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Safe digital stabilization of nonlinear systems with an application to glucose control

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Abstract— **In this paper, we propose a framework for the semiglobal practical safe stabilization of nonlinear continuous-time systems based on limited amount of information. This approach requires the availability of a continuous-time control law ensuring global asymptotic stability and a robust safety condition. Following an emulation-based approach, we introduce time sampling and quantizations on the input and state signals. We show that sufficiently high sampling frequency and small quantization error guarantee safety preservation and practical state stabilization to an arbitrarily small neighborhood of the origin while keeping the state within a safe region during the whole system evolution, which is essential in safety-critical applications. Numerical simulations on a glucose control problem in a non-ideal setting show the effectiveness of the approach.**

I. Introduction

Safety-critical systems are complex control systems characterized by high performance goals and tight computational constraints, often in presence of limited resources [1]. Examples in engineering and industry are given by air traffic control [2], nuclear plants [3], electrical vehicles [4] and biomedical applications [5].

From a system-theoretical viewpoint, the safety property can be formalized in terms of forward invariance in a prescribed set [6] and can be characterized by means of *barrier functions* (or certificates) [7], which play a similar role as Lyapunov functions with respect to stability [8]. The control versions of the two notions, called Control Lyapunov Functions (CLF) [9] and Control Barrier Functions (CBF), can be jointly applied in the context of optimization-based control, as shown e.g. in the excellent survey paper [10]. More recently, the safe stabilization problem has been tackled in [11] in the more complex case of time-delay systems. In both papers [10], [11], the safe stabilization problem is treated in absence of digital non-idealities (sampling and quantization), which we aim at addressing in this work. In this regard, the recent conference paper [12] tackles the problem of sampled-data practical safety, without considering stability and in absence of quantization, with a different approach (using approximate discrete-time models) than the one followed in this work.

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In this paper, we pursue the goal of safe digital (sampled and quantized) control and consider this problem in the framework of the stabilization in the sample-and-hold sense [13], [14], following an *emulation-based approach* [15]. In more detail, as a first step, a globally asymptotically stabilizing state-feedback control law is designed or is assumed to be known, jointly guaranteeing safety of the closed-loop system with respect to a prescribed set when applied in a continuous-time fashion, i.e. ignoring network and/or digital non-idealities and assuming perfect measuring and actuation capabilities. Such an ideal setting is then embedded into a completely digital framework, accounting for time sampling and quantization of input and state [16], [17], so that the communication among the different components of the control loop is characterized by exchange of limited information [18] over a finite bandwidth, a problem also tackled in Networked Control Systems [19], preserving to some extent the properties of the closed-loop system.

With respect to classical nonlinear state-feedback design, the proposed solution considers at once the two most important digital non-idealities (sampling and quantization), which, to the best of our knowledge, have not been jointly considered so far to tackle stabilization and safety problems in a unified setting. We also show that a major difference in dealing with safety with respect to stability in digital control is that it can be enforced exactly on a prescribed set $\mathcal{C}$ (which is crucial in safety-critical applications) by assuming a robust safety condition in continuous time, i.e. ensuring invariance on an subset $\bar{\mathcal{C}}$ approximating $\mathcal{C}$ with arbitrary precision.

The paper is organized as follows. Section II recalls some notation and basic notions. Section III sets up the model formulation and assumptions. Section IV includes the main result of the paper, regarding the safe stabilization in the sampled-and-hold sense of nonlinear systems with quantization. Section V shows an example of application of the developed method to the important problem of glucose control and hypoglycemia management in diabetes. Section VI offers concluding remarks and ideas for future work.

Notation and preliminaries: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of non-negative reals $[0, +\infty)$, $\mathbb{R}^n$ denotes the n-dimensional euclidean space, $\mathbb{Z}^+$ denotes the set of non-negative integer numbers. The symbol $|\cdot|$ stands for the Euclidean norm of a real vector. For a given positive integer n and a given positive real h, the symbol $\mathcal{B}_h^n$ denotes the subset $\{x \in \mathbb{R}^n : \ |x| \leq h\}$. Given a set $\mathcal{A} \subseteq \mathbb{R}^n$, we denote by $\delta\mathcal{A}$ its boundary.

Let us here recall that a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is: of class $\mathcal{P}_0$ if $\gamma(0) = 0$; of class $\mathcal{P}$ if it is of class $\mathcal{P}_0$ and

$\gamma(s) > 0$, $s > 0$; of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Furthermore, a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ is of extended class $\mathcal{K}$ if it is strictly increasing with $\gamma(0) = 0$. It is odd if $\gamma(-x) = -\gamma(x)$ for all $x \in \mathbb{R}$.

