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A structure exploiting SDP solver for robust controller synthesis

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***Abstract*— In this paper, we revisit structure exploiting SDP solvers dedicated to the solution of Kalman-Yakubovic-Popov semi-definite programs (KYP-SDPs). These SDPs inherit their name from the KYP lemma and they play a crucial role in e.g. robustness analysis, robust state feedback synthesis, and robust estimator synthesis for uncertain dynamical systems. Off-the-shelve SDP solvers require $O(n^6)$ arithmetic operations per Newton step to solve this class of problems, where n is the state dimension of the dynamical system under consideration. Specialized solvers reduce this complexity to $O(n^3)$. However, existing specialized solvers do not include semi-definite constraints on the Lyapunov matrix, which is necessary for controller synthesis. In this paper, we show how to include such constraints in structure exploiting KYP-SDP solvers.**

I. INTRODUCTION

Let $\mathbb{S}^n$ denote the set of symmetric matrices of dimension n. In this work, we study optimization problems with semi-definite constraints of the form

$$\underset{\lambda\in\mathbb{R}^p, P\in\mathbb{S}^n}{\text{minimize}} \quad c^\top\lambda - \operatorname{trace}(\Sigma P) \tag{1a}$$

$$\text{s.t.} \quad \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^\top \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \begin{pmatrix} Q(\lambda) & S(\lambda) \\ S^\top(\lambda) & R(\lambda) \end{pmatrix} \prec 0, \tag{1b}$$

$$N(\lambda) \succ 0, \tag{1c}$$

$$P \succ 0, \tag{1d}$$

where $H(\lambda) := H_0 + \sum_{i=1}^p \lambda_i H_i$ for $H \in \{N, Q, S, R\}$ are affine matrix valued functions of λ. The matrix parameters are chosen to be of compatible dimensions, i.e., $Q_i \in \mathbb{S}^n$, $S_i \in \mathbb{R}^{n\times m}$, $R_i \in \mathbb{S}^m$, $N_i \in \mathbb{S}^r$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $c \in \mathbb{R}^p$ for $i = 0, \ldots, p$ and positive integers $n, m, p, r \in \mathbb{N}$. Finally, we assume that the matrix pair (A, B) is controllable and that the matrix $\Sigma \in \mathbb{S}^n$ is positive semi-definite. Linear matrix inequalities of the form (1b) frequently appear in control and signal processing and are related to the celebrated KYP lemma. A partial list of applications for the SDP (1) includes robustness analysis, robust controller synthesis, and robust estimator synthesis for linear dynamical systems. Today, the solution of online SDPs, e.g., for online data-driven control, or the solution of large SDPs, e.g., for estimating the Lipschitz constant of recurrent neural networks, should be of particular interest.

In many scenarios where the SDP (1) appears, the state dimension n is larger than the number p of multipliers λ_i. In this case, the $O(n^2)$ decision variables in the matrix P typically dominate the computational effort for solving (1). Many off-the-shelve solvers work in the canocical representation of LMIs $F(x) = F_0 + x_1 F_1 + \ldots$, for which the the cost of one Newton step scales with n^6. LMILab and STRUL [7] can beneficially exploit matrix variables by cleverly assembling the involved Hessian matrix. The KYP-LMI (1b) fits well into this structure (7) in [7], but there are further possibilities to solve optimization (1a)-(1c) beyond cleverly assembling the involved Hessian within a standard scheme [3], [15], [11], [8], [7], [21], [20].

Further structure exploiting algorithms for (1a)-(1c) include cutting plane methods [22], [6], [10], [1]. These methods optimize over P in an inner loop, whereas cutting planes for λ are constructed in an outer loop. Such a splitting approach enables a more efficient optimization over P, e.g., by solving Riccati equations. Consequently, cutting plane methods are effective when the number of variables λ_i is small, but according to [12] probably less effective when this number is moderate.

Alternatively to optimizing over P in (1), one can also approach (1a)-(1c) by solving the equivalent frequency domain inequality. This is considered in [14], where the frequency domain inequality, which involves an infinite number of semi-definite constraints, is solved using a sampling approach. The latter can reduce the computational effort for Newton iterations, but produces only a lower bound on the optimal value. For this reason, in [12], a barrier function for the frequency domain inequality over all frequencies is constructed. Evaluating this barrier function requires solving Riccati and Lyapunov equations in each inner loop iteration. In addition, [12] differentiates through the Riccati and Lyapunov equation to enable efficient optimization also for moderate numbers of λ_i using second-order optimization algorithms.

In the present paper, we extend the problem (1a)-(1c) studied in the cited KYP-SDP literature with the semi-definite constraint (1d). This constraint enables, for example, robust state-feedback synthesis. Methodologically, we employ a second-order optimization algorithm

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to minimize a barrier function relaxation of (1) similarly to [12]. To this end, our key step is introducing a convex barrier function for the existence of a solution to a Riccati equation.

