2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Composite Nonlinear Feedback Control for Cooperative Output Regulation of Linear Multi-Agent Systems With Input Saturation

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Abstract— **This paper investigates the cooperative output regulation problem of multi-agent systems. Each agent is modeled as a general linear system with input saturation, and the network topology among agents is represented by a directed graph containing a directed spanning tree. A distributed dynamic control law based on composite nonlinear feedback (CNF) control technique is developed, which consists of a distributed dynamic compensator and a controller with a linear feedback law leading to small damping ratio and a nonlinear feedback law making the system to be highly damped as the tracking error decreases to reduce the overshoot. It is shown that the cooperative output regulation problem can be solved and the transient performance of the multi-agent systems can be improved by properly tuning the parameters of the nonlinear feedback. The effectiveness of the theoretical results is illustrated by a numerical example.**

I. Introduction

Cooperative control of multi-agent systems has attracted much attention due to its extensive applications on unmanned aerial vehicles, mobile robots, and distributed sensor networks [1]–[3]. Due to its general problem description, the cooperative output regulation problem which aims at designing a controller such that a group of agents are able to asymptotically track prescribed trajectory, and reject external disturbances, becomes a hot topic of interest [4]–[6]. On the other hand, it is important to take into account the input saturation for the multi-agent systems, since it is one of the most typical actuator constraints in practical systems. In fact, actuator saturation in control design has also been widely investigated from individual systems to multi-agent systems, please refer to [7] for more details on this issue. Cooperative control of multi-agent systems with input saturation started with simple agent dynamics, such as single-integrator dynamics [8], double-integrator dynamics [9], neutrally stable agent dynamics [10], etc. For agent of which the dynamics are asymptotically null controllable with bounded controls, various forms of semi-global consensus results can also be achieved with actuator saturation, see, e.g., [11]–[15]. However, most of these results were based on low-gain feedback design techniques, and thus the control input would move away from the maximum allowable value as the states approach the origin. Therefore, the closed-loop system cannot be operated at full capacity, which would degrade the transient performance.

To overcome this drawback and improve transient performance, a low-and-high gain technique was developed in [16] to solve the semi-global stabilization problem. A limitation of this method is that, when the controlled output reaches a specified value, there is a hard switching between the low-gain control law and the high-gain control law. To improve the transient performance of the system and make the control law more smoothly, in [17], the composite nonlinear feedback (CNF) control method was first proposed for a second-order linear system with input saturation. The CNF control law consists of a linear part and a nonlinear part. The linear part is designed to yield a closed-loop system with small damping ratio to obtain a fast response. The nonlinear part is used to increase the damping ratio of the closed-loop system to reduce the overshoot caused by the linear part. This idea was used for multi-variable systems in [18]. In [19], a more general class of linear systems with measurement feedback was comprehensively investigated using the CNF control technique. The extension of this result to multi-variable systems was reported in [20]. Later on, the CNF control has been devoted to the output regulation problem [21], [22], path-following [23], [24], transient behavior improvement in nonlinear systems [25], and descriptor systems [26]. In fact, few works have tried CNF in cooperative control of multi-agent systems. In [27], a bounded composite nonlinear feedback-based formation controller with hyperbolic functions is developed. In [28], a distributed controller design strategy based on CNF control is applied to the robot's formation control problem.

In this paper, we investigate the cooperative output regulation problem of multiple general linear systems with input saturation. The network topology among agents is represented by a directed graph containing a directed spanning tree. The CNF control method is addressed to improve the transient performance of the cooperative output regulation problem. It is shown that the CNF control technique will not destroy the solvability conditions of the cooperative output regulation problem. Thus, by appropriately designing the CNF control law, the transient performance of each agent can be significantly improved.

The main contribution of this paper can be highlighted as follows. First, the cooperative output regulation of multi-agent systems in [5] is revisited by considering the constraint of each agent with input saturation. The solvability conditions of the cooperative output regulation problem of multi-agent systems with input saturation are established. Second,

This work was supported in part by National Natural Science Foundation of China under Grants 62173283 and 62273285, and in part by Natural Science Foundation of Fujian Province of China under Grant 2021J01051.
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the CNF control technique is introduced to improve the transient performance of the cooperative output regulation of multi-agent systems. A CNF control law is designed for the cooperative output regulation problem of multi-agent systems by constructing a distributed observer. The transient performance can be improved by tuning the parameters of the CNF control law.

