\mathbb{I}_{n\times n}$. Given $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ means that P is symmetric and positive definite. The Euclidean norm of a vector $x \in \mathbb{R}^n$ weighted by a matrix $P \in \mathbb{R}^{n\times n}$ is denoted by $\|x\|_P := \sqrt{x^\top P x}$. Given $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, we compactly denote $(x, y) = (x^\top, y^\top)^\top$.

II. PROBLEM FORMULATION

In this letter, we focus on a class of bilinear systems in which the control signal acts additively and multiplicatively simultaneously, i.e.,

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i x + b_i)\, u_i + G, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ is the control input and $A, B_i \in \mathbb{R}^{n\times n}, b_i, G \in \mathbb{R}^n$.[1]

In practical applications, it is usually desired that the system state and input satisfy predefined constraints. Namely, for some compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ of the state and the input, respectively, we require $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$, $\forall t \geq 0$. While satisfying the above constraints, the goal of this work is to stabilize (1) at an equilibrium point. For stabilization of bilinear systems, it is standard in the literature to impose some stability assumption on the open-loop system; see e.g. [4], [6]–[9], [11] and the references therein. Accordingly, in this letter, we assume that there exists at least one $u^* \in \mathcal{U}$ such that $A + \sum_{i=1}^{m} B_i u_i^*$ is Hurwitz. Also, such u^* and the unique corresponding steady state x^*, given by

$$x^* = -\left(A + \sum_{i=1}^{m} B_i u_i^*\right)^{-1} \left(\sum_{i=1}^{m} b_i u_i^* + G\right), \tag{2}$$

satisfy $(x^*, u^*) \in \mathcal{X} \times \mathcal{U}$. To state this formally, we define the following set:

$$\mathcal{D} := \left\{(x, u) \in \mathcal{X} \times \mathcal{U} \,|A + \sum_{i=1}^{m} B_i u_i^* \text{ is Hurwitz}\right\}. \tag{3}$$

Assumption 1: There exists $u^* \in \mathcal{U}$ such that u^* and the unique corresponding x^* in (2) *satisfy $(x^*, u^*) \in \mathcal{D}$, with $\mathcal{D}$ in* (3).

Note that Assumption 1 is equivalent to supposing that for any $Q = Q^\top \succ 0$, there exists $P = P^\top \succ 0$ such that the following Lyapunov equation holds

$$\left(A + \sum_{i=1}^{m} B_i u_i^*\right)^\top P + P\left(A + \sum_{i=1}^{m} B_i u_i^*\right) = -Q. \tag{4}$$

Now, we are ready to state the main problem studied in this letter.

Problem 1: Design a controller for (1) *such that the closed-loop system is asymptotically stable at $(x^*, u^*) \in \mathcal{D}$; and find a* feasible *region of attraction $\Omega \subset \mathcal{X} \times \mathcal{U}$ such that any trajectory starting from Ω stays within the* feasible *set $\mathcal{X} \times \mathcal{U}$.*

Note that the open-loop controller $u(t) = u^*$ is a trivial solution to Problem 1. However, open-loop controllers are usually less robust than feedback controllers. Therefore, we design in the next section a dynamic feedback controller.

III. DYNAMIC CONTROL DESIGN

In this section, we design a dynamic controller to solve Problem 1 based on integral actions of both input and performance output, where the performance output is defined later. We show asymptotic stability by constructing a suitable Lyapunov function which is also used to estimate a region of attraction.

The proposed controller recovers three simpler controllers as special cases: 1) a controller with the integral action of the input only; 2) one with the integral action of the performance output only; and 3) a controller without integral action, i.e., a static state-feedback controller. The first controller is totally new in the literature of bilinear systems, while the other two recover the controllers proposed in [6] and [10], respectively.