Throughout the paper, GAS stands for Globally Asymptotically Stable or Global Asymptotic Stability.

II. Continuous-time characterizations of stability and safety notion

It is convenient to recall here the following Lyapunov stability theorem (see Theorem 4.18 in [8], and references therein).

Theorem 1: A system described by

$$\dot{x}(t) = \bar{f}(x(t)), \tag{1}$$

with $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz and satisfying $\bar{f}(0) = 0$, is GAS if there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$ and α_3 of class $\mathcal{K}$, such that the following conditions hold for all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{2}$$

$$\frac{\partial V}{\partial x}\bar{f}(x) \leq -\alpha_3(|x|). \tag{3}$$

The safety property of a system in the form (1) with respect to a set is equivalent to the definition of a forward invariant set for the system (1).

Definition 1: [10] The set $\mathcal{C}$ is forward invariant for a system in the form (1) if, for every $x_0 \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for $x(0) = x_0$ and all $t \geq 0$. A system in the form (1) is safe with respect to the set $\mathcal{C}$ if the set $\mathcal{C}$ is forward invariant for (1).

Safety can be characterized in a similar way to the stability property, by means of the so-called Barrier functions (see [10] and references therein), which we recall hereafter.

Theorem 2: (adapted from [10]) A system described by Eq. (1), with $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz and satisfying $\bar{f}(0) = 0$, is safe with respect to a set $\mathcal{C}$ defined as the sub-level set of a smooth function $H : \mathbb{R}^n \to \mathbb{R}$, i.e. $\mathcal{C} = \{x \in \mathbb{R}^n : H(x) \leq 0\}$, with $\frac{\partial H(x)}{\partial x} \neq 0$ for all $x \in \delta\mathcal{C}$, if there exists a function α of extended class $\mathcal{K}$, such that the following condition holds for all $x \in \mathbb{R}^n$:

$$\frac{\partial H}{\partial x}\bar{f}(x) \leq -\alpha(H(x)). \tag{4}$$

We highlight that in [10] and in some other previous work, safety is usually characterized with respect to a super-level set of a function $h(x) \geq 0$, i.e. $\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\}$, and by the invariance condition

$$\frac{\partial h}{\partial x}\bar{f}(x) \geq -\alpha(h(x)),$$

which is equivalent to (4), by defining the opposite function $H(x) = -h(x)$, and assuming that α is an odd function. Our choice (in line with the one in [20]) allows to write symmetrical notation for safety and stability conditions, in particular with the same direction ($\leq$) of the inequalities in (3) and (4), respectively.

In order to fill, later in this paper, the gap between continuous-time and sampled-data guarantees, we need to enforce a robust safety property in continuous time. In particular, if $\mathcal{C}$ is a sub-level set of a smooth function $H(x)$, i.e. $\mathcal{C} = \{x \in \mathbb{R}^n : H(x) \leq 0\}$, we can impose a stronger invariance condition with respect to (4), i.e. there exists $\bar{H} \in \mathbb{R}^+$ such that

$$\frac{\partial H}{\partial x}\bar{f}(x) \leq -\alpha(H(x) + \bar{H}), \tag{5}$$

which is equivalent to imposing safety in the set $\bar{\mathcal{C}}$ defined as the sub-level set of function $H(x) + \bar{H}$, i.e.

$$\bar{\mathcal{C}} = \{x \in \mathbb{R}^n : H(x) \leq -\bar{H}\}. \tag{6}$$

III. Model formulation and assumptions

We consider a continuous-time system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{7}$$

where: x_0, $x(t) \in \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$ is the input signal, assumed piecewise-continuous; n and m are positive integers; f is a locally Lipschitz function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, satisfying $f(0,0) = 0$.

We introduce here the following assumption, which ensures the existence of a continuous-time control law globally asymptotically stabilizing the closed-loop system and also ensuring safety of the same system with respect to a subset $\bar{\mathcal{C}} \subseteq \mathcal{C}$, approximating the original prescribed safe set $\mathcal{C}$ with arbitrarily good accuracy.

