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Synthesis of Robust State Estimation Algorithms under Unknown Sensor Inputs

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***Abstract*— The problem of estimating the state of a dynamical system using sensor measurements becomes challenging when some of the measurements are modified by unknown inputs, which can arise due to sensor faults, modeling errors, or adversarial data injection attacks. To solve this problem, several authors have developed robust state estimation algorithms by assuming that the unknown input follows a known dynamical or probabilistic model. However, to the best of our knowledge, the stability of the existing algorithms under arbitrary unknown input sequences (which may violate the assumed dynamical or probabilistic model) has not been studied in the literature. In this paper, we address this limitation by proposing and analyzing a class of robust state estimation algorithms which unifies the existing algorithms. We derive stability guarantees that are applicable to a wider range of unknown input sequences, including (but not limited to) the ones considered in the literature. Through a numerical example, it is demonstrated that the proposed robust state estimation method achieves better state estimation performance than the existing algorithms in the presence of unknown inputs.**

I. Introduction

State estimation refers to the problem of reconstructing the state of a dynamical system by processing noisy sensor measurements. In many practical applications, the measurements obtained from some of the sensors may be corrupted by a variety of unknown inputs, which can occur due to sensor bias (due to malfunction or miscalibration of sensors), multipath errors in range-based sensors, and adversarial data injection attacks [1], [2]. A seminal work in the field is the Simultaneous Input and State Estimation (SISE) algorithm [3]–[5], in which the robust state estimation algorithm is derived by combining a Kalman filter with a minimum-variance estimator to estimate the unknown input. While the SISE algorithm is derived under the assumption that no prior knowledge about the unknown input is available, other works have considered the case where the probability density function (pdf) of the unknown input is known. In the latter case, the system state and the unknown input can be jointly estimated using Bayesian estimation theory [6]. It has been shown that the SISE algorithm arises as a special case of the Bayesian approach, when the unknown input is modeled using a flat (uninformative) pdf [7]–[9].

In each of the above works, the input is assumed to be independent and identically distributed (i.i.d.) at each timestep. An alternative scenario is that of a constant or a slowly-varying input, in which case the input at a given timestep is correlated with the estimated input at the previous timestep. Based on this observation, the authors in [10], [11] and [12] proposed robust state estimation algorithms wherein the unknown input is mitigated through batch processing of a finite history of measurement data.

In the algorithms discussed thus far, the unknown input estimator does not maintain an internal state, i.e., the input is re-estimated at each timestep. We refer to this approach as *static* input estimation. In contrast, *dynamic* (recursive) estimation of the unknown input uses a dynamical model to propagate the estimated input between successive measurement updates. Dynamic input estimation has been considered in [13]–[15], and [16], under the assumption that the input evolves according a known dynamical model. It is shown numerically in [1] that dynamic unknown input estimation can perform better than the SISE algorithm even when a dynamical model of the input is not available. However, the authors do not discuss the stability of the dynamic input and state estimation algorithm, or analyze its convergence properties in a mathematically rigorous way.

As the dynamics of the state estimation error and the estimated input are coupled, a rigorous stability analysis of this class of algorithms is necessary before they can be implemented. Moreover, as the dynamical model of the unknown input may not be known in practice, it is desirable to design a new robust state estimation algorithm that can adaptively reconstruct the system state under a wide range of unknown input scenarios. To this end, we propose a class of robust state estimation algorithms which unifies and generalizes the above algorithms, including the ones based on static input estimation and those based on dynamic input estimation. By conducting a stochastic stability analysis of the algorithm, the conditions for its stability are derived in terms of the system matrices and design parameters, which facilitates the design of new robust state estimation algorithms based on a combination of the above approaches. Through a numerical example, it is shown that the proposed robust state estimation method can adaptively mitigate the impact of unknown inputs on the state estimation performance when the probabilistic or dynamical model of the input is not available.

Notation: In the paper, I and $\mathbf{0}$ denote the identity matrix and the matrix of all zeros, respectively, of appropriate dimensions. $\mathcal{K}$ and $\mathcal{KL}$ are classes of comparison functions, whose definitions can be found in [17]. For matrices A and B, $A \succ B$ ($A \succcurlyeq B$) means that $A - B$ is positive definite (semidefinite). $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix and $\|\cdot\|$ denotes the 2-norm and induced 2-norm for vectors and

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matrices, respectively.