A. Main Results

Besides solving Problem 1, we consider the additional goal of regulating a performance output $y = Cx$, see, e.g. [16]. Namely, for some reference $y^* := Cx^* \in \mathbb{R}^q$ $(q \leq m)$, we additionally require $\lim_{t\to\infty} y(t) = y^*$. Achieving this tracking objective requires the closed-loop system to be robust with respect to (small) perturbations in the system parameters [17]. If there is no parameter uncertainty, the regulation of the performance output is guaranteed by stabilizing the system.

To guarantee the solvability of the output regulation problem, we assume that C has the following property.

Assumption 2: For given $(x^*, u^*) \in \mathcal{D}$, $C \in \mathbb{R}^{q\times n}$ is selected such that

$$\text{rank}\left(M \begin{bmatrix} B_1 x^* + b_1 & \cdots & B_m x^* + b_m \end{bmatrix}\right) = q, \tag{5a}$$

$$M := C\left(A + \sum_{i=1}^{m} B_i u_i^*\right)^{-1}. \tag{5b}$$

Now, under the above assumption, we design a controller based on a suitable Lyapunov candidate defined as a quadratic function on the state, input and output errors, depending on the equilibrium point (see the proof of Theorem 1). Then, a Lyapunov analysis leads to the following controller:

$$K\dot{u} = -H(u - u^*) - \varepsilon_1 R(x) + \varepsilon_2 S(x, z) \tag{6a}$$

$$\dot{z} = y - y^* = C(x - x^*), \tag{6b}$$

with $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$, $K, H \succ 0$ $(K, H \in \mathbb{R}^{m\times m})$, and

$$R(x) := \begin{bmatrix} (x - x^*)^\top P (B_1 x + b_1) \\ \vdots \\ (x - x^*)^\top P (B_m x + b_m) \end{bmatrix} \tag{6c}$$

[1]For example, for a counter-current heat exchange system (see Section IV), G is the input thermal power to the system.

$$S(x,z) := \begin{bmatrix} (z - M(x-x^*))^\top \theta M(B_1 x + b_1) \\ \vdots \\ (z - M(x-x^*))^\top \theta M(B_m x + b_m) \end{bmatrix}, \quad (6d)$$

where $P \succ 0$ is the solution to the Lyapunov equation (4) for arbitrary given $Q \succ 0$, and $\theta \in \mathbb{R}^{q\times q}$ is any arbitrary matrix such that $\theta \succ 0$. From (6a) and (6b), $K^{-1}H$ determines the time constant of the dynamics of the input error $u - u^*$. Also, $\varepsilon_1 K^{-1}$ and $\varepsilon_2 K^{-1}$ respectively specify the importance of information of $R(x)$ and $S(x,z)$, where $R(x)$ is to evaluate the state error $x - x^*$, and $S(x,z)$ is for output regulation through the integral action of the output error $y - y^*$. By tuning the controller parameters, one can balance these three components.

Now, we are ready to state the main result of this letter.

Theorem 1: Let Assumptions 1 and 2 hold. Given $(x^*, u^*) \in \mathcal{D}$, *consider* (1) *in closed-loop with* (6). *Then, the following statements hold:*

(i) the closed-loop system is globally asymptotically stable and locally exponentially stable at $(x,u,z) = (x^*, u^*, 0)$, *and thus* $y(t) \to y^*$ *as* $t \to \infty$;

(ii) for any initial condition satisfying $\varepsilon_2 \|z(0) - M(x(0) - x^*)\|_\theta^2 / 2\varepsilon_1 \le \varepsilon$, *the following set*

$$\Omega_c := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : \\ \|x - x^*\|_P^2 + \|u - u^*\|_K^2/\varepsilon_1 \le 2c\},\ c > 0 \quad (7)$$

is a feasible region of attraction if $\Omega_{c+\varepsilon} \subset \mathcal{X} \times \mathcal{U}$.