Assumption 1: There exist a locally Lipschitz state feedback $k : \mathbb{R}^n \to \mathbb{R}^m$, a smooth function $V : \mathbb{R}^n \to \mathbb{R}^+$, and a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ satisfying $\frac{\partial H(x)}{\partial x} \neq 0$ for all $x \in \delta\bar{\mathcal{C}}$, with $\bar{\mathcal{C}} = \{x \in \mathbb{R}^n : H(x) \leq -\bar{H}\}$, for some $\bar{H} \in \mathbb{R}^+$, such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{8}$$

$$\frac{\partial V}{\partial x}f(x, k(x)) \leq -\alpha_3(|x|), \tag{9}$$

$$\frac{\partial H}{\partial x}f(x, k(x)) \leq -\alpha_4\left(H(x) + \bar{H}\right), \tag{10}$$

for any $x \in \mathbb{R}^n$, for some some class-$\mathcal{K}_\infty$ functions α_1, α_2, some class-$\mathcal{K}$ function α_3, and some extended class-$\mathcal{K}$ odd function α_4.

Note that, by virtue of Assumption 1, Theorems 1–2 hold for the closed-loop system $\dot{x}(t) = \bar{f}(x(t)) := f(x(t), k(x(t)))$, implying from (3)–(4) that the closed-loop system (7) with $u(t) = k(x(t))$ is GAS and safe with respect to $\bar{\mathcal{C}}$.

We also define, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, the quantities

$$D^+V(x, u) := \frac{\partial V(x)}{\partial x}f(x, u), \tag{11}$$

$$D^+H(x, u) := \frac{\partial H(x)}{\partial x}f(x, u), \tag{12}$$

denoting the directional derivatives $D^+V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $D^+H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of the functions V and H along the dynamics f, respectively. Hence Eqs. (9)–(10) can be rewritten as

$$D^+V(x,k(x)) \le -\alpha_3(|x|), \tag{13}$$
$$D^+H(x,k(x)) \le -\alpha_4\left(H(x)+\bar{H}\right). \tag{14}$$

The reader can refer to the recent paper [20] for the construction of state feedbacks as in Assumption 1, by compatible CLFs and CBFs, as well as for a related literature overview and discussion.

IV. Safe Sample-and-Hold Quantized Stabilization

We recall here the notion of partition of $[0,+\infty)$ [13].

Definition 2: A partition $\pi = \{t_i\}_{i\in\mathbb{Z}^+}$ of $[0,+\infty)$ is a countable, strictly increasing sequence t_i, with $t_0 = 0$, such that $t_i \to +\infty$ as $i \to +\infty$. The diameter of π, denoted $diam\,(\pi)$, is defined as $\sup_{i\ge 0} t_{i+1}-t_i$. The dwell time of π, denoted $dwell\,(\pi)$, is defined as $\inf_{i\ge 0} t_{i+1}-t_i$. For any positive real $a \in (0,1]$, $\delta > 0$, $\pi_{a,\delta}$ is any partition π with $a\delta \le dwell\,(\pi) \le diam\,(\pi) \le \delta$.
The real $a \in (0,1]$, in Definition 2, is introduced in order to allow for non-uniform sampling and to guarantee a minimum inter-event time between two consecutive sampling instants at least equal to $a\delta$ (absence of Zeno behavior) [17].

We now define input and state quantizer operators as:

$$[\cdot]_{\mu_u} : \mathbb{R}^m \to \mathcal{Q}_u \qquad [\cdot]_{\mu_x} : \mathbb{R}^n \to \mathcal{Q}_x \tag{15}$$

where $\mathcal{Q}_u$ and $\mathcal{Q}_x$ are suitable finite subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. These quantizers are characterized by the following implications (see [11], [12]):

$$|u| \le E_u \to |u-[u]_{\mu_u}| \le \mu_u \tag{16}$$
$$|x| \le E_x \to |x-[x]_{\mu_x}| \le \mu_x \tag{17}$$

for some positive reals E_u, E_x, and μ_u, μ_x, called ranges and error bounds of the quantizers, respectively [17], [18].

In the following, we consider the following more compact notation for the sake of readability:

$$u^x(t) = k([x(t)]_{\mu_x}) \tag{18}$$
$$u^*(t) = [u^x(t)]_{\mu_u} \tag{19}$$

Next, the quantized sampled-data controller is presented. For given positive reals r, R, with $0 < r < R$, let E, $\bar{E}$, E_U, $\bar{E}_U$ be positive reals such that:

$$0 < r < R < E, \qquad \alpha_1(E) > \alpha_2(R), \tag{20}$$
$$\bar{E} = E+1, \quad E_U = \sup_{x\in\mathcal{B}^n_{\bar{E}}} |k(x)|, \quad \bar{E}_U = E_U + 1, \tag{21}$$

where functions α_1 and α_2 are defined in (8).