The remainder of this paper is organized as follows. In Section II, we start by presenting some basic facts from graph theory, and introduce the cooperative output regulation problem of multi-agent systems. In Section III, we give a CNF control law with a distributed observer, and our main results are presented. The design of the CNF control law is illustrated on an example in Section IV. Finally, we draw some conclusions in Section V.

II. PRELIMINARIES

A. Graph Theory

We first begin with the introduction on graph theory, which can be found in [5], or [29] for more details. The network consisting of N agents can be represented by a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$. In this directed graph, $\mathcal{V}=\{1,2,\ldots,N\}$ is a non-empty finite node set of N nodes, where each node v_i represents one agent. The set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ containing ordered pairs of nodes represents the neighboring relationships among agents. Let (i,j) denote the edge in $\mathcal{E}$, where node i is the parent node and node j is the child node, indicating a connection from node i to node j. Node i is also referred to as a neighbor of the node j. We use $\mathcal{N}_i$ to denote the subset of $\mathcal{V}$, including all the neighbors of node i. A directed tree is a directed graph in which each node has only one parent except for a node called the root node and from which all other nodes are reachable. A directed graph $\mathcal{G}$ contains a directed spanning tree if and only if there exists at least one node that can reach all other nodes.

The directed graph $\mathcal{G}$ can be completely characterized by its Laplacian matrix $\mathcal{L}=\mathcal{D}-\mathcal{A}$ where $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix defined as

$$a_{ii}=0, a_{ij}>0\Leftrightarrow(j,i)\in\mathcal{E}$$

and $\mathcal{D}$ is a diagonal matrix, in which the ith diagonal element is the in-degrees of node i. The Laplacian matrix has at least one zero eigenvalue with the corresponding eigenvector $\mathbf{1}_N$, which is an $N\times 1$ column vector with all elements being 1.

B. Problem Formulation

Consider the following cooperative output regulation of linear multi-agent systems which consist of N single-input single-output (SISO) agents with input saturation,

$$\begin{aligned}\dot{x}_i&=A_ix_i+B_i\,\mathrm{sat}\left(u_i\right)+E_iv\\ e_i&=C_ix_i+F_iv.\quad i=1,...,N\end{aligned}\tag{1}$$

where $x_i\in\mathbb{R}^n$ is the state, $u_i\in\mathbb{R}$ is the control input, $e_i\in\mathbb{R}^p$ is the tracking error and $v\in\mathbb{R}^q$ is the exogenous signal representing the reference input to be tracked or the disturbance to be rejected and is assumed to be generated by exogenous system (exo-system) as

$$\dot{v}=Sv\tag{2}$$

A_i,B_i,E_i,C_i,F_i, and S are constant matrixes with respective appropriate dimensions, and sat: $\mathbb{R}\to\mathbb{R}$ represents the input saturation defined as:

$$\mathrm{sat}\left(u_i\right)=\mathrm{sgn}\left(u_i\right)\min\left(u_{\max\ i},\left|u_i\right|\right)\tag{3}$$

with $u_{\max\ i}$ being the saturation level of the input for each agent i.

Definition 2.1: (Linear Cooperative Output Regulation Problem) Given the multi-agent systems (1), exo-system (2), and the directed graph $\bar{\mathcal{G}}$, design a distributed CNF control law in the form of

$$\begin{aligned}\dot{\hat{v}}_i&=\varphi\left(v,\hat{v}_i\right)\\ u_i&=\phi\left(x_i,\hat{v}_i\right)\end{aligned}\tag{4}$$

which will be shown in detail later, such that

1) The system matrix of the overall closed-loop system is Hurwitz.

2) For all $x_i\left(0\right)\in X_0$, $v\left(0\right)\in V_0$, $\hat{v}_i\left(0\right)\in\hat{V}_0$, the tracking error

$$\lim_{t\to\infty}e_i\left(t\right)=0,\quad i=1,...,N.$$

where $X_0\subset\mathbb{R}^n$, $V_0\subset\mathbb{R}^q$, and $\hat{V}_0\subset\mathbb{R}^q$ are some compact sets containing the origin of $\mathbb{R}^n$, $\mathbb{R}^q$, and $\mathbb{R}^q$ respectively.