Proof: The system equation (1) at (x^*, u^*) yields

$$G = -Ax^* - \sum_{i=1}^m (B_i x^* + b_i) u_i^*.$$

Substituting this expression of G into (1) leads to

$$\begin{aligned} \dot{x} &= A(x - x^*) + \sum_{i=1}^m B_i(x u_i - x^* u_i^*) + \sum_{i=1}^m b_i(u_i - u_i^*) \\ &= \left(A + \sum_{i=1}^m B_i u_i^*\right)(x - x^*) + \sum_{i=1}^m (B_i x + b_i)(u_i - u_i^*). \end{aligned} \quad (8)$$

Now, we select a Lyapunov candidate as

$$V(x,u,z) := \frac{1}{2}\|x - x^*\|_P^2 + \frac{1}{2\varepsilon_1}\|u - u^*\|_K^2 + \frac{\varepsilon_2}{2\varepsilon_1}\|z - M(x - x^*)\|_\theta^2. \quad (9)$$

Its time derivative along the closed-loop trajectory satisfies

$$\begin{aligned} \dot{V}(x,u,z) &= (x - x^*)^\top P\left(A + \textstyle\sum_i B_i u_i^*\right)(x - x^*) \\ &\quad + (x - x^*)^\top P \textstyle\sum_i (B_i x + b_i)(u_i - u_i^*) \\ &\quad + \tfrac{\varepsilon_2}{\varepsilon_1}(z - M(x - x^*))^\top \theta(\dot{z} - M\dot{x}) - \tfrac{1}{\varepsilon_1}\|u - u^*\|_H^2 \\ &\quad - (u - u^*)^\top R(x) + \tfrac{\varepsilon_2}{\varepsilon_1}(u - u^*)^\top S(x,z). \end{aligned}$$

Considering $(u - u^*)^\top R(x) = \sum_{i=1}^m (u_i - u_i^*) R_i(x)$, it follows from (4) and (6c) that

$$\begin{aligned} \dot{V}(x,u,z) = &-\frac{1}{2}\|x - x^*\|_Q^2 - \frac{1}{\varepsilon_1}\|u - u^*\|_H^2 \\ &+ \frac{\varepsilon_2}{\varepsilon_1}(z - M(x - x^*))^\top \theta(\dot{z} - M\dot{x}) \\ &+ \frac{\varepsilon_2}{\varepsilon_1}(u - u^*)^\top S(x,z), \end{aligned}$$

where the term $\dot{z} - M\dot{x}$ in the second line can be rearranged as follows (see (5b), (6b), and (8)):

$$\begin{aligned} \dot{z} - M\dot{x} &= C(x - x^*) - M\left(A + \sum_{i=1}^m B_i u_i^*\right)(x - x^*) \\ &\quad - M\sum_{i=1}^m (B_i x + b_i)(u_i - u_i^*) \\ &= -M\sum_{i=1}^m (B_i x + b_i)(u_i - u_i^*). \end{aligned}$$

This and (6d) lead to

$$\begin{aligned} &\dot{V}(x,u,z) \\ &= -\frac{1}{2}\|x - x^*\|_Q^2 - \frac{1}{\varepsilon_1}\|u - u^*\|_H^2 + \frac{\varepsilon_2}{\varepsilon_1}(u - u^*)^\top S(x,z) \\ &\quad - \frac{\varepsilon_2}{\varepsilon_1}(z - M(x - x^*))^\top \theta M \sum_{i=1}^m (B_i x + b_i)(u_i - u_i^*) \\ &= -\frac{1}{2}\|x - x^*\|_Q^2 - \frac{1}{\varepsilon_1}\|u - u^*\|_H^2. \end{aligned}$$

We show now asymptotic stability based on LaSalle's invariance principle. Since $V(x,u,z)$ in (9) is radially unbounded and positive definite, then its level set

$$\bar{\Omega}_c := \{(x,u,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q : V(x,z,u) \le c\}$$

is not only positively invariant but also compact for any $c > 0$. Let us define the following set

$$\begin{aligned} \Upsilon &:= \left\{(x,u,z) \in \bar{\Omega}_c : \dot{V}(x,u,z) = 0\right\} \\ &= \left\{(x,u,z) \in \bar{\Omega}_c : x = x^*, u = u^*\right\}. \end{aligned}$$