We further impose the sub-linearity condition:

$$\alpha_4(s) \le s \qquad \forall s \ge 0, \tag{22}$$

where function α_4 is defined in (10).

Furthermore, let L, K_V and K_H be positive reals such that the following inequalities hold:

$$|k(x_1)-k(x_2)| \le L|x_1-x_2|, \tag{23}$$
$$|D^+V(x_1,u_1)-D^+V(x_2,u_2)| \le K_V\left(|x_1-x_2|+|u_1-u_2|\right), \tag{24}$$
$$|D^+H(x_1,u_1)-D^+H(x_2,u_2)| \le K_H\left(|x_1-x_2|+|u_1-u_2|\right), \tag{25}$$

$\forall x_1,x_2 \in \mathcal{B}^n_{\bar{E}}$, $\forall u_1,u_2 \in \mathcal{B}^m_{\bar{E}_U}$, where the maps D^+V and D^+H are defined in (11)–(12).

We here state the main result of the paper.

Theorem 3: Let $\mathcal{C} = \{x \in \mathbb{R}^n : H(x) \le 0\}$ be a prescribed safe set, whose points are characterized by the safety condition $H(x) \le 0$, with $\frac{\partial H(x)}{\partial x} \ne 0$ for all $x \in \delta\mathcal{C}$. Let $a \in (0,1]$. Then, $\forall\, r,R \in \mathbb{R}^+$, with $0<r<R$, for any E, E_U, satisfying (20)–(21), there exist positive reals δ, T, μ_u, μ_x such that: for any partition $\pi_{a,\delta} = \{t_j,\ j=0,1,...\}$ of $[0,+\infty)$, for any input and state quantizers with error bounds μ_u, μ_x and ranges E_U, E, respectively, for any $x_0 \in \mathcal{B}^n_R \cap \mathcal{C}$, the solution of system (7) starting from $x(0) = x_0$ and with the sampled-data quantized control law $u(t) = u^*(t_j)$, $t \in [t_j,t_{j+1})$, $j = 0,1,...$ (see (19)), exists $\forall t \ge 0$ and, furthermore, satisfies:

$$|x(t)| \le E \qquad \forall t \ge 0; \qquad |x(t)| \le r \qquad \forall t \ge T; \tag{26}$$
$$x(t)\in\mathcal{C} \qquad \forall t \ge 0. \tag{27}$$

Proof: Taking into account Assumption 1 and Theorems 1–2, let $V : \mathbb{R}^n \to \mathbb{R}^+$ and $H : \mathbb{R}^n \to \mathbb{R}$ be smooth functions, let α_1, α_2 be functions of class $\mathcal{K}_\infty$, α_3 be a function of class $\mathcal{K}$, α_4 be an extended class-$\mathcal{K}$ function fulfilling (22), and let $\bar{H} \in \mathbb{R}^+$ s.t. conditions (8)–(10) hold.

Let r, R, be any positive reals, $0<r<R$. Let $a \in (0,1]$ be arbitrarily fixed. Let $x_0 \in \mathcal{B}^n_R \cap \mathcal{C}$, let e_1, e_2 be positive reals satisfying $e_2 < e_1 < r$ and $\alpha_1(r) > \alpha_2(e_1)$, and let E be a positive real satisfying the inequalities in (20)–(21), where the increased bounds $\bar{E}$ and $\bar{E}_U$ are defined to account for the further uncertainty involved in the quantization of state and input, respectively. In particular, it can be readily seen that $x \in \mathcal{B}^n_E$ implies that $[x]_{\mu_x} \in \mathcal{B}^n_{\bar{E}}$, finally leading to $u^* \in \mathcal{B}^m_{\bar{E}_U}$ (see also [17]).

Taking into account that V and H are smooth functions, let M, L, K_V, K_H be positive reals such that conditions (23)–(25) and the following inequality hold:

$$|f(x_1,u_1)| \le M, \tag{28}$$

$\forall x_1 \in \mathcal{B}^n_{\bar{E}}$, $\forall u_1 \in \mathcal{B}^m_{\bar{E}_U}$. Let $\eta = \alpha_3(e_2)$. Let δ,μ_u,μ_x be positive reals such that:

$$0 < \delta \le 1, \qquad e_2 + \delta M < e_1, \qquad R + \delta M < E, \tag{29}$$
$$0 < \mu_u \le 1, \qquad 0 < \mu_x \le 1, \tag{30}$$
$$\alpha_1(r) > \alpha_2(e_1) + \frac{2}{3}\eta\delta, \qquad \frac{\eta}{3} \ge K_V(M\delta+\mu_u+L\mu_x), \tag{31}$$
$$K_H(M\delta+\mu_u+L\mu_x) \le \alpha_4\left(\frac{\bar{H}}{2}\right). \tag{32}$$

Let us consider a partition $\pi_{a,\delta} = \{t_j\}_{j\in\mathbb{Z}^+}$ (see Definition 2). Following the reasoning developed e.g. in [14], [17], it results that the solution exists in $[0, +\infty)$ (Claim 1 in [17]) and that $x(t) \in \mathcal{B}^n_E$, $t \geq 0$.