II. A Class of Robust State Estimators

Consider the following stochastic discrete-time linear time-varying models for the system state and measurements:

$$x(k+1) = A_k x(k) + v(k) \tag{1}$$

$$y(k) = C_k x(k) + D_k u(k) + w(k) \tag{2}$$

where $x(k) \in \mathbb{R}^{n_x}$ and $y(k) \in \mathbb{R}^{n_y}$ are the system state and measurement at timestep k, respectively, and n_x and n_y denote their dimensions. The initial condition $x(0)$ is assumed to have a multivariate Gaussian distribution with mean x_0 and variance $\Sigma_{x,0}$. The unknown input is denoted as $u(k) \in \mathbb{R}^{n_u}$. $v(k) \in \mathbb{R}^{n_x}$ and $w(k) \in \mathbb{R}^{n_y}$ are the process and measurement noise, sampled from zero mean white Gaussian noise processes having the covariance matrices $\Sigma_{v,k}$ and $\Sigma_{w,k}$, respectively, at timestep k. The matrix $D_k \in \mathbb{R}^{n_y \times n_u}$ is assumed to be known; the column space of D_k can encode the known vulnerability of sensors to unknown inputs [2]. Without loss of generality, we may assume that $n_y \geq n_u$ and that the columns of D_k are linearly independent ([3]). It is also assumed that (A_k, C_k) satisfies the time-varying observability condition [18, Definition 4.2]. Furthermore, it is assumed that $\mathbb{E}[u(k)v(k)^\top] = \mathbb{E}[u(k)w(k)^\top] = \mathbf{0}$. Note that this assumption is weaker than the one used to derive the Simultaneous Input and State Estimation (SISE) algorithm [8, Assumption 2]. In particular, our assumption allows for $u(k)$ to be a function of the state $x(k)$, whereas in [8] it is assumed that $u(k)$ is independent of $x(k)$.

The objective of robust state estimation is to obtain an accurate estimate of the system state at timestep k using the observed measurement data till timestep k. Furthermore, it is desirable to process the measurement data recursively, rather than in a batch, in order to minimize the computational effort required at each timestep. To this end, consider the following recursions for the robust state estimation algorithm:

$$\hat{x}(k+1) = A_k \hat{x}(k) + A_k K_k \big(y(k) - C_k \hat{x}(k) - D_k \hat{u}(k)\big) \tag{3}$$

$$\hat{u}(k) = E_k \hat{u}(k-1) + F_k \big(y(k) - C_k \hat{x}(k)\big) \tag{4}$$

where $\hat{x}(k)$ is the estimate of the system state $x(k)$. E_k and F_k are designed such that $\hat{u}(k)$ tracks the unknown input $u(k)$. The algorithm is initialized as $\hat{u}(-1) = \hat{u}(0) = \mathbf{0}$ and $\hat{x}(0) = \hat{x}_0$, where $\hat{x}_0 \in \mathbb{R}^{n_x}$ is the initial guess of $x(0)$, which is chosen independently of the process and measurement noise.

The intuition behind this update model is twofold: (i) as shown in Table I, several existing robust state estimators (including the SISE algorithm) use the recursions (3) and (4), and (ii) when an accurate estimate of the unknown input is available, it can be subtracted from the measurements as in (3), resulting in a Kalman filter-like algorithm. The main difference between the algorithms in Table I is in the assumptions or constraints that they place on the unknown input, $u(k)$. In the following section, we derive the conditions on the design matrices K_k, E_k and F_k which guarantee the stability of this class of algorithms.

III. Sufficient Conditions for Stability

Let $e(k) := x(k) - \hat{x}(k)$ denote the state estimation error. Using equations (1) to (4), the state estimation error dynamics can derived as

$$\begin{aligned} e(k+1) &= A_k(I - K_k \tilde{F}_k C_k) e(k) + v(k) \\ &\quad - A_k K_k \big(\tilde{F}_k D_k u(k) + \tilde{F}_k w(k) - D_k E_k \hat{u}(k-1)\big) \end{aligned} \tag{5}$$

where $\tilde{F}_k = I - D_k F_k$. On the other hand, (4) can be rewritten as

$$\hat{u}(k) = E_k \hat{u}(k-1) + F_k \big(C_k e(k) + D_k u(k) + w(k)\big) \tag{6}$$

Thus, the dynamics of the estimated input, $\hat{u}(k)$, is coupled with that of the state estimation error.

To study the joint stability of $e(k)$ and $\hat{u}(k)$, define $\eta(k) := \begin{bmatrix} e(k)^\top & \hat{u}(k-1)^\top \end{bmatrix}^\top$. Using (5) and (6), the dynamics of $\eta(k)$ can be written concisely in the block matrix form, as

$$\begin{aligned} \eta(k+1) &= \underbrace{\begin{bmatrix} A_k(I - K_k \tilde{F}_k C_k) & A_k K_k D_k E_k \\ F_k C_k & E_k \end{bmatrix}}_{G_k} \eta(k) \\ &+ \underbrace{\begin{bmatrix} -A_k K_k \tilde{F}_k D_k \\ F_k D_k \end{bmatrix}}_{H_k} u(k) + \underbrace{\begin{bmatrix} -A_k K_k \tilde{F}_k \\ F_k \end{bmatrix}}_{L_k} w(k) + \underbrace{\begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}}_{M_k} v(k) \end{aligned} \tag{7}$$

which is a stochastic linear time-varying system subject to the input $u(k)$. To derive the stability conditions for this system, the following lemmas are introduced.