II. Problem statement

Since the problem (1) can be expensive to solve by off-the-shelve SDP solvers, we study the alternative problem

$$\begin{aligned} \underset{\lambda\in\mathbb{R}^p}{\text{minimize}} \quad & c^\top\lambda - \operatorname{trace}\Sigma P_+(\lambda) \\ \text{s.t.} \quad & N(\lambda) \succ 0, P_+(\lambda) \succ 0, \lambda \in \mathcal{D}. \end{aligned} \tag{2}$$

Here, the value of the function $\lambda \mapsto P_+(\lambda)$ is defined as the unique anti-stabilizing solution P of the Riccati equation $F(P,\lambda) = 0$, where F is defined as

$$F(P,\lambda) = A^\top P + PA + Q - (PB+S)R^{-1}(PB+S)^\top \tag{3}$$

and where we abbreviate $Q = Q(\lambda)$, $S = S(\lambda)$ and $R = R(\lambda)$. Further, $\mathcal{D}$ is defined as the set of all $\lambda \in \mathbb{R}^p$ with $R(\lambda) \prec 0$ for which $F(\cdot,\lambda) = 0$ has an anti-stabilizing solution. This problem formulation is motivated by the following extended version of the KYP lemma [5].

Lemma 2.1: Consider a fixed $\lambda \in \mathbb{R}^p$ and suppose that $\operatorname{eig}(A)\cap i\mathbb{R} = \varnothing$ and that (A,B) is controllable. Then the following statements are equivalent.

1) $\begin{pmatrix}(A-i\omega I)^{-1}B \\ I\end{pmatrix}^* \begin{pmatrix} Q & S \\ S^\top & R\end{pmatrix}\begin{pmatrix}(A-i\omega I)^{-1}B \\ I\end{pmatrix} \prec 0$ for all $\omega \in \mathbb{R}\cup\{\infty\}$.
2) There exist some symmetric P satisfying (1b).
3) $R(\lambda) \prec 0$ and there exists $P \in \mathbb{S}^n$ with $F(P,\lambda) \prec 0$.
4) $R(\lambda) \prec 0$ and the Riccati equation $F(P,\lambda) = 0$ has an anti-stabilizing solution $P_+(\lambda) \in \mathbb{S}^n$.

We mention that standard solvers are based on 2), [12], [14] are based on 1), whereas we utilize 4).

In the course of our exposition, we show that our formulation (2) provides the same numerical advantages as [12], but additionally allows us to consider the constraint (1d) and the cost term $-\operatorname{trace}\Sigma P$. The challenge is to handle the constraint $\lambda \in \mathcal{D}$, i.e., the feasibility of the Riccati equation, and the nonlinear function $\lambda \mapsto P_+(\lambda)$. We address these challenges by deriving a convex barrier function for the feasibility of the Riccati equation and by showing that the mapping $\lambda \mapsto P_+(\lambda)$ is concave (in the sense of Hermitian valued functions).

We conclude the section with an equivalence theorem for (1) and (2) which is proven in Section IV.

Theorem 2.2 (Equivalence of (1) *and* (2)*):* Problem (1) and problem (2) are equivalent, i.e., the optimal values coincide and $\lambda \in \mathbb{R}^p$ is feasible for (2) if and only if there exists $P \in \mathbb{S}^n$ such that (λ, P) is feasible for (1).

III. An interior point method for (2)

To solve (2), we propose to employ a path-following barrier method similar to [12]. For this purpose, a barrier for the constraint $\lambda \in \mathcal{D}$ is given by $\lambda \mapsto -\log\det(-R(\lambda)) - \log\det\Delta(\lambda)$ where $\Delta(\lambda) := P_+(\lambda) - P_-(\lambda)$ is the difference between the stabilizing solution $P_-(\lambda)$ and the anti-stabilizing solution $P_+(\lambda)$ of the Riccati equation. That this yields a suitable barrier is proven in Section IV. For the remaining semi-definite constraints, we utilize the standard $\log\det$ barrier function. Overall, for an increasing sequence of t, we minimize

$$\begin{aligned} v_t(\lambda) = {} & t(c^\top\lambda - \operatorname{trace}\Sigma P_+(\lambda)) - \log\det N(\lambda) \\ & - \log\det P_+(\lambda) - \log\det(-R(\lambda)) - \log\det\Delta(\lambda) \end{aligned} \tag{4}$$

as a function of λ. To solve this optimization problem, we need to determine first- and second-order derivatives of the solutions $P_+(\lambda)$, $P_-(\lambda)$ of the Riccati equation $F(P,\lambda) = 0$. To simplify the notation we drop the argument λ in our matrix-valued functions sometimes.

Theorem 3.1: Given $\lambda_0 \in \mathbb{R}^p$ and $P_0 \in \mathbb{S}^n$ with $F(P_0,\lambda_0) = 0$, if $A - BK$ for $K := R^{-1}(P_0B + S)^\top$ has no eigenvalues on the imaginary axis, then there exist a neighbourhood $\mathcal{N}$ of λ_0 and an arbitrarily often differentiable function $P : \mathcal{N} \to \mathbb{S}^n$ with $P(\lambda_0) = P_0$, such that $F(P(\lambda),\lambda) = 0$ for all $\lambda \in \mathcal{N}$. Moreover, the partial derivative $\partial_{\lambda_i}P$ is the solution of the Lyapunov equation

$$\begin{aligned} 0 = {} & \partial_{\lambda_i}P(A-BK) + (A-BK)^\top\partial_{\lambda_i}P \\ & + \begin{pmatrix} I \\ -K\end{pmatrix}^\top \begin{pmatrix} Q_i & S_i \\ S_i^\top & R_i\end{pmatrix}\begin{pmatrix} I \\ -K\end{pmatrix}. \end{aligned} \tag{5}$$

Furthermore, the second order partial derivative $\partial_{\lambda_i}\partial_{\lambda_j}P$ is the unique solution of the Lyapunov equation

$$\begin{aligned} 0 = {} & (A-BK)^\top\partial_{\lambda_i}\partial_{\lambda_j}P + \partial_{\lambda_i}\partial_{\lambda_j}P(A-BK) \\ & -\partial_{\lambda_j}K^\top R\partial_{\lambda_i}K - \partial_{\lambda_i}K^\top R\partial_{\lambda_j}K, \end{aligned} \tag{6}$$

where $\partial_{\lambda_i}K := R^{-1}(B^\top\partial_{\lambda_i}P + S_i^\top - R_iK)$.