We here provide only the safety part of the proof, omitting the stability part, which is a particular case of the one given in [17] for systems with state delays (see also Corollary 1 of the same paper). Let

$$B(t) = H(x(t)), \tag{33}$$

at all times t, where $x(t)$ is the solution of the closed-loop system described by the state equation (7) with control law $u(t) = u^*(t_j)$, $t \in [t_j, t_{j+1})$, $j = 0, 1, ...$ (see (19)).

The statement $x(t) \in \mathcal{C}$ is equivalent to $H(x(t)) = B(t) \leq 0$, hence our goal is to show, for any $j \geq 0$, that $B(t_j) \leq 0$ implies $B(t) \leq 0$ for all $t \in (t_j, t_{j+1}]$, hence preserving safety at all times.

Then, for any fixed $t \in (t_j, t_{j+1}]$, $j \geq 0$, for some $t^\star \in [t_j, t]$, by virtue of the Mean Value Theorem for integrals, one can write:

$$\begin{aligned} B(t) &= B(t_j) + \int_{t_j}^{t} D^+H(x(\theta), u^*(t_j))d\theta \\ &= B(t_j) + D^+H(x(t^\star), u^*(t_j))(t - t_j) \\ &\leq B(t_j) + D^+H(x(t_j), k(x(t_j)))(t - t_j) \\ &\quad + |D^+H(x(t^\star), u^*(t_j)) - D^+H(x(t_j), k(x(t_j)))|(t - t_j). \end{aligned} \tag{34}$$

By conditions (14) and (25) and the definition of $B(t)$, we have:

$$D^+H(x(t_j), k(x(t_j))) \leq -\alpha_4(B(t_j) + \bar{H}), \tag{35}$$

$$\begin{aligned} &|D^+H(x(t^\star), u^*(t_j)) - D^+H(x(t_j), k(x(t_j)))| \\ &\quad \leq K_H\left(|x(t^\star) - x(t_j)| + |u^*(t_j) - k(x(t_j))|\right) \\ &\quad \leq K_H\big(|x(t^\star) - x(t_j)| + |u^*(t_j) - u^x(t_j)| \\ &\qquad + |u^x(t_j) - k(x(t_j))|\big) \end{aligned} \tag{36}$$

where

$$|x(t^\star) - x(t_j)| \leq M(t^\star - t_j) \leq M\delta \tag{37}$$

is implied by (28) and by $t^\star - t_j \leq \delta$. Furthermore:

$$|u^*(t_j) - u^x(t_j)| = |\left[u^x(t_j)\right]_{\mu_u} - u^x(t_j)| \leq \mu_u, \tag{38}$$

$$\begin{aligned} |u^x(t_j) - k(x(t_j))| &= |k([x(t_j)]_{\mu_x}) - k(x(t_j))| \\ &\leq L|x(t_j) - [x(t_j)]_{\mu_x}| \leq L\mu_x. \end{aligned} \tag{39}$$

So, by accounting that $t - t_j \leq \delta \leq 1$ in agreement with the first inequality in (29), we get from (34):

$$B(t) \leq B(t_j) - \alpha_4(B(t_j) + \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right). \tag{40}$$

Accounting that $B(t_j) \leq 0$ and $\bar{H} > 0$, we can distinguish two alternative cases:

a) $B(t_j) + \bar{H} \leq 0$,

b) $B(t_j) + \bar{H} > 0$, implying $-\bar{H} < B(t_j) \leq 0$,

where case b) can be further decomposed into

b1) $-\bar{H} < B(t_j) \leq -\frac{\bar{H}}{2}$,

b2) $-\frac{\bar{H}}{2} < B(t_j) \leq 0$ implying $-B(t_j) - \bar{H} < -\frac{\bar{H}}{2}$.

Next we prove that $B(t) \leq 0$ in all cases, starting from (40), and exploiting (22) and (32).