Lemma 1 (Input-to-State Stability in Probability [20]): Let $\{\zeta_k\}_{k\geq 0}$ denote the trajectory of a stochastic dynamical system subject to a sequence of bounded inputs, denoted as $\{\upsilon(k)\}_{k\geq 0}$. Suppose there exist continuous functions $\{V_k(\cdot)\}_{k\geq 0}$ satisfying

$$\underline{v}\|\zeta_k\|^2 \leq V_k(\zeta_k) \leq \bar{v}\|\zeta_k\|^2 \tag{8}$$

for some $\bar{v}, \underline{v} > 0$, and

$$\mathbb{E}\left[V_{k+1}(\zeta_{k+1})|\zeta_k\right] - V_k(\zeta_k) \leq \mu - \alpha V_k(\zeta_k) + \sigma\big(\|\upsilon(k)\|\big) \tag{9}$$

for some $\mu \geq 0$, $\alpha \in (0, 1]$ and $\sigma \in \mathcal{K}$, then for any $\epsilon > 0$, there exist $\beta(\cdot, k) \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that for all k,

$$P\Big(\|\zeta_k\| < \beta\big(\|\zeta_0\|, k\big) + \gamma\big(\sup_{k'<k} \|\upsilon(k')\|\big)\Big) \geq 1 - \epsilon \tag{10}$$

Lemma 2 (Almost-Sure Exponential Stability [18]): If, in addition to the conditions of Lemma 1, $\upsilon(k) \equiv \mathbf{0}$, then there exists $\tilde{\mu} \geq 0$ such that the following holds almost-surely for all k:

$$\mathbb{E}\left[\|\zeta_k\|^2\right] \leq \tilde{\mu} + \frac{\bar{v}}{\underline{v}}\, \mathbb{E}\left[\|\zeta_0\|^2\right](1-\alpha)^k \tag{11}$$

Lemma 1 ensures the stability of a stochastic dynamical system under arbitrary unknown but bounded input sequences, even when the input is state-dependent. Usually, a bound on the unknown input may arise as a physical constraint

TABLE I: Appropriate choices of E_k and F_k based on the unknown input model

No.	Unknown Input Model	Design Matrices E_k	Design Matrices F_k	Remarks and Reference
1.	$u(k) \equiv \mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	Equivalent to the Kalman filter
2.	Arbitrary†	$\mathbf{0}$	$(D_k^\top R_k^{-1} D_k)^{-1} D_k^\top R_k^{-1}$ where $R_k = C_k P_k C_k^\top + \Sigma_{w,k}$	Corresponds to the SISE algorithm [3]
3.	Uniformly Bounded† $\lVert u(k) \rVert \leq \bar{u}$	$\mathbf{0}$	$\left(\frac{1}{\bar{u}^2} I + D_k^\top R_k^{-1} D_k\right)^{-1} D_k^\top R_k^{-1}$	Corresponds to L_2-regularized (ridge) estimation of $u(k)$ [9]
4.	Zero-mean Gaussian† $u(k) \sim \mathcal{N}(0, \Sigma_{u,k})$	$\mathbf{0}$	$\left(\hat{\Sigma}_{u,k}^{-1} + D_k^\top \Sigma_{w,k}^{-1} D_k\right)^{-1} D_k^\top \Sigma_{w,k}^{-1}$	$\hat{\Sigma}_{u,k}$ is updated using a Riccati equation [6]
5.	Gaussian Random Walk $u(k+1) = u(k) + \tilde{v}(k)$	$(I - F_k D_k)$	$\left(\hat{\Sigma}_{u,k} D_k^\top + \hat{\Sigma}_{ux,k} C_k^\top\right) \Sigma_{w,k}^{-1}$	See [16], as well as [13], [19]
6.	General Linear Model $u(k+1) = \Phi_k u(k) + \tilde{v}(k)$	$\Phi_k (I - N_k)$	See reference for N_k and F_k	Assumes that Φ_k is known [14]

† Also require that $\mathbb{E}\left[u(k) e(k)^\top\right] = \mathbf{0}$, where $e(k) = x(k) - \hat{x}(k)$.

of the sensor, or can be explicitly enforced by using a residual test [2]. In addition, Lemma 2 may be used to ensure that the robust state estimation algorithm is exponentially stable in the absence of unknown inputs, which is essential for designing robust state estimation algorithms that do not compromise the state estimation performance. Note that, given (11), $\mathbb{E}\lVert\zeta_k\rVert$ can also be bounded by using Jensen's inequality, as $\mathbb{E}\lVert\zeta_k\rVert \leq \sqrt{\mathbb{E}[\lVert\zeta_k\rVert^2]}$.