For the proof, we refer to [12], [17].

Theorem 3.1 enables us to differentiate the solutions $P_+(\cdot)$ and $P_-(\cdot)$ of the Riccati equation. As a consequence, we can formulate the path-following interior point method Algorithm 1 for solving (2). Derivatives of the barrier functions are derived using standard formulas as provided in Algorithm 1.

Remark 3.2 (Initial feasible points): To generate an initial interior point for Algorithm 1, we apply a standard procedure found in [4] and extend our decision variable to $\tilde\lambda := \begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_p\end{pmatrix}^\top$, the matrices Q, S and R to

$$\begin{pmatrix}\widetilde{Q}(\tilde\lambda) & \widetilde{S}(\tilde\lambda) \\ \widetilde{S}(\tilde\lambda)^\top & \widetilde{R}(\tilde\lambda)\end{pmatrix} := \begin{pmatrix} Q(\lambda) & S(\lambda) \\ S(\lambda)^\top & R(\lambda)\end{pmatrix} - \lambda_0\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix},$$

and N to $\widetilde{N}(\tilde\lambda) := N(\lambda) + \lambda_0 I$. Then $\tilde\lambda$ is an interior point for the modified problem (1) with $\widetilde{Q}(\cdot), \widetilde{R}(\cdot), \widetilde{S}(\cdot), \widetilde{N}(\cdot)$ replacing $Q(\cdot), R(\cdot), S(\cdot), N(\cdot)$ if λ_0 is sufficiently large. An interior point for the original problem can thus be found by minimizing λ_0 as the objective for the modified problem. If the infimum of this auxiliary problem is larger than or equal to zero, then the original problem is infeasible.

Algorithm 1 Solver for (2)

Input: $\varepsilon, t_{\max}$, initial feasible point λ of (2).
while $t \le t_{\max}$ **do**
  $P_- \leftarrow$ stabilizing solution of $F(P,\lambda) = 0$
  $P_+ \leftarrow$ anti-stabilizing solution of $F(P,\lambda) = 0$
  $(Q,S,R,N) \leftarrow (Q(\lambda), S(\lambda), R(\lambda), N(\lambda))$
  $v_t \leftarrow -\log\det P_+ - \log\det(-R) - \log\det N$
    $-\log\det\Delta + t(c^\top\lambda - \operatorname{trace}\Sigma P_+)$
  $(\nabla v_t)_i \leftarrow -\operatorname{trace}\Delta^{-1}\partial_{\lambda_i}\Delta + t(c_i - \operatorname{trace}\Sigma\partial_{\lambda_i}P_+)$
    $-\operatorname{trace}P_+^{-1}\partial_{\lambda_i}P_+ - \operatorname{trace}R^{-1}R_i - \operatorname{trace}N^{-1}N_i$
  $(H_{v_t})_{ij} \leftarrow \operatorname{trace}P_+^{-1}(\partial_{\lambda_i}P_+P_+^{-1}\partial_{\lambda_j}P_+ - \partial_{\lambda_i}\partial_{\lambda_j}P_+)$
    $+\operatorname{trace}R_iR^{-1}R_jR^{-1} + \operatorname{trace}N_iN^{-1}N_jN^{-1}$
    $+\operatorname{trace}\Delta^{-1}(\partial_{\lambda_i}\Delta\Delta^{-1}\partial_{\lambda_j}\Delta - \partial_{\lambda_i}\partial_{\lambda_j}\Delta)$
    $-t\operatorname{trace}\Sigma\partial_{\lambda_i}\partial_{\lambda_j}P_+$
  $d \leftarrow -H_{v_t}^{-1}\nabla v_t$ (Newton search direction)
  $\alpha \leftarrow$ line search for $\operatorname{argmin}_\alpha v_t(\lambda + \alpha d)$
  $\lambda \leftarrow \lambda + \alpha d$
  if (stopping criterion) **then** $t \leftarrow 10t$
end while
return (P_+, λ)

IV. Convexity and equivalence result of the reformulation

Algorithm 1 relies on the equivalence of (1) and (2), and the fact that (4) is a convex barrier function. We prove this fact in this section.

Lemma 4.1: Suppose $R \prec 0$ and (A,B) is controllable. Then $P \in \mathbb{S}^n$ with $F(P,\lambda) \prec 0$ exists if and only if P_+ and P_- exist. If one of these conditions holds, then the following facts are true:

1) $\forall P \in \mathbb{S}^n : F(P,\lambda) \preceq 0 \Rightarrow P_- \preceq P \preceq P_+$,
2) $\forall \varepsilon > 0 \exists P \in \mathbb{S}^n : F(P,\lambda) \prec 0$ and $P_+ - \varepsilon I \prec P \prec P_+$,
3) $\Delta = P_+ - P_- \succ 0$.