Case a). Since α_4 is an odd function, we can write

$$\begin{aligned} B(t) &\leq B(t_j) - \alpha_4(B(t_j) + \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &= B(t_j) + \alpha_4(-B(t_j) - \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &\leq B(t_j) - B(t_j) - \bar{H} + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &= -\bar{H} + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &\leq -\alpha_4\left(\frac{\bar{H}}{2}\right) + K_H\left(M\delta + \mu_u + L\mu_x\right) \leq 0. \end{aligned} \tag{41}$$

Case b1).

$$\begin{aligned} B(t) &\leq B(t_j) - \alpha_4(B(t_j) + \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &< B(t_j) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &\leq -\frac{\bar{H}}{2} + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &\leq -\alpha_4\left(\frac{\bar{H}}{2}\right) + K_H\left(M\delta + \mu_u + L\mu_x\right) \leq 0. \end{aligned} \tag{42}$$

Case b2).

$$\begin{aligned} B(t) &\leq B(t_j) - \alpha_4(B(t_j) + \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &= B(t_j) + \alpha_4(-B(t_j) - \bar{H}) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &< B(t_j) + \alpha_4\left(-\frac{\bar{H}}{2}\right) + K_H\left(M\delta + \mu_u + L\mu_x\right) \\ &\leq -\alpha_4\left(\frac{\bar{H}}{2}\right) + K_H\left(M\delta + \mu_u + L\mu_x\right) \leq 0. \end{aligned} \tag{43}$$

So we proved that $B(t) \leq 0 \; \forall t \geq 0$, concluding the proof. ∎

V. Application to Safe Digital Glucose Control

We consider the following compact model of the glucose-insulin system

$$\dot{G}(t) = -K_{xgi}G(t)I(t) + \frac{T_{gh}}{V_G} + \frac{v_g(t)}{V_G} \tag{44}$$

$$\dot{I}(t) = -K_{xi}I(t) + \frac{T_{iG\max}}{V_I}\varphi(G(t)) + \frac{v_i(t)}{V_I} \tag{45}$$

where $G(t)$ [mmol/L] and $I(t)$ [pmol/L] are the plasma glucose and insulin concentrations, respectively, $v_i(t)$ [(pmol/kgBW)/min] is the exogenous intra–venous insulin delivery rate (i.e. the insulin control input) and $\varphi(G) = \frac{(G/G^*)^\gamma}{1+(G/G^*)^\gamma}$ models the endogenous pancreatic insulin delivery rate. The model (44)–(45) is a modified version of the delay differential model [21], already exploited in [22], where we neglect the delay in the function φ and we consider an additional glucose control input $v_g(t)$ [(mmol/kgBW)/min] in the glucose dynamics. This additional input is an emergency treatment consisting in a rate of fast-acting carbohydrates (for example provided by a sugar cube) useful to prevent the occurrence of hypoglycemic episodes in type-2 diabetic patients in the context of the Artificial Pancreas [23]. Such dangerous conditions, characterized by too low levels of blood glucose concentrations, can lead to short-term issues including, in the worst cases, coma and death.

For lack of space, we omit an explanation of the meaning of parameters and the choice of their numeric values, for which the interested reader is referred to [24].

The system equilibrium (G_b, I_b) with zero input satisfies

$$\frac{T_{gh}}{V_G} - K_{xgi} G_b I_b = 0, \tag{46}$$

$$\frac{T_{iG\max}}{V_I}\varphi(G_b) - K_{xi} I_b = 0. \tag{47}$$

We assume $G_b = 8.85$ to be corresponding to the basal value of a diabetic patient ($G_b > 7$) and we want to practically stabilize the closed-loop system to an euglycemic (healthy) equilibrium $G_{eu} = 5$, which uniquely determines the corresponding insulinemia equilibrium value I_{eu} and the constant stationary input $\bar{v}_{i,eu}$, as follows:

$$\frac{T_{gh}}{V_G} - K_{xgi} G_{eu} I_{eu} = 0 \Longrightarrow I_{eu} = \frac{T_{gh}}{V_G K_{xgi} G_{eu}} = 81.12 \tag{48}$$

$$- K_{xi} I_{eu} + \frac{T_{iG\max}}{V_I}\varphi(G_{eu}) + \frac{\bar{v}_{i,eu}}{V_I} = 0 \Longrightarrow \tag{49}$$
$$\bar{v}_{i,eu} = V_I K_{xi} I_{eu} - T_{iG\max}\varphi(G_{eu}) = 0.7706.$$