Given the multi-agent system (1) and an exo-system (2), define a nonnegative matrix $\bar{\mathcal{A}}=[a_{ij}],i,j=0,1,\ldots,N$ that satisfy $a_{ii}=0,i=0,1,\ldots,N$. If the control u_i has access to the exogenous signal v, then $a_{i0}>0$ for $i=1,\ldots,N$. We can define a directed graph $\bar{\mathcal{G}}=\left(\bar{\mathcal{V}},\bar{\mathcal{E}}\right)$ with $\bar{\mathcal{A}}$, where $\bar{\mathcal{V}}=\{0,1,\ldots,N\}$ and node 0 is associated with the exo-system while all other nodes represent the N agents. The edge $(i,j)\in\bar{\mathcal{E}}$ exists if and only if $a_{ji}>0$. Define $\mathcal{G}=(\mathcal{V},\mathcal{E})$ as subgraph of $\bar{\mathcal{G}}$, where $\mathcal{V}=\{1,\cdots,N\}$, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$.

Let Δ be an $N\times N$ nonnegative diagonal matrix, with the ith diagonal element being denoted by a_{i0}, $i=1,\ldots,N$. The Laplacian $\bar{\mathcal{L}}$ of the directed graph $\bar{\mathcal{G}}$ is given by

$$\bar{\mathcal{L}}=\begin{pmatrix}\sum\limits_{j=1}^{N}a_{0j} & [a_{01},\cdots,a_{0N}]\\ -\mathbf{\Delta}\mathbf{1}_N & H\end{pmatrix}$$

where, for all $j=1,\ldots,N$, $a_{0j}>0$ if $(j,0)\in\bar{\mathcal{E}}$ and $a_{0j}=0$ otherwise. Then, $H\mathbf{1}_N=(\mathcal{L}+\Delta)\mathbf{1}_N=\mathbf{\Delta}\mathbf{1}_N$ since $\bar{\mathcal{L}}\mathbf{1}_{N+1}=0$. The property of H is described by the following lemma which is proved in [5].

Lemma 2.1: All the nonzero eigenvalues of H, if any, have positive real parts. Furthermore, H is nonsingular if and only if the directed graph $\bar{\mathcal{G}}$ contains a directed spanning tree with the node 0 as its root.

To solve the linear cooperative output regulation problem of multi-agent systems with input saturation, the following assumptions are needed.

Assumption 1: The pairs (A_i,B_i) are stabilizable.

Assumption 2: All the eigenvalues of S are on the imaginary axis and S is neutrally stable.

Assumption 3: There exist solution pairs (Π_i, Γ_i) that solve the regulator equations (5) respectively.

$$\begin{aligned} \Pi_i S &= A_i \Pi_i + B_i \Gamma_i + E_i \\ 0 &= C_i \Pi_i + F_i. \end{aligned} \tag{5}$$

Assumption 4: The directed graph $\bar{\mathcal{G}}$ contains a directed spanning tree with the node 0 as its root.

Remark 1: Assumptions 1–3 are the standard assumptions for the solvability of the classical linear output regulation problem [30], [31]. Assumption 4 is necessary for cooperative output regulation problem of multi-agent systems [5].

III. Main Results

Now, we are ready to introduce a distributed dynamic compensator, designed as

$$\dot{\hat{v}}_i = S\hat{v}_i + \varepsilon \left(\sum_{j \in \mathcal{N}_i} a_{ij} (\hat{v}_j - \hat{v}_i) + a_{i0} (v - \hat{v}_i) \right) \tag{6}$$

where $i = 1, 2, ..., N$, $\hat{v}_i \in \mathbb{R}^q$, and ε is some positive real number such that $\lambda_i(S) - \varepsilon \lambda_j(H) < 0, \forall i, j$, that is,

$$\tilde{S} = (I_N \otimes S) - \varepsilon (H \otimes I_q) \tag{7}$$

is Hurwitz, with $\lambda_i(S)$ and $\lambda_j(H)$ being the eigenvalues of S and H, respectively. Such an ε exists if the directed graph $\bar{\mathcal{G}}$ contains a directed spanning tree with the node 0 as the root, because all the real parts of $\lambda_j(H)$ are positive by Lemma 2.1. We call (6) a distributed observer since the dynamics of $\hat{v}_i$ also depends on $\hat{v}_j$, $j \in \mathcal{N}_i$.