Proof: According to [18], $R \prec 0$ implies that $P \in \mathbb{S}^n$ with $F(P,\lambda) \prec 0$ exists if and only if P_+ and P_- exist.

1) Due to $R \prec 0$, this fact can be found in [18].

2) Since P_+ exists, $(A - BK_+)$ is anti-stable, where $K_+ = R^{-1}(P_+B + S)^\top$. Therefore, the Lyapunov equation

$$(A - BK_+)^\top H + H(A - BK_+) = I$$

has a solution $H \succ 0$. Now, for $\varepsilon > 0$, consider

$$\begin{aligned} F(P_+ - \varepsilon H, \lambda) &= A^\top(P_+ - \varepsilon H) + (P_+ - \varepsilon H)A + Q \\ &\quad - (S + (P_+ - \varepsilon H)B)R^{-1}(S + (P_+ - \varepsilon H)B)^\top \\ &= -\varepsilon(A - BK_+)^\top H - \varepsilon H(A - BK_+) \\ &\quad - \varepsilon^2 HBR^{-1}B^\top H \\ &= -\varepsilon I - \varepsilon^2 HBR^{-1}B^\top H. \end{aligned}$$

This expansion proves that there exists an $\varepsilon_0 > 0$, such that $P := P_+ - \varepsilon H$ is feasible for (1b) for all $\varepsilon \in]0, \varepsilon_0[$.

3) Due to 2), there exists $P \in \mathbb{S}^n$ with $P \prec P_+$. Hence, by 1), we have $P_- \preceq P \prec P_+$. ∎

Proof of Theorem 2.2: Let (λ, P) be any feasible point of (1). Due to Lemma 2.1 this implies $R \prec 0$ and the existence of P_+, i.e., $\lambda \in \mathcal{D}$. Furthermore, $P \preceq P_+$ holds (Lemma 4.1) implying $P_+ \succ 0$ and $c^\top\lambda - \operatorname{trace}\Sigma P \ge c^\top\lambda - \operatorname{trace}\Sigma P_+$. Hence, λ is feasible for (2) and the optimal value of (2) is smaller than or equal to the optimal value of (1).

Now let $\lambda \in \mathbb{R}^p$ be any feasible point of (2). Then $R \prec 0$ holds true and the anti-stabilizing solution P_+ exists implying (Lemma 4.1) the existence of $P \in \mathbb{S}^n$ with $P_+ - \varepsilon I \preceq P \prec P_+$ and $F(P,\lambda) \prec 0$ for any $\varepsilon > 0$. Hence, we can choose ε so small that $P \succ 0$ is guaranteed and we can perform a Schur complement showing that (P,λ) also satisfies (1b). Furthermore, P can be moved arbitrarily close to P_+ by letting $\varepsilon \to 0$, in which case the objective value of (1) for (λ, P) converges to $c^\top\lambda - \operatorname{trace}P_+$. ∎

Theorem 2.2 already implies that the feasible set of (2) is convex since it is the projection of the convex feasible set of (1) onto the λ variable. However, we are also able to show that all the constraint functions and the objective function of (2) are convex. To this end, we show the convexity (concavity) of the Hermitian valued functions $\lambda \mapsto P_-(\lambda)$ and $\lambda \mapsto P_+(\lambda)$. Such functions are called convex with respect to the cone of positive semi-definite matrices, if $P_-(\alpha\lambda_1 + (1-\alpha)\lambda_2) \preceq \alpha P_-(\lambda_1) + (1-\alpha)P_-(\lambda_2)$ holds for all $\alpha \in [0,1]$ or concave, if $P_+(\alpha\lambda_1 + (1-\alpha)\lambda_2) \succeq \alpha P_+(\lambda_1) + (1-\alpha)P_+(\lambda_2)$ holds for all $\alpha \in [0,1]$ ([4] p. 109).

Lemma 4.2: The mapping $\mathcal{D} \to \mathbb{S}^n, \lambda \mapsto P_-(\lambda)$ is convex and the mapping $\mathcal{D} \to \mathbb{S}^n, \lambda \mapsto P_+(\lambda)$ is concave. Furthermore, the mappings $\lambda \mapsto -\log\det P_+(\lambda)$, $\lambda \mapsto -\log\det\Delta(\lambda)$, and $\lambda \mapsto -\operatorname{trace}\Sigma P_+(\lambda)$ are convex.

Proof: W.l.o.g. consider $\lambda \mapsto P_+(\lambda)$ and two arbitrary $\lambda_1, \lambda_2 \in \mathcal{D}$. Then $P_1 = P_+(\lambda_1)$ and $P_2 = P_+(\lambda_2)$ are solutions of the Riccati equation and thus both satisfy the non-strict version of (1b). Since (1b) is a convex constraint in both λ and P, also $\lambda_\alpha = \alpha\lambda_1 + (1-\alpha)\lambda_2$ and $P_\alpha = \alpha P_1 + (1-\alpha)P_2$ satisfy the non-strict (1b) for any $\alpha \in [0,1]$. Consequently, P_α satisfies the Riccati inequality $F(P_\alpha, \lambda) \preceq 0$ for $\lambda = \lambda_\alpha$ implying by 1) of Lemma 4.1 that

$$\alpha P_+(\lambda_1) + (1-\alpha)P_+(\lambda_2) = P_\alpha \preceq P_+(\alpha\lambda_1 + (1-\alpha)\lambda_2)$$

holds. This shows the concavity of $P_+(\cdot)$.