The largest invariant set E contained in Υ satisfies

$$\begin{aligned} E &\subset \{(x,u,z) \in \Upsilon : x = x^*, u = u^*, \dot{u} = 0\} \\ &= \{(x,u,z) \in \Upsilon : x = x^*, u = u^*, \\ &\qquad z^\top \theta M(B_i x^* + b_i) = 0,\ i = 1, \cdots, m\}. \end{aligned}$$

From (5a) and $\theta \succ 0$, we have $E \subset \{(x^*, u^*, 0)\}$, and $(x^*, u^*, 0)$ is an equilibrium point of the closed-loop system. Therefore, $(x^*, u^*, 0)$ is globally asymptotically stable. Furthermore, it is locally exponentially stable because the linearization of the closed-loop system at the equilibrium point is exponentially stable. Again, this can be shown with the Lyapunov function V in (9). Details are omitted for space reasons.

Now, we prove item (ii). From the structures of Ω_c in (7) and $\bar{\Omega}_c$ with $V(x,u,z)$ in (9), if $(x(0), u(0)) \in \Omega_c$ and $\varepsilon_2\|z(0) - M(x(0) - x^*)\|_\theta^2/2\varepsilon_1 \le \varepsilon$, then $(x(0), u(0), z(0)) \in \bar{\Omega}_{c+\varepsilon}$. Since $\bar{\Omega}_{c+\varepsilon}$ is positively invariant, $(x(t), u(t), z(t)) \in \bar{\Omega}_{c+\varepsilon}$ for all $t \ge 0$. We also have $(x(t), u(t)) \in \Omega_{c+\varepsilon}$ for all $t \ge 0$. Therefore, if $c > 0$ is selected such that $\Omega_{c+\varepsilon} \subset \mathcal{X} \times \mathcal{U}$, then Ω_c is a feasible region of attraction. ■

For given P and K/ε_1, to obtain the largest estimation of the region of attraction contained in $\mathcal{X} \times \mathcal{U}$ by Theorem 1, we only have to find the maximum $c > 0$ such that $\Omega_{c+\varepsilon} \subset \mathcal{X} \times \mathcal{U}$. If such $c > 0$ exists, this is unique, since $\mathcal{X} \times \mathcal{U}$ is compact. The shape of Ω_c can be changed by selecting different P and K/ε_1, where P is determined by Q from (4).

Next, we briefly mention robustness thanks to the output integral action, by applying results in [17]. In particular, we show that under small parameter perturbations, the closed-loop system preserves the existence of an asymptotically stable equilibrium at which output regulation $y = y^*$ is achieved.

Corollary 1: Consider the controller (6) *applied to the perturbed model*

$$\dot{x} = \tilde{A}x + \sum_{i=1}^{m} \left(\tilde{B}_i x + \tilde{b}_i\right) u_i + \tilde{G}. \tag{10}$$

For any compact set $X \times U \times Z \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$ of initial conditions there exists $\delta > 0$ such that, if

$$\|\tilde{A} - A\| \leq \delta,\ \|B_i - \tilde{B}_i\| \leq \delta,\ \|b_i - \tilde{b}_i\| \leq \delta,\ \|G - \tilde{G}\| \leq \delta,$$

for any $i = 1, \ldots, m$, then the system (10) *in closed-loop with* (6) *admits an equilibrium $(\tilde{x}^*, \tilde{u}^*, \tilde{z}^*)$ which is locally exponentially stable and asymptotically stable with a domain of attraction including the set $X \times U \times Z$. Furthermore, on such an equilibrium one has $C\tilde{x}^* = y^*$.*