Then, the dynamic state feedback distributed CNF control law is proposed as

$$\begin{cases} u_i = K_{i1} x_i + K_{i2} \hat{v}_i + \rho_i(\hat{e}_i) B_i^T P_i (x_i - \Pi_i \hat{v}_i) \\ \dot{\hat{v}}_i = S\hat{v}_i + \varepsilon \left(\sum_{j \in \mathcal{N}_i} a_{ij} (\hat{v}_j - \hat{v}_i) + a_{i0} (v - \hat{v}_i) \right) \end{cases}$$
$$i = 1, 2, ..., N. \tag{8}$$

Under Assumption 1, there exists K_{i1} such that $A_i + B_i K_{i1}$ is Hurwitz, and let K_{i2} be

$$K_{i2} = \Gamma_i - K_{i1} \Pi_i$$

where Γ_i, Π_i are the solution of (5). For convenience, we let $u_{li} = K_{i1} x_i + (\Gamma_i - K_{i1} \Pi_i) \hat{v}_i$ and $u_{ni} = \rho_i(\hat{e}_i) B_i^T P_i (x_i - \Pi_i \hat{v}_i)$. Denote P_i as the positive definite solution to the following Lyapunov function equation

$$(A_i + B_i K_{i1})^T P_i + P_i (A_i + B_i K_{i1}) = -W_i \tag{9}$$

for some $W_i > 0$. The nonlinear function $\rho_i(\hat{e}_i)$ is a non-positive function Lipschitz in $\hat{e}_i = C_i x_i + F_i \hat{v}_i$, and it can be designed in the following form

$$\rho_i(\hat{e}_i) = -\beta_i e^{-\alpha_i \gamma_i |\hat{e}_i(t)|} \tag{10}$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are tuning parameters, and γ_i is

$$\gamma_i = \begin{cases} |\hat{e}_i(0)|^{-1}, & \hat{e}_i(0) \neq 0 \\ 1, & \hat{e}_i(0) = 0 \end{cases}$$

More design details of this class of nonlinear function can be found in [32].

Remark 2: When the tracking error is large, $|\rho_i(\hat{e}_i)|$ is small, and the nonlinear control variable of the CNF control law is also small. As a result, the nonlinear part has a very limited effect and is negligible for fast response caused by the linear part. As the tracking error decreases, $|\rho_i(\hat{e}_i)|$ becomes larger and larger, thus the nonlinear part will become effective.

Let us start with some simple denotations. Let $A =$ block diag $(A_1, \ldots, A_N)$, $B =$ block diag $(B_1, \ldots, B_N)$, $C =$ block diag $(C_1, \ldots, C_N)$, $E =$ block diag $(E_1, \ldots, E_N)$, $F =$ block diag $(F_1, \ldots, F_N)$, $\Pi =$ block diag $(\Pi_1, \ldots, \Pi_N)$, $\Gamma =$ block diag $(\Gamma_1, \ldots, \Gamma_N)$, $P =$ block diag $(P_1, \ldots, P_N)$, $W =$ block diag $(W_1, \ldots, W_N)$, $K_1 =$ block diag $(K_{11}, \ldots, K_{N1})$, $\rho(\hat{e}) =$ block diag $(\rho_1(\hat{e}_1), \ldots, \rho_N(\hat{e}_N))$. Then, (5) implies

$$\begin{aligned} \Pi(I_N \otimes S) &= A\Pi + B\Gamma + E \\ 0 &= C\Pi + F \end{aligned} \tag{11}$$

where $\otimes$ stands for the Kronecker product, and let $x = [x_1^T, \ldots, x_N^T]^T$, $\hat{v} = [\hat{v}_1^T, \ldots, \hat{v}_N^T]^T$, $u = [u_1^T, \ldots, u_N^T]^T$, $u_l = [u_{l1}^T, \ldots, u_{lN}^T]^T$, $u_n = [u_{n1}^T, \ldots, u_{nN}^T]^T$, $e = [e_1^T, \ldots, e_N^T]^T$, $\hat{e} = [\hat{e}_1^T, \ldots, \hat{e}_N^T]^T$, $\omega = \text{sat}(u_l + u_n) - u_l = [\omega_1^T, \ldots, \omega_N^T]^T$, $\bar{v} = \mathbf{1}_N \otimes v$.

Then, the main result of this paper is stated as follows.

Theorem 3.1: Consider the linear multi-agent systems (1) and the exo-system (2) under Assumptions 1–4. Given a positive-definite matrix $W_Q \in \