To recast the system in the form (7) with the desired equilibrium at the origin for the error variables, we set $x_1 = G(t) - G_{eu}$, $x_2 = I(t) - I_{eu}$, $u_1(t) = v_g(t)$, $u_2(t) = v_i(t) - \bar{v}_{i,eu}$, $x(t) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $u(t) = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$, to get (we omit time dependencies):

$$\dot{x} = f(x,u) = \tilde{f}(x) + \begin{bmatrix} u_1/V_G \\ u_2/V_I \end{bmatrix}, \tag{50}$$

with

$$\begin{aligned} \tilde{f}(x) &= \begin{bmatrix} \tilde{f}_1(x) \\ \tilde{f}_2(x) \end{bmatrix} \\ &= \begin{bmatrix} -K_{xgi}(x_1 + G_{eu})(x_2 + I_{eu}) + \dfrac{T_{gh}}{V_G} \\ -K_{xi}(x_2 + I_{eu}) + \dfrac{T_{iG\max}}{V_I}\varphi(x_1 + G_{eu}) + \dfrac{\bar{v}_{i,eu}}{V_I} \end{bmatrix}. \end{aligned} \tag{51}$$

In the following, we exploit the additional control input to preliminarily construct a stabilizer in the form $u_2 = \bar{k}_2(x, u_1)$, depending on input u_1 (to design later with safety guarantees), allowing to ensure GAS of the closed-loop system for any value of u_1. This will allow to decouple the design of the state-feedback safe stabilizer $u = k(x) = \begin{bmatrix} k_1(x) & k_2(x) \end{bmatrix}^T$ with $k_2(x) := \bar{k}_2(x, k_1(x))$, so that the design of the safe sub-controller $u_1 = k_1(x)$ can be indipendently computed without affecting the stability guarantees.

To this end, the quadratic CLF $V(x) = x^T P x$ and a modification of Sontag's universal controller [25] are employed to ensure closed-loop GAS (in agreement with Theorem 1), simplyfying the design already proposed in [21] in presence of state delay, hence obtaining

$$\bar{k}_2(x, u_1) = \begin{cases} -\frac{a(x,u_1) + \sqrt{a^2(x,u_1) + b^4(x)}}{b(x)} & \text{if } b(x) \neq 0 \\ 0 & \text{if } b(x) = 0 \end{cases} \tag{52}$$

with

$$a(x, u_1) = 2(1+\eta) x^T P \begin{bmatrix} \tilde{f}_1(x) + \frac{u_1}{V_G} \\ \tilde{f}_2(x) \end{bmatrix} + x^T \eta \mu P x,$$

$$b(x) = 2(1+\eta) x^T P \begin{bmatrix} 0 \\ \frac{1}{V_I} \end{bmatrix},$$

and matrix P and parameters chosen as in [21].

We now proceed with the design of the safety sub-controller $k_1(x)$, for which we define a safe set $\mathcal{C}$ defined by the condition $G(t) \geq G_{hypo}$ at all times t, where $G_{hypo} = 3.3$ is a hypoglycemic value. In the system variables, we get

$$x_1 + G_{eu} = G(t) \geq G_{hypo} \Longrightarrow H(x) := G_{hypo} - G_{eu} - x_1 \leq 0$$

so that the safe set $\mathcal{C} = \left\{ x \in \mathbb{R}^2 : x_1 + G_{eu} \geq G_{hypo} \right\}$ is the sublevel set of the CBF $H(x)$. This implies

$$D^+ H(x,u) = \begin{bmatrix} -1 & 0 \end{bmatrix} f(x,u) = -\left(\tilde{f}_1(x) + \frac{u_1}{V_G} \right). \tag{53}$$

We set $\alpha_4(s) = k_4 s$, with $k_4 \in (0, 1]$, which is of extended $\mathcal{K}$, is odd and satisfies the sublinearity assumption (22). To obtain a sub-controller guaranteeing robust safety, we choose

$$\begin{aligned} k_1(x) &= -V_G \tilde{f}_1(x) - V_G k_4 x_1 \\ &\geq -V_G \tilde{f}_1(x) - V_G k_4 x_1 - V_G k_4 (G_{eu} - G_{hypo} - \bar{H}) \\ &= -V_G \tilde{f}_1(x) + V_G k_4 \left(H(x) + \bar{H} \right), \end{aligned} \tag{54}$$

with $\bar{H} \in [0, G_{eu} - G_{hypo}]$, so that condition (14) is satisfied. Since Assumption 1 holds with the choice $k(x) = \begin{bmatrix} k_1(x) & \bar{k}_2(x, k_1(x)) \end{bmatrix}^T$, with k_1 and $\bar{k}_2$ defined in (54) and (52), respectively, then Theorems 1–2 hold for the closed-loop system $\dot{x}(t) = \bar{f}(x(t)) := f(x(t), k(x(t)))$, implying that the hypotheses of Theorem 3 hold, and safe digital control described in Section IV can be applied.