Proof: Consider two Lyapunov level sets $\Omega_i = \{(x, u, z) : V(x, u, z) \leq c_i\}$ with V given by (9) such that $X \times U \times Z \subset \Omega_1 \subset \Omega_2$. They are forward invariant compact sets contained in the domain of attraction of the equilibrium (x^*, u^*, z^*). Hence all the assumptions of [17, Lemma 5] are verified. We conclude that by allowing small parameter perturbations of the matrices A, B_i, b_i, G, the resulting closed-loop dynamics are close-enough to the nominal ones for all $(x, z, u) \in \Omega_2$. The statement of the corollary follows from the total stability properties claimed in [17, Lemma 5]. Similar details are also given in [4, Theorem 1]. ■

Note that M in (5b) is defined in terms of the nominal values of A, B_i, C, and u_i^*. Thus, M is not affected by perturbations, and we have shown in Corollary 1 that by using the nominal M, the proposed controller guarantees robustness against perturbations. Let x^*_{per} denote the equilibrium point of the perturbed system, which is obtained by (2) with $\tilde{A} = A + \Delta A$, $\tilde{B}_i = B_i + \Delta B_i$, $\tilde{b}_i = b_i + \Delta b_i$, and $\tilde{G} = G + \Delta G$. To estimate a feasible set (7) for the perturbed system, we need to describe $\|x - x^*_{\text{per}}\|$ as a function of ΔA, ΔB_i, Δb_i, and ΔG, which is left for future work.

B. Special Cases

In this subsection, we examine closely the special cases of the proposed controller (6). First, we consider the integral action of the input only (i.e., we remove the term $S(x, z)$ for the output integral action), which yields

$$K\dot{u} = -H\left(u - u^*\right) - \varepsilon_1 R\left(x\right). \tag{11}$$

With this controller, we can show the exponential stability of the closed-loop system.

Corollary 2: Given $(x^*, u^*) \in \mathcal{D}$, consider (1) *in closed-loop with* (11). *Then, the following statements hold:*

(i) the closed-loop system is globally exponentially stable at (x^*, u^*);

(ii) Ω_c in (7) *is a feasible region of attraction if $\Omega_c \subset \mathcal{X} \times \mathcal{U}$.*

Proof: We select a Lyapunov candidate as

$$\tilde{V}\left(x, u\right) := \frac{1}{2}\|x - x^*\|_P^2 + \frac{1}{2\varepsilon_1}\|u - u^*\|_K^2. \tag{12}$$

From (8), its time derivative along trajectories of the closed-loop system satisfies

$$\begin{aligned}\dot{\tilde{V}}\left(x, u\right) = {} & (x - x^*)^\top P\left(A + \sum_{i=1}^{m} B_i u_i^*\right)(x - x^*) \\ & + (x - x^*)^\top P \sum_{i=1}^{m}(B_i x + b_i)(u_i - u_i^*) \\ & - \frac{1}{\varepsilon_1}\|u - u^*\|_H^2 - (u - u^*)^\top R\left(x\right).\end{aligned}$$

Then, it follows from (4) and (6c) that

$$\dot{\tilde{V}}\left(x, u\right) = -\frac{1}{2}\|x - x^*\|_Q^2 - \frac{1}{\varepsilon_1}\|u - u^*\|_H^2. \tag{13}$$

Since $Q \succ 0$ and $H \succ 0$, there exists some $\alpha > 0$ such that $\dot{\tilde{V}} \leq -\alpha\tilde{V}$. Thus, the closed-loop system is globally exponentially stable at (x^*, u^*). Also the second statement holds, since Ω_c is a level set of $\tilde{V}(x, u)$ that is positively invariant and a feasible region of attraction. ■