The convexity of the $\log\det$ functions and the cost function of (2) follows from the composition theorem ([4] page 110) for convex functions. ∎

A key role in our barrier function (4) is played by the difference Δ between the stabilizing and anti-stabilizing solution of the Riccati equation. This difference can be obtained by solving the Riccati equation twice or, more efficiently, it can be obtained from only one solution P_+ of the Riccati equation and then solving a Lyapunov equation, according to the following lemma.

Lemma 4.3: Let P_1, P_2 denote two solutions of the Riccati equation $F(P,\lambda) = 0$ and $K_1 = R^{-1}(S + P_1B)^\top$ the controller gain of P_1. If the difference $Y = P_2 - P_1$ is invertible, then it satisfies the Lyapunov equation

$$Y^{-1}(A - BK_1)^\top + (A - BK_1)Y^{-1} = BR^{-1}B^\top \qquad (7)$$

and $(A - BK_1)$ has no eigenvalues on the imaginary axis.

Proof: Since P_2 is a solution of the Riccati equation, $F(P_2,\lambda)=0$ holds true. Substituting P_1+Y for P_2 yields

$$\begin{aligned}0 &= A^\top(P_1+Y)+(P_1+Y)A+Q\\ &\quad-(S+(P_1+Y)B)R^{-1}(S+(P_1+Y)B)^\top.\end{aligned}$$

By rearranging terms and using $F(P_1,\lambda)=0$ we obtain

$$0=A^\top Y+YA-YBK_1-K_1^\top B^\top Y-YBR^{-1}B^\top Y.$$

Multiplying this equation from both sides by Y^{-1} yields the claimed Lyapunov equation. Next, we show that $(A-BK_1)$ has no imaginary eigenvalues. To this end, assume that w is an eigenvector of $(A-BK_1)^\top$ with imaginary eigenvalue μ. Multiplying (7) from both sides by w yields

$$w^*BR^{-1}B^\top w=w^*Y^{-1}(\mu w)+(\mu w)^*Y^{-1}w=0.$$

Since $BR^{-1}B^\top\preceq 0$, this implies $BR^{-1}B^\top w=0$. The latter cannot be true, since (A,B) is controllable implying that $((A-BK_1)^\top, BR^{-1}B^\top)$ is observable. This prevents the existence of an eigenvector of $(A-BK_1)^\top$ with $BR^{-1}B^\top w=0$ by the Hautus Lemma. ∎

Both the Riccati equation (3) and the Lyapunov equation (7) also appear in [12]. There, these equations are solved to obtain the factorization of a transfer matrix involved in their barrier function. Our arguments show that solving this Riccati and Lyapunov equation corresponds to computing both solutions of the Riccati equation (3).

Finally, we conclude in the following lemma that (4) is indeed a suitable barrier function for the problem (2).

Theorem 4.4: Let $t>0$ be fixed. If (λ_k) is a sequence of feasible values for (2) with a limit $\bar\lambda$ on the boundary of the feasible set, then $v_t(\lambda_k)$ converges to infinity.

Proof: Since $\bar\lambda$ is on the boundary of the feasible set of (2), we can perturb the problem data Q,R,N as in Remark 3.2 to $\widetilde{Q}(\lambda)=Q(\lambda)-\lambda_0 I$, $\widetilde{R}(\lambda)=R(\lambda)-\lambda_0 I$ and $\widetilde{N}(\lambda)=N(\lambda)+\lambda_0 I$ with $\lambda_0>0$. For the perturbed problem, all λ_k and $\bar\lambda$ are feasible and P_+ and P_- satisfy the strict Riccati inequality. Consequently, P_+ and P_- satisfy, by Lemma 4.1, the inequality

$$\widetilde{P}_-(\lambda_k)\preceq P_-(\lambda_k)\preceq P_+(\lambda_k)\preceq\widetilde{P}_+(\lambda_k)$$

for all $k\in\mathbb{N}$, where $\widetilde{P}_-(\lambda_k)$ and $\widetilde{P}_+(\lambda_k)$ are the solutions of the perturbed Riccati equations. Since $\widetilde{P}_-(\lambda_k)$ and $\widetilde{P}_+(\lambda_k)$ converge to $\widetilde{P}_-(\bar\lambda)$ and $\widetilde{P}_+(\bar\lambda)$, the sequences $P_+(\lambda_k)$ and $P_-(\lambda_k)$ are bounded and, consequently, all log-determinants in (4) are bounded from below.

If $R(\bar\lambda)\nprec 0$, then, by continuity, we infer $R(\bar\lambda)\preceq 0$ and $\det R(\bar\lambda)=0$ which implies that one of the terms in (4) goes to infinity while the others are bounded from below.