In the following, we show a 3-hour simulation of the closed-loop system, starting from constant initial condition equal to the basal pair (G_b, I_b), with feedback k applied both in its pure stabiling version ($k_1(x) = 0$) and following the safe stabilizing approach developed above. Only the positive part of the computed control laws is considered, since it is not possible to deliver negative glucose and insulin rates. The design parameters are chosen equal to $\delta = 5$ $[min]$, $\mu_x = 0.05$, $\mu_u = 0.005$, $\bar{H} = 0.5$, $k_4 = 0.1$. Preliminary simulations on the closed-loop system show that, in nominal conditions, the target state (G_{eu}, I_{eu}) is reached from the initial state (G_b, I_b) without any glucose undershoots, so the safety controller does not take an active role. So, in the following we show how the safe digital stabilizer is more robust and reliable than the classical digital stabilizer with respect to the hypoglycemia management in non-ideal working conditions.

In Figs. 1–2 we consider a random variation of the parameters (up to $\pm 25\%$ of the nominal values, sampled from a uniform distribution) in the simulated patient with respect to those exploited in the computation of the control law. In this case, we can observe that the digital stabilizer is not able to counteract the hypoglycemic behavior, differently

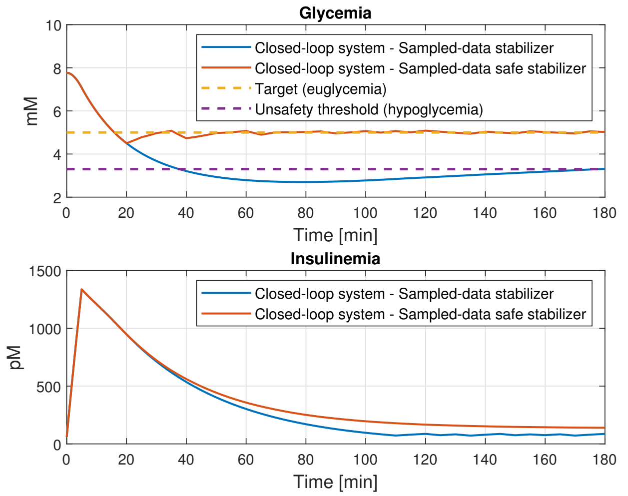

Fig. 1. State variables: glycemia (top panel), insulinemia (bottom panel).

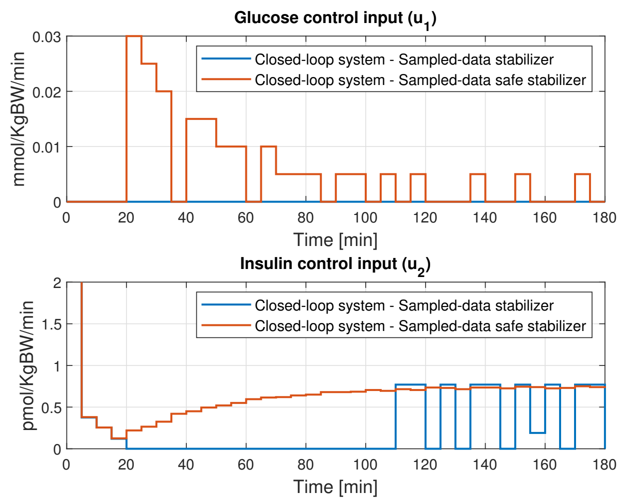

Fig. 2. Control input (digital stabilizer vs. safe digital stabilizer): glucose delivery rate (top panel), insulin delivery rate (bottom panel).

from its safe counterpart exploiting the additional glucose input, which also leads to a faster practical convergence to the healthy target state.

VI. Conclusions and Open Issues

In this work, we addressed the topic of digital (sampled and quantized) safe stabilization of nonlinear systems. In the spirit of the emulation-based approach, we assumed the existence of a controller ensuring global asymptotic stability and robust safety in continuous time, and derived conditions for jointly preserving stability (in a semiglobal practical sense) and safety of the closed-loop system at all times, in spite of the presence of digital non-idealities. Preliminary validation results of the proposed framework in a non-ideal setting are encouraging for its application in safety-critical control systems. The extension of this framework to the emulation of output-feedback controllers and to the infinite-dimensional case of nonlinear time-delay systems, by also including event-based design, will be object of future work.

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