Next, we consider the output integral action only (i.e. we select $\dot{u} = 0$), which recovers the controller proposed in [6] (without the saturation for the input), i.e.,

$$u = u^* - H^{-1}\left(\varepsilon_1 R\left(x\right) - \varepsilon_2 S\left(x, z\right)\right), \tag{14a}$$

$$\dot{z} = C(x - x^*). \tag{14b}$$

According to [6], this controller achieves global asymptotic stabilization of $(x^*, 0)$. This can be confirmed by using the following Lyapunov candidate:

$$\overline{V}\left(x, z\right) = \frac{1}{2}\|x - x^*\|_P^2 + \frac{\varepsilon_2}{2\varepsilon_1}\|z - M\left(x - x^*\right)\|_\theta^2.$$

To satisfy the input constraint, i.e., $u(t) \in \mathcal{U}$ for all $t \geq 0$, a saturation mechanism is applied in [4], [6] instead of estimating a feasible set. However, to satisfy the state constraint, an additional analysis is needed.

As a further special case, removing $S(x, z)$ from (14) leads to the following static controller:

$$u = u^* - \varepsilon_1 H^{-1} R(x), \tag{15}$$

which globally asymptotically stabilizes x^*, see e.g. [10]. This can be shown based on the Lyapunov candidate $\hat{V}\left(x\right) = \|x - x^*\|_P^2/2$. In order to deal with the input constraint, we need to analyze the controller equation (15) containing the quadratic function $R(x)$.

In contrast to these two existing controllers, by the proposed controller (6) (or its special case (11)), a feasible set

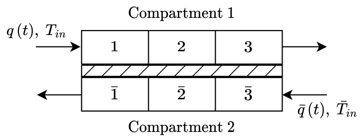

Fig. 1. Schematic of a counter-current heat exchanger with three cells in two layers

TABLE I
VALUES OF SYSTEM'S PARAMETERS

Parameter	Value	Parameter	Value
λ	10 (J/K/s)	ρ	997 (kg/m^3)
v	0.002 (m^3)	c_ρ	4185 (J/kgK)
T_{in}	360 (K)	$\bar{T}_{in}$	300 (K)
$\bar{q}$	0.02 (kg/s)	$u_{\min}/u_{\max}$	0/0.05 (kg/s)

Ω_c can be intuitively shaped by selecting K/ε_1. This aspect is further emphasized below by designing a controller for a heat exchanger.

IV. COUNTER-CURRENT HEAT EXCHANGE SYSTEM

A heat exchanger is a device in which two fluid streams exchange heat without mixing. Here we consider, in particular, a counter-current heat exchanger, a diagram of which is shown in Fig. 1. Each of the exchanger's compartments is split into three cells in which temperature is assumed to be uniform. We denote by T_i and $\bar{T}_i$ the temperature of the ith cell at compartments 1 and 2, respectively. Analogous descriptions follow for the stream volumetric flow rates $q(t)$ and $\bar{q}(t)$, and the stream temperatures T_{in} and $\bar{T}_{in}$ at the inlet of each compartment.

Let $x = (T_1, T_2, T_3, \bar{T}_1, \bar{T}_2, \bar{T}_3)$ be the system state and $u(t) = q(t)$ the control input. If $\bar{q}$, T_{in}, and $\bar{T}_{in}$ are assumed to be positive constants, then the evolution of x is determined by the following model [6]:

$$\dot{x} = Ax + (Bx + b)\,u + G, \tag{16}$$

$$A = \frac{1}{\beta}\left[\begin{array}{c|c} -\alpha\mathbb{I}_{3\times3} & \alpha\mathbb{I}_{3\times3} \\ \hline -\alpha\mathbb{I}_{3\times3} & \begin{smallmatrix} -\bar{q}-\alpha & \bar{q} & 0 \\ 0 & -\bar{q}-\alpha & \bar{q} \\ 0 & 0 & -\bar{q}-\alpha \end{smallmatrix} \end{array}\right]$$

$$B = \frac{1}{\beta}\left[\begin{array}{c|c} \begin{smallmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{smallmatrix} & 0_{3\times3} \\ \hline 0_{3\times3} & 0_{3\times3} \end{array}\right], \quad b = \frac{1}{\beta}\begin{bmatrix} T_{in} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$G = \frac{1}{\beta}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{q}\bar{T}_{in} \end{bmatrix}^\top,$$

and $\alpha = \lambda/c_\rho > 0$, $\beta = \rho v > 0$. The constant parameters λ, c_ρ, ρ, and v denote the heat transfer coefficient, heat capacity, mass density of the fluid, and volume of the fluid in the heat exchanger, respectively.