Hence, suppose $R(\bar\lambda)\prec 0$. If $\det\Delta(\lambda_k)\to 0$, then $v_t(\lambda_k)$ also goes to infinity. If $\det\Delta(\lambda_k)$ does not converge to zero, then there exist accumulation points $\overline{P}_-$ and $\overline{P}_+$ of $P_-(\lambda_k)$ and $P_+(\lambda_k)$ with $\det(\overline{P}_+-\overline{P}_-)\neq 0$, since $P_-(\lambda_k)$ and $P_+(\lambda_k)$ are bounded sequences. By continuity we infer $\overline{P}_+-\overline{P}_-\succ 0$ and that $\overline{P}_+$ and $\overline{P}_-$ solve the Riccati equation. Hence, by Lemma 4.3, the eigenvalues of $(A-B\overline{K}_+)$ and $(A-B\overline{K}_-)$ cannot lie on the imaginary axis, implying that $\overline{P}_-$ and $\overline{P}_+$ are (anti-)stabilizing solutions of $F(\cdot,\bar\lambda)=0$. In this case, we infer $\bar\lambda\in\mathcal{D}$ implying that $\bar\lambda$ can only be infeasible if $P_+(\bar\lambda)\nsucc 0$. Then, also $v_t(\lambda_k)\to\infty$. ∎

V. On the complexity of Algorithm 1

The complexity of one (Newton) iteration of Algorithm 1 is dominated by evaluating the $n\times n$ matrices P_+ and P_- and by computing the derivatives $\partial_{\lambda_i}P_+$, $\partial_{\lambda_i}P_-$ and $\partial_{\lambda_i}\partial_{\lambda_j}P_+, \partial_{\lambda_i}\partial_{\lambda_j}P_-$ of these matrices for $i,j=1,\dots,p$. To this end, q_1 Riccati equations need to be solved, where q_1 is the number of line search iterations, and $q_1+p(p+3)$ Lyapunov equations need to be solved. Here, two times p Lyapunov equations are required for the first order derivatives and two times $p(p+1)/2$ Lyapunov equations are required for the second order derivatives of P_+ and P_-. Using the Schur method, the leading term of the multiplication/division operations required for the Riccati equation is $45q_2n^3$, where q_2 is the *average number of double QR-iterations required to make a sub diagonal element equal to zero* [16]. For Lyapunov equations, the leading term of the complexity bound can be reduced to $5q_2n^3$.

Summing this up leads to an asymptotic complexity of $5n^3q_2(10q_1+p(p+3))$ per Newton step for solving Riccati and Lyapunov equations. In addition, there is a computational effort of $O(q_1p(n^2+m^2+r^2))$ for evaluating $Q(\lambda), S(\lambda), R(\lambda)$ and $N(\lambda)$, of $O(p^2(n+m+r)^2+p(n+m+r)^3)$ for evaluating the log-determinant and its derivatives, of $O(p^2(nm^2+n^2))$ for setting up the Riccati and Lyapunov equations, and of $O(p^3)$ for solving the Newton system. However, these should be dominated by the complexity of Riccati and Lyapunov equations.

VI. Application example: Robust state feedback design

Unlike the prior works we cited in the introduction, Algorithm 1 enables the solution of KYP-SDPs for state-feedback synthesis. Thus, we consider as a benchmark a robust LQR synthesis task for dynamical systems

$$\begin{pmatrix}\dot x(t)\\ z(t)\end{pmatrix}=\begin{pmatrix}\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2\\ \mathcal{C} & \mathcal{D}_1 & \mathcal{D}_2\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\\ w(t)\end{pmatrix}. \tag{8}$$

In this state space description, $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, and $w(t)\in\mathbb{R}^d$ and $z(t)\in\mathbb{R}^l$ are the input and output of an uncertain system component. We assume that this uncertain component satisfies, for all times, the family of quadratic constraints

$$\begin{pmatrix}\mathcal{C}x+\mathcal{D}_1u+\mathcal{D}_2w\\ w\end{pmatrix}^\top M(\lambda)^{-1}\begin{pmatrix}\mathcal{C}x+\mathcal{D}_1u+\mathcal{D}_2w\\ w\end{pmatrix}\succeq 0 \tag{9}$$

for all $\lambda\in\mathbb{R}^p$ with $N(\lambda)\succ 0$. Our goal is finding a robust performance control Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}, x\mapsto x^\top P^{-1}x$ with positive definite $P=P^\top$ such that

$$\min_{u\in\mathbb{R}^m}\nabla V(x)^\top(\mathcal{A}x+\mathcal{B}_1u+\mathcal{B}_2w)+x^\top\mathcal{Q}x+u^\top\mathcal{R}u\leq 0 \tag{10}$$

holds true for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^d$ satisfying (9). In (10), $\mathcal{Q}$ and $\mathcal{R}$ are positive definite matrices and $(x,u) \mapsto x^\top \mathcal{Q} x + u^\top \mathcal{R} u$ is a stage cost function. As we show in Appendix A using standard techniques from robust control, such a Lyapunov function can be found by solving the SDP

$$\underset{P \in \mathbb{S}^n, \lambda \in \mathbb{R}^p}{\text{minimize}} \quad -\operatorname{trace} P \tag{11}$$

subject to $P \succ 0$, $N(\lambda) \succ 0$ and

$$0 \succ \left(\begin{array}{c|cc} \mathcal{A}^\top & I & \mathcal{C}^\top \\ \hline I & 0 & 0 \end{array}\right)^\top \left(\begin{array}{c|c} 0 & P \\ \hline P & 0 \end{array}\right) \left(\begin{array}{c|cc} \mathcal{A}^\top & I & \mathcal{C}^\top \\ \hline I & 0 & 0 \end{array}\right) - \tag{12}$$

$$(\star)^\top \begin{pmatrix} \mathcal{Q}^{-1} & 0 & & \\ 0 & \mathcal{R}^{-1} & & \\ & & M_{11}(\lambda) & M_{12}(\lambda) \\ & & M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} \left(\begin{array}{c|cc} 0 & -I & 0 \\ \mathcal{B}_1^\top & 0 & \mathcal{D}_1^\top \\ 0 & 0 & -I \\ \mathcal{B}_2^\top & 0 & \mathcal{D}_2^\top \end{array}\right)$$

if the family of multipliers satisfies the conditions

$$\begin{pmatrix} -I \\ \mathcal{D}_2^\top \end{pmatrix}^\top \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} \begin{pmatrix} -I \\ \mathcal{D}_2^\top \end{pmatrix} \succ 0, \; M_{22}(\lambda) \prec 0 \tag{13}$$

for all $\lambda \in \mathbb{R}^p$ with $N(\lambda) \succ 0$.