It is shown in Theorem 1 that both notions of stability can be satisfied simultaneously using the same discrete-time dynamic Lyapunov equation. To prove Theorem 1, we make the assumption that the entries of each of the matrices G_k, H_k and L_k are finite and uniformly bounded.

Theorem 1: If there exist bounded sequences of positive definite matrices $\{P_k\}_{k\geq 0}$ and $\{Q_k\}_{k\geq 0}$, such that

$$G_k^\top P_{k+1} G_k - P_k \preccurlyeq -Q_k \tag{12}$$

then the trajectory of system (7) is both input-to-state stable in probability as well as almost-surely exponentially stable in the absence of inputs.

Proof: Consider a quadratic Lyapunov function as a candidate for satisfying conditions (8) and (9), $V_k(\eta(k)) = \eta(k)^\top P_k \eta(k)$. The value of the Lyapunov function at timestep $k+1$ is

$$V_{k+1}\big(\eta(k+1)\big) = \eta(k+1)^\top P_{k+1} \eta(k+1) \tag{13}$$

By substituting (7) in (13) and taking the conditional expectation, we have

$$\begin{aligned}\mathbb{E}&\big[V_{k+1}\big(\eta(k+1)\big)\big|\eta(k)\big]\\ &= \mathbb{E}\left[\eta(k)^\top G_k^\top P_{k+1} G_k \eta + 2\eta(k)^\top G_k^\top P_{k+1} H_k u(k)\right.\\ &\quad + u(k)^\top H_k^\top P_{k+1} H_k u(k) + w(k)^\top L_k^\top P_{k+1} L_k w(k)\\ &\quad \left. + v(k)^\top M_k^\top P_{k+1} M_k v(k) \;\middle|\; \eta(k)\right]\end{aligned} \tag{14}$$

Let $\Delta V_k \coloneqq \mathbb{E}\left[V_{k+1}\big(\eta(k+1)\big)\middle|\eta(k)\right] - V_k\big(\eta(k)\big)$. Using (12) in (14), we get

$$\begin{aligned}\Delta V_k \leq \mathbb{E}&\left[-\eta(k)^\top Q_k \eta(k) + 2\eta(k)^\top G_k^\top P_{k+1} H_k u(k)\right.\\ &+ u(k)^\top H_k^\top P_{k+1} H_k u(k) + w(k)^\top L_k^\top P_{k+1} L_k w(k)\\ &\left. + v(k)^\top M_k^\top P_{k+1} M_k v(k) \;\middle|\; \eta(k)\right]\end{aligned} \tag{15}$$

As it is assumed that the entries of all the matrices are finite and bounded, the equivalence of matrix norms can be used to establish upper bounds (in terms of the induced 2-norm) for the terms in (15). Thus, we have

$$\begin{aligned}\Delta V_k &\leq -\lambda_{\min}(Q_k)\lVert\eta(k)\rVert^2\\ &+ 2\lVert G_k^\top P_{k+1} H_k\rVert\lVert\eta(k)\rVert\lVert u(k)\rVert + \lVert H_k^\top P_{k+1} H_k\rVert\lVert u(k)\rVert^2\\ &+ \mathbb{E}\left[w(k)^\top L_k^\top P_{k+1} L_k w(k) + v(k)^\top M_k^\top P_{k+1} M_k v(k)\right]\end{aligned} \tag{16}$$

By rearranging the terms, we have

$$\begin{aligned}\Delta V_k &\leq -\tfrac{1}{2}\lambda_{\min}(Q_k)\lVert\eta(k)\rVert^2\\ &+ \left(\lVert H_k^\top P_{k+1} H_k\rVert + \tfrac{2}{\lambda_{\min}(Q_k)}\lVert G_k^\top P_{k+1} G_k\rVert^2\right)\lVert u(k)\rVert^2\\ &- \left(\sqrt{\tfrac{\lambda_{\min}(Q_k)}{2}}\lVert\eta(k)\rVert - \sqrt{\tfrac{2}{\lambda_{\min}(Q_k)}}\lVert G_k^\top P_{k+1} G_k\rVert\lVert u(k)\rVert\right)^2\\ &+ \mathbb{E}\left[w(k)^\top L_k^\top P_{k+1} L_k w(k) + v(k)^\top M_k^\top P_{k+1} M_k v(k)\right]\end{aligned} \tag{17}$$

In order to evaluate the last term of (17), note that

$$\mathbb{E}\left[v(k)^\top M_k^\top P_{k+1} M_k v(k)\right]$$
$$= \mathbb{E}\left[\operatorname{Tr}\left(v(k)^\top M_k^\top P_{k+1} M_k v(k)\right)\right] \tag{18}$$
$$= \operatorname{Tr}\left(M_k^\top P_{k+1} M_k\, \mathbb{E}[v(k)v(k)^\top]\right) \tag{19}$$
$$= \operatorname{Tr}(M_k^\top P_{k+1} M_k \Sigma_{v,k}) \tag{20}$$