From a physical viewpoint, it is desirable that the temperature T_i or $\bar{T}_i$ of each cell remains in a suitable interval all the time. Moreover, it is practically sensible that the input should be constrained by specified lower and upper bounds. Therefore, considering the analysis reported in [6], we propose to respectively restrict x and u to the following compact sets:

$$\mathcal{X} = \left\{x \in \mathbb{R}^6 : \bar{T}_{in} \le x_i \le T_{in},\ i = 1, \cdots, 6\right\}, \tag{17a}$$
$$\mathcal{U} = \{u \in \mathbb{R} : u_{\min} \le u \le u_{\max}, \}. \tag{17b}$$

To apply the proposed controller (6), we verify Assumptions 1 and 2. For any $u^* > 0$, it is possible to show that $A + Bu^*$ is Hurwitz, i.e., Assumption 1 holds.

For the output regulation problem, following [6], we choose x_4 as the system output, obtaining the following output matrix $C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. In this case, for any given u^* and x^* satisfying (2) except for $x_1^* = x_2^* = x_3^* = T_{in}$, it is possible to show that (5a) holds [6]. Therefore, the controller (6) globally asymptotically stabilizes the equilibrium $(x^*, u^*, 0)$. Also, a feasible set $\Omega_c \subset \mathcal{X} \times \mathcal{U}$ in (7) can be estimated. For the considered heat exchanger, P satisfying (4) can be selected as diagonal because for any $u^* > 0$, $A + Bu^*$ is Hurwitz and Metzler; see references on positive systems, e.g., [18, Proposition 1]. Similarly, the controller (11) only with the input integral action can be implemented for making (x^*, u^*) exponentially stable while estimating a feasible set.

Simulation

We report now the results of a number of MATLAB simulations on the considered heat exchanger model (16) in closed-loop with the controllers proposed in section III. Table I contains the heat exchanger parameters used in simulation.

First, we apply the dynamic controller (11) without output regulation. The matrix P is set as

$$P = \text{diag}\left([3.60, 0.82, 0.42, 0.42, 0.42, 1.21]\right).$$

Also, we choose $\varepsilon_1 = \varepsilon_2 = 1$ such that the convergence speeds of $x - x^*$ and z become similar. Then, we select $K = 1.032 \times 10^5$ to shape the estimated feasible set Ω_c such that it becomes very close to the actual feasible set $\mathcal{X} \times \mathcal{U}$. Finally, we determine H such that $K^{-1}H = 0.1$ based on the discussion in Section III-A about the relation among the convergence speeds of $u - u^*$, $x - x^*$, and z.

The simulation result is shown in Fig. 2. The initial conditions are

$$x(0) = [355.56, 351.09, 346.57, 314.76, 309.89, 304.97]^\top$$

and $u(0) = 0.022$. The desired state and input equilibrium pair (x^*, u^*) is switched over time as follows:

$$\begin{aligned}
x^* &= [356.94, 353.76, 350.45, 315.5, 310.54, 305.38]^\top, \\
u^* &= 0.03246, \qquad \text{when } 0 \le t < 1000(s), \\
x^* &= [357.52, 354.90, 352.14, 315.81, 310.83, 305.54]^\top, \\
u^* &= 0.04024, \qquad \text{when } 1000 \le t < 2000(s), \\
x^* &= [356.35, 352.60, 348.75, 315.18, 310.26, 305.20]^\top, \\
u^* &= 0.27, \qquad \text{when } t \ge 2000(s),
\end{aligned}$$

We note that convergence is achieved in all the cases and that $x \in \mathcal{X}$ and $u \in \mathcal{U}$ all the time. It is worth noting that all pairs (x^*, u^*) in this section satisfy Assumption 1.