In order to consider realistic control systems, we use the database [13] to select the system matrices $\mathcal{A}$ and $\mathcal{B}_1$. To model uncertainty (which is not available in [13]), we assume that the actuators of our controller are subject to a parametric multiplicative uncertainty of 25%. This model assumption can be implemented by choosing the matrices $\mathcal{C} = 0$, $\mathcal{D}_1 = I$, $\mathcal{D}_2 = 0$ and $\mathcal{B}_2 = \mathcal{B}_1$. Furthermore, M and N can be chosen as $N(\lambda) = \operatorname{diag}(\lambda_1, \dots, \lambda_p)$ and

$$M(\lambda) = \operatorname{diag}(\gamma^2\lambda_1, \dots, \gamma^2\lambda_p, -\lambda_1, \dots, -\lambda_p),$$

where $\gamma = 0.25$. For these system matrices and multiplier matrix, we solve the KYP-SDP (11) using Algorithm 1, the method STRUL [7], which incorporates structure exploitation for matrix variables into SDPT3, and the off-the-shelve SDP solvers SeDuMi [19] and Mosek [2]. Solution times for multiple discretizations of an Euler Bernoulli Beam (EB) system, a heat flow (HF) system, and a cable mass (CM) model can be found in Table I. Our implementation, as well as the statistics for all the other models featured in [13], are provided on github (`https://github.com/SphinxDG/KYP-SDP`).

VII. Conclusion

We have presented a new solver for KYP-SDPs. To exploit the structure of these LMI optimization problems, we formulate an equivalent problem where the Lyapunov matrix of the KYP-LMI is eliminated by solving a Riccati equation instead. This step removes $O(n^2)$ decision variables from the SDP and preserves the convexity of the original problem. As seen in Table I, this approach achieves a significant speed-up compared to off-the-shelve solvers and enables us to solve larger problems.

TABLE I: Computation times for four solvers. The number of system states is n the number of multipliers (control inputs) is $p = m$. "-" means that a solver did not solve the problem within $10^4 s$.

Problem	n	p	STRUL	SeDuMi	Mosek	Algo 1
EB1	10	1	0.184s	0.116s	0.0717s	0.0188s
EB2	10	1	0.189s	0.112s	0.0700s	0.0171s
EB3	10	1	0.197s	0.114s	0.0708s	0.0172s
EB4	20	1	0.269s	0.404s	0.130s	0.0575s
EB5	40	1	1.10s	4.18s	0.682s	0.405s
EB6	160	1	319s	-	270s	7.53s
HF2D3	4489	2	-	-	-	8579s
HF2D4	2025	2	-	-	-	715s
HF2D5	4489	2	-	-	-	8670s
HF2D6	2025	2	-	-	-	690s
CM1	20	1	2.20s	0.604s	1.03s	0.130s
CM2	60	1	19.4s	41.3s	4.20s	0.90s
CM3	120	1	483s	2770s	71.6s	2.59s
CM4	240	1	-	-	2073s	19.9s
CM5	480	1	-	-	-	92.0s
CM6	960	1	-	-	-	404s

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Appendix

A. Recap of elimination for robust LQR-synthesis

Denote by $\widetilde{M}$ the matrix

$$\begin{pmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} := \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}^{-1}.$$

The first step to derive the KYP-LMI (12) is a multiplier relaxation of the constraint (10). Namely, if there exists a $\lambda \in \mathbb{R}^p$ with $N(\lambda) \succ 0$ and a $\mathcal{K} \in \mathbb{R}^{m\times n}$, such that

$$\nabla V(x)^\top((\mathcal{A}+\mathcal{B}_1\mathcal{K})x+\mathcal{B}_2 w) + \begin{pmatrix} z \\ w \end{pmatrix}^\top \widetilde{M}(\lambda) \begin{pmatrix} z \\ w \end{pmatrix} + x^\top(\mathcal{Q}+\mathcal{K}^\top\mathcal{R}\mathcal{K})x \tag{14}$$

is non-positive for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w \in \mathbb{R}^{d_1}$, then this implies (10). The new constraint (14) means that

$$(\star)^\top \begin{pmatrix} 0 & P^{-1} & & & & \\ P^{-1} & 0 & & & & \\ & & \mathcal{Q} & 0 & & \\ & & 0 & \mathcal{R} & & \\ & & & & \widetilde{M}_{11} & \widetilde{M}_{12} \\ & & & & \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A}+\mathcal{B}_1\mathcal{K} & \mathcal{B}_2 \\ I & 0 \\ \mathcal{K} & 0 \\ \mathcal{C}+\mathcal{D}_1\mathcal{K} & \mathcal{D}_2 \\ 0 & I \end{pmatrix} \tag{15}$$

must be negative definite. This constraint is non-convex due to $\mathcal{K}$. Hence, we apply the following elimination lemma ([9], Theorem 2) to eliminate the variable $\mathcal{K}$.