Furthermore, for matrices Z_1 and Z_2, $\operatorname{Tr}(Z_1^\top Z_2)$ is the Frobenius inner product for matrices, so we can use the Cauchy–Schwarz inequality to establish that

$$\operatorname{Tr}(Z_1^\top Z_2) \leq \lVert Z_1\rVert_F \lVert Z_2\rVert_F \tag{21}$$

where $\lVert Z\rVert_F = \sqrt{\operatorname{Tr}(Z^\top Z)}$ is the Frobenius norm. Using (20) and (21) in (17), we see that the condition

$$\Delta V_k \leq \mu - \alpha V_k(\eta(k)) + \sigma(\lVert u(k)\rVert) \tag{22}$$

is satisfied with

$$\mu = n_x \bar{v}\bar{\sigma}_v + \sqrt{n_x n_y} n_u \bar{l}^{\,2} \bar{v}\bar{\sigma}_w, \quad \alpha = \frac{q}{2\bar{v}}$$

$$\sigma(\lVert u(k)\rVert) = \left(\lVert H_k^\top P_{k+1} H_k\rVert + \frac{2}{\underline{q}}\lVert G_k^\top P_{k+1} G_k\rVert^2\right)\lVert u(k)\rVert^2$$

where $\bar{l}$, $\bar{\sigma}_v$ and $\bar{\sigma}_w$ are the uniform bounds (in $\lVert\cdot\rVert$) of L_k, $\Sigma_{v,k}$ and $\Sigma_{w,k}$, respectively, and we have used the equivalence of matrix norms to replace $\lVert\cdot\rVert_F$ with $\lVert\cdot\rVert$. ∎

In the proof, it is assumed that $\mathbb{E}[u(k)|\eta(k)] = u(k)$, which means that $u(k)$ does not have any randomness of its own. This assumption is satisfied even if $u(k)$ is state-dependent; for instance, we may have $u(k) = g\big(x(k)\big)$, where $g\colon \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ is a continuous function. When $u(k)$ has additional randomness, the randomness of $u(k)$ can be subsumed in the measurement noise, $w(k)$. Thus, the assumption $\mathbb{E}[u(k)|\eta(k)] = u(k)$ is not restrictive, and can accommodate state-dependent and random unknown inputs.

In order to design a stable state estimator, it is desirable to rewrite (12) in terms of the system and design matrices. To accomplish this, we may restrict P_k to be block diagonal, as shown in Corollary 1, to obtain a more conservative stability condition.

Corollary 1: If there exist bounded sequences of positive definite matrices $\{P_{1,k}\}_{k\geq 0}$, $\{P_{2,k}\}_{k\geq 0}$, $\{Q_{1,k}\}_{k\geq 0}$ and $\{Q_{2,k}\}_{k\geq 0}$, such that

$$\Phi_{11} \preccurlyeq \mathbf{0}, \quad \Phi_{22} - \Phi_{12}^\top \Phi_{11}^{-1} \Phi_{12} \preccurlyeq \mathbf{0} \tag{23}$$

where

$$\begin{aligned}
\Phi_{11} &= \tilde{A}_k^\top P_{1,k+1} \tilde{A}_k + C_k^\top F_k^\top P_{2,k+1} F_k C_k - P_{1,k} + Q_{1,k} \\
\Phi_{12} &= \tilde{A}_k^\top P_{1,k+1} A_k K_k D_k E_k + C_k^\top F_k^\top P_{2,k+1} E_k \\
\Phi_{22} &= E_k^\top D_k^\top K_k^\top A_k^\top P_{1,k+1} A_k K_k D_k E_k + E_k^\top P_{2,k+1} E_k \\
&\quad - P_{2,k} + Q_{2,k}
\end{aligned}$$

and $\tilde{A}_k = A_k(I - K_k \tilde{F}_k C_k)$, then the trajectory of system (7) is bounded in the sense of Lemmas 1 and 2.

Proof: We use the following result for positive definiteness of block matrices [21, p. 298]:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^\top & Z_{22} \end{bmatrix} \succcurlyeq \mathbf{0} \iff \begin{array}{c} Z_{11} \succcurlyeq \mathbf{0}, \text{ and} \\ Z_{22} - Z_{12}^\top Z_{11}^{-1} Z_{12} \succcurlyeq \mathbf{0} \end{array} \tag{24}$$

Consider a block-diagonal form for the matrices P_k and Q_k in Theorem 1, i.e.,

$$P_k = \begin{bmatrix} P_{1,k} & \mathbf{0} \\ \mathbf{0} & P_{2,k} \end{bmatrix} \qquad Q_k = \begin{bmatrix} Q_{1,k} & \mathbf{0} \\ \mathbf{0} & Q_{2,k} \end{bmatrix}$$

Substituting these matrices in the stability condition (12) and using the identity (24), we obtain (23). ∎

IV. Estimator Design Procedure

While E_k and F_k are designed to track the unknown input, the Kalman gain K_k is designed based on the minimum mean squared error (MMSE) criterion, where the mean squared error (MSE) is

$$\mathbb{E}[\|e(k)\|^2] = \mathrm{Tr}\left(\Sigma_{x,k}\right)$$

where $\Sigma_{x,k} \coloneqq \mathbb{E}\left[e(k)e(k)^\top\right]$. In the MMSE design criterion, K_k is chosen to minimize the MSE at the next timestep. However, a rigorous application of the MMSE design criterion requires the dynamical model or prior distribution of the unknown input. When the model of the unknown input is not known, a consistent, sub-optimal estimator may be designed instead, by substituting the unknown terms with their corresponding upper bounds [22]. The following propositions show how a stable robust state estimator can be designed by combining the MMSE design criterion with the results of the previous section:

Proposition 1: Let K_k be computed such that the following MMSE design criterion is satisfied:

$$K_k = \operatorname*{argmin}_{K_k} \mathrm{Tr}(\Sigma_{x,k+1}) \tag{25}$$

There exist $\bar{e}, \bar{f} > 0$ such that, given $\|E_k\| \leq \bar{e}$ and $\|F_k\| \leq \bar{f}$, the robust state estimation algorithm (3) and (4) is stable in the sense of Lemmas 1 and 2.

Proof: Recall that $\tilde{F}_k = I - D_k F_k$. The upper bound $\bar{f}$ of the proposition can be chosen to be small enough to ensure that $\tilde{F}_k$ is full rank, and thus, $(A_k, \tilde{F}_k C_k)$ is observable. Using Lemma 3.1 of [18], it can be shown that if $(A_k, \tilde{F}_k C_k)$ is observable and K_k is chosen as per (25), then there exists $\tilde{\alpha} > 0$ satisfying

$$\begin{aligned}
(A_k - A_k K_k \tilde{F}_k C_k)^\top \Sigma_{x,k+1}^{-1} (A_k - A_k K_k \tilde{F}_k C_k) - \Sigma_{x,k}^{-1} \\
\preccurlyeq -\tilde{\alpha} \Sigma_{x,k}^{-1}
\end{aligned} \tag{26}$$

Consider now the matrix inequality,

$$C_k^\top F_k^\top P_{2,k+1} F_k C_k + Q_{1,k} \preccurlyeq \tilde{\alpha} \Sigma_{x,k}^{-1} \tag{27}$$

Observe that adding (27) to (26) results in the first LMI of (23), where the matrices $P_{1,k}$ of Corollary 1 are chosen as $\Sigma_{x,k}^{-1}$. This shows that (27) is a sufficient condition for satisfying the first LMI of (23), for the given choice of K_k. Moreover, as $Q_{1,k}$ can be chosen as any small (in terms of $\|\cdot\|$) positive definite matrix, (27) can always be satisfied by choosing a small enough F_k. In fact, (27) gives a conservative upper bound on $\|F_k\|$ for ensuring the stability of the robust state estimation algorithm.

Similarly, observe that the second LMI of (23) may be trivially satisfied when $E_k \equiv \mathbf{0}$. Thus, given the choice of K_k as (25), for each feasible choice of $P_{2,k}$, $Q_{1,k}$ and $Q_{2,k}$, there exist corresponding upper-bounds (in $\|\cdot\|$) for E_k and F_k which guarantee the satisfaction of both the LMIs of Corollary 1. ∎

Proposition 2: Let K_k be computed as (25), then a necessary condition for the stability (in the sense of Lemmas 1 and 2) of the robust state estimation algorithm (3) and (4) is that $(A_k, \tilde{F}_k C_k)$ must be observable.

Moreover, if $E_k \equiv \mathbf{0}$, then the foregoing condition is both sufficient and necessary for stability.

Proof: *(Sufficiency)* Note that by setting E_k to $\mathbf{0}$, the second LMI of (23) is satisfied by arbitrarily small (in terms of $\|\cdot\|$) choices of $P_{2,k}$ and $Q_{2,k}$. Thus, the LMI (27) can be satisfied by choosing a small $Q_{1,k}$ as well, with the only requirement on F_k being that $(A_k, \tilde{F}_k C_k)$ should be observable.

(Necessity) In the case of the SISE algorithm, observe that the matrix F_k in this case (given in the second entry of Table I) is such that $D_k F_k$ is a projection onto the range space of D_k. $\tilde{F}_k = I - D_k F_k$ is a projection onto the orthogonal complement of the range space of D_k, which filters out the

unknown input from the innovation vector. More generally, the recursions (3) and (4) can be rewritten as

$$\hat{x}(k+1) = A_k(I - K_k\tilde{F}_kC_k)\hat{x}(k) + A_kK_k\tilde{F}_ky(k) \\ - A_kK_kD_kE_k\hat{u}(k-1) \quad (28)$$

i.e., the measurement is pre-multiplied by $\tilde{F}_k$. Observe that (28) resembles a Kalman filter-like recursion, wherein the observation matrix C_k is replaced with $\tilde{F}_kC_k$. As any information about the state which is in the kernel of $\tilde{F}_k$ is discarded, the estimation performance of the algorithm (3) and (4) is dominated by that of the Kalman filter designed for the system $(A_k, \tilde{F}_kC_k)$, which is the best possible linear estimator for this system. When $(A_k, \tilde{F}_kC_k)$ is not observable, the Kalman filter is not stable, so neither is any other linear estimator which uses the same measurement data. We have shown the contrapositive of the desired result. ∎

V. Numerical Example

In this section, we numerically simulate the proposed algorithm to demonstrate how dynamic input estimation can improve the overall performance of robust state estimators. Consider the simulation scenario of a vehicle, whose state is a concatenated vector comprising of 2D position and velocity vectors. The vehicle traverses in a straight line, makes a left turn and continues straight. Its motion is described by the dynamical model (1), with

$$A_k = \begin{cases} \begin{bmatrix} I_2 & hI_2 \\ \mathbf{0} & I_2 \end{bmatrix} & \textit{if } k = 1, 2, \ldots, 450, 551, \ldots, 1000 \\ \begin{bmatrix} I_2 & hI_2 \\ \mathbf{0} & R \end{bmatrix} & \textit{if } k = 451, \ldots, 550 \end{cases}$$

$$\Sigma_{v,k} = \text{diag}(\begin{bmatrix} 0.005 & 0.005 & 0.25 & 0.25 \end{bmatrix})$$

where $h = 0.01s$ is the sampling period, I_2 is the 2×2 identity matrix, R is the rotation matrix which rotates by $0.9°$, and $\text{diag}(\cdot)$ embeds a vector into the diagonal entries of a diagonal matrix. The vehicle is able to measure its position and velocity using GPS and inertial navigation system (INS) sensors, respectively (at $100Hz$ each), where the GPS measurements are subject to an unknown time-varying bias. Every 15 timesteps, an additional set of position measurements is available, for e.g., through communication with a base station. Thus, the measurement model is given by (2), with

$$C_k = \begin{cases} \begin{bmatrix} I_2 & \mathbf{0} & I_2 \\ \mathbf{0} & I_2 & \mathbf{0} \end{bmatrix}^\top & \textit{if } k = 15, 30, \ldots \\ I_4 & \textit{otherwise} \end{cases} \qquad D_k = \begin{bmatrix} I_2 \\ \mathbf{0} \end{bmatrix}$$

$$\Sigma_{w,k} = \begin{cases} \text{diag}([1\ 1\ 5\ 5\ 25\ 25]) & \textit{if } k = 15, 30, \ldots \\ \text{diag}([1\ 1\ 5\ 5]) & \textit{otherwise} \end{cases}$$

The unknown input $u(k)$ is given by

$$u(k) = \begin{bmatrix} 5\left(\cos\frac{\pi k}{1000} - 1\right) & 5\sin\frac{\pi k}{1000} \end{bmatrix}^\top$$

i.e., the unknown input is $\mathbf{0}$ at $k = 0$, and grows in magnitude as k increases. In Figure 1, the solid black line depicts the locus of (i.e., the set of points traced by) $u(k)$ as k varies from 1 to 1000, which is a semicircle of radius $5m$.

The position and velocity of the vehicle are estimated using the Kalman Filter (KF), the SISE algorithm and the regularized SISE (R-SISE) algorithm given in the third entry of Table 1 (with $\bar{u}$ chosen as 10), which are special cases of the class of algorithms analyzed in the preceding sections. In addition, we design a new Dynamic Input and State Estimation (DISE) algorithm by combining the E_k of the fifth algorithm in Table I with F_k chosen as per the R-SISE algorithm. For these choices of E_k and F_k, it can be seen that $\|E_k\| \leq 1$ and $\|F_k\|$ is proportional to $\bar{u}$. By setting $\bar{u}$ to 1.25, we ensure the stability of the algorithm (as per Proposition 1) while ensuring that $\tilde{F}_k$ is full rank, making $(A_k, \tilde{F}_kC_k)$ observable (as per Proposition 2). The initial state of the vehicle is $x(0) = [0m\ 0m\ 25ms^{-1}\ 0ms^{-1}]^\top$, and the initial estimate of the algorithms is sampled from the multivariate Gaussian distribution centred at $x(0)$, having the covariance $\text{diag}([20\ 20\ 10\ 10])$.

Figure 1 shows the estimates of the input $u(k)$ computed by each algorithm. The Kalman filter is considered as a special case of the proposed recursion (3) and (4), with $E_k, F_k = \mathbf{0}$, and therefore its estimate of $u(k)$ is $\mathbf{0}$ at all timesteps. It can be seen that DISE can track the unknown input best. SISE underperforms in this regard, as it does not account for the slowly-varying dynamics of the unknown input. As R-SISE computes a regularized estimate of the unknown input, it underestimates the input. Figure 2 shows the true trajectory of the vehicle as well as the estimated trajectory computed using each algorithm; the state estimates computed by SISE and DISE track the true system state quite well, whereas the estimates of KF and R-SISE are biased.

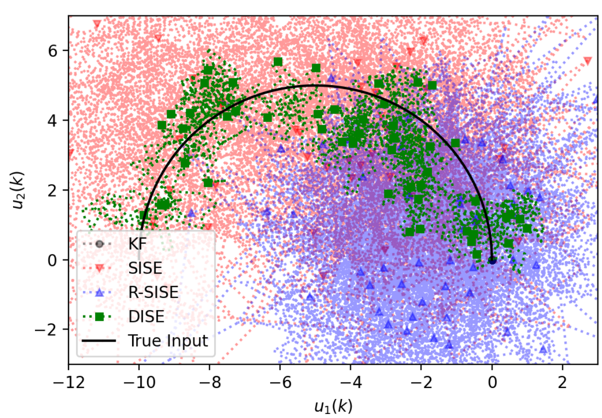

Fig. 1: The locus of the unknown input vectors $u(k) = [u_1(k)\ u_2(k)]^\top$, plotted alongside the estimated inputs $\hat{u}(k)$.

Further insight about the performance of these algorithms can be gained through Fig. 3, which shows the MSE of each algorithm averaged over 1000 Monte Carlo trials. It can be seen that DISE is able to adapt to the variation in the unknown input; when the unknown input is small, DISE incorporates the GPS measurements in its estimate, leading

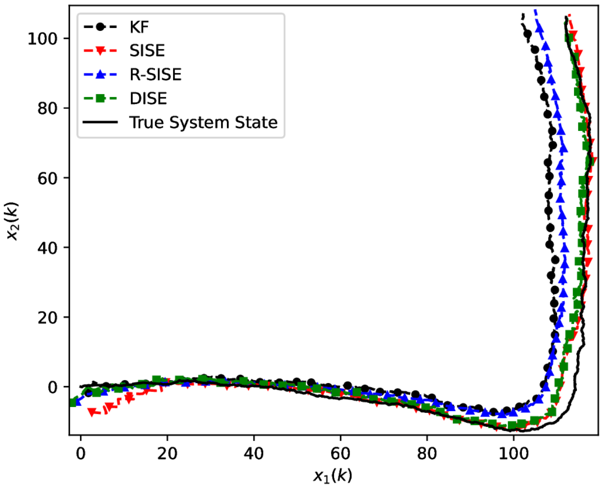

Fig. 2: The true trajectory of the vehicle, plotted alongside the state estimates computed by each algorithm.

to faster convergence speed. As the unknown input grows larger, DISE is able to achieve state estimation performance close to the SISE algorithm (which completely filters out GPS measurements). In this way, dynamic input estimation is able to adapt from one unknown input scenario to another, whereas SISE and R-SISE are effective when the assumptions placed on the unknown input are met.

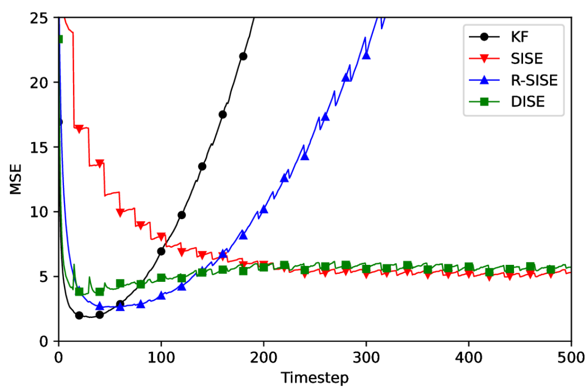

Fig. 3: The mean squared error (MSE) of each algorithm, averaged over 1000 Monte Carlo trials.

VI. Conclusion

In this paper, we proposed a class of robust state estimation algorithms which subsumes various existing algorithms. In addition, new robust state estimation algorithms can be designed using the proposed approach, which combines the advantages of existing algorithms to handle a wider range of unknown inputs. Various stability conditions were derived, characterizing the convergence properties of the proposed class of algorithms both in the presence and absence of unknown inputs. A simulation scenario was used to demonstrate the practicality of our analysis for designing robust state estimation algorithms. Future work on this topic will focus on deriving the stability conditions under unbounded inputs.

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