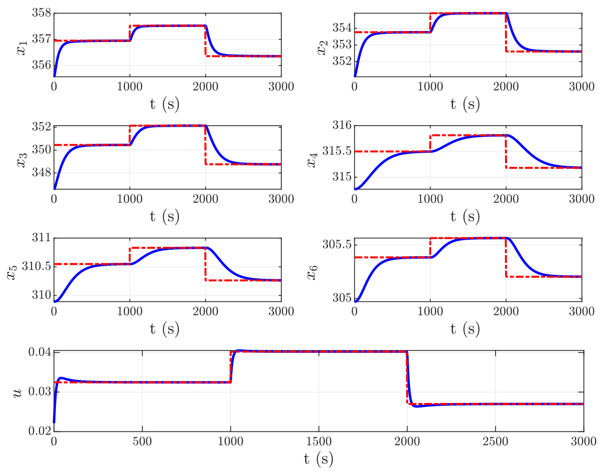

Fig. 2. Evolution of the temperatures of the heat exchanger cells and the control effort with input regulation process.

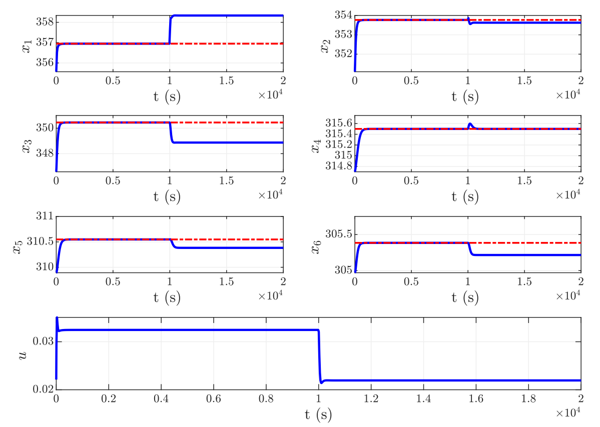

Fig. 3. Evolution of the temperatures of the the heat exchanger cells and the control effort with input and output regulation processes.

The second scenario concerns the designed output regulator (6). Here, we investigate the ability of the controller to regulate the output $y = x_4$ to a desired setpoint in presence of uncertainty in the system parameters. The controller parameter θ is chosen as $\theta = 0.1$, and the others are the same as above. The closed-loop system behavior is shown in Fig. 3, where the initial conditions are the same as in the previous scenario, and the desired equilibrium is the first of those listed above. Both x and u converge to their equilibrium values for $t < 10000$ (s). At the time 10000 (s) a perturbation is introduced by changing the value of T_{in} from 360 to 363 (K). In this case, the output converges to the desired value $y^* = x_4^*$, while the stability properties of the system are preserved and $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ for all $t \geq 0$. Compared to other controllers, the static controller in [10] does not satisfy the feasibility constraints and also fails to achieve output regulation under perturbations. Also, since the proposed controller does not contain a saturation, the control signal does not cause (high frequency) switching that may be experienced in saturated-based controllers such as the one in [6].

V. Conclusion and Future Works

In this letter, we have studied stabilizing control design for a class of bilinear systems under state and input constraints. Our controller is based on integral actions on the input and output to balance input and output regulations. Closed-loop stability has been proven based on Lyapunov analysis, which also gives a sufficient condition to satisfy input and state constraints. Finally, our approach has been applied to the counter-current heat exchanger system, and the effectiveness of the proposed method has been illustrated by numerical simulations. Future work includes estimating a tighter feasible region of attraction and also a range of perturbations in which output regulation is guaranteed.

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