Lemma 1.1: Consider the matrix inequality

$$\begin{pmatrix} I_k \\ U^\top\mathcal{K}V+W \end{pmatrix}^\top \mathcal{P} \begin{pmatrix} I_k \\ U^\top\mathcal{K}V+W \end{pmatrix} \prec 0 \tag{16}$$

and assume that $\mathcal{P} = \mathcal{P}^\top$ is invertible with exactly k negative eigenvalues. Let $U_\perp, V_\perp$ be basis matrices of $\ker(U), \ker(V)$ respectively. Then there exists a $\mathcal{K} \in \mathbb{R}^{m\times n}$ such that (16) is satisfied if and only if

$$V_\perp^\top \begin{pmatrix} I \\ W \end{pmatrix}^\top \mathcal{P} \begin{pmatrix} I \\ W \end{pmatrix} V_\perp \prec 0 \ \& \ U_\perp^\top \begin{pmatrix} W^\top \\ -I \end{pmatrix}^\top \mathcal{P}^{-1} \begin{pmatrix} W^\top \\ -I \end{pmatrix} U_\perp \succ 0.$$

Note that (15) satisfies the eigenvalue condition, since

$$\begin{pmatrix} 0 & P^{-1} \\ P^{-1} & 0 \end{pmatrix}$$

has n positive and n negative eigenvalues, the matrix $\widetilde{M}$ has d_1 negative and d_2 positive eigenvalues due to (13), and $\mathcal{Q}$ and $\mathcal{R}$ have n and m positive eigenvalues. This makes a total number of $n+d_1$ negative eigenvalues. Next, we reorder terms in (15) to bring it to the form (16) and enable the application of Lemma 1.1. This yields

$$(\star)^\top \begin{pmatrix} 0 & & P^{-1} & & & \\ & \widetilde{M}_{22} & & & & \widetilde{M}_{21} \\ P^{-1} & & 0 & & & \\ & & & \mathcal{Q} & 0 & \\ & & & 0 & \mathcal{R} & \\ & \widetilde{M}_{12} & & & & \widetilde{M}_{11} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & I_{d_1} \\ \mathcal{A}+\mathcal{B}_1\mathcal{K} & \mathcal{B}_2 \\ I & 0 \\ \mathcal{K} & 0 \\ \mathcal{C}+\mathcal{D}_1\mathcal{K} & \mathcal{D}_2 \end{pmatrix}.$$

We choose $\mathcal{P}$ as the middle matrix of this product and

$$U^\top\mathcal{K}V + W = \begin{pmatrix} \mathcal{B}_1 \\ 0 \\ I \\ \mathcal{D}_1 \end{pmatrix} \mathcal{K} \begin{pmatrix} I & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{A} & \mathcal{B}_2 \\ I & 0 \\ 0 & 0 \\ \mathcal{C} & \mathcal{D}_2 \end{pmatrix}.$$

The basis matrices of the kernels can be chosen as

$$U_\perp = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\mathcal{B}_1^\top & 0 & -\mathcal{D}_1^\top \\ 0 & 0 & I \end{pmatrix}, \qquad V_\perp = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Next, by computing the products

$$\begin{pmatrix} W^\top \\ -I \end{pmatrix} U_\perp = \begin{pmatrix} \mathcal{A}^\top & I & \mathcal{C}^\top \\ \mathcal{B}_2^\top & 0 & \mathcal{D}_2^\top \\ -I & 0 & 0 \\ 0 & -I & 0 \\ \mathcal{B}_1^\top & 0 & \mathcal{D}_1^\top \\ 0 & 0 & -I \end{pmatrix}, \qquad \begin{pmatrix} I \\ W \end{pmatrix} V_\perp = \begin{pmatrix} 0 \\ I_{d_1} \\ \mathcal{B}_2 \\ 0 \\ 0 \\ \mathcal{D}_2 \end{pmatrix}$$

and applying Lemma 1.1, we can see that (15) is negative definite if and only if

$$(\star)^\top \begin{pmatrix} 0 & & P^{-1} & & & \\ & \widetilde{M}_{22} & & & & \widetilde{M}_{21} \\ P^{-1} & & 0 & & & \\ & & & \mathcal{Q} & 0 & \\ & & & 0 & \mathcal{R} & \\ & \widetilde{M}_{12} & & & & \widetilde{M}_{11} \end{pmatrix} \begin{pmatrix} 0 \\ I_{d_1} \\ \mathcal{B}_2 \\ 0 \\ 0 \\ \mathcal{D}_2 \end{pmatrix} \tag{17}$$

is negative definite and

$$(\star)^\top \begin{pmatrix} 0 & & P & & & \\ & M_{22} & & & & M_{21} \\ P & & 0 & & & \\ & & & \mathcal{Q}^{-1} & 0 & \\ & & & 0 & \mathcal{R}^{-1} & \\ & M_{12} & & & & M_{11} \end{pmatrix} \begin{pmatrix} \mathcal{A}^\top & I & \mathcal{C}^\top \\ \mathcal{B}_2^\top & 0 & \mathcal{D}_2^\top \\ -I & 0 & 0 \\ 0 & -I & 0 \\ \mathcal{B}_1 & 0 & \mathcal{D}_1 \\ 0 & 0 & -I \end{pmatrix} \tag{18}$$

is positive definite. Rearranging terms again in (18) yields (12) and multiplying out (17) shows that this constraint is included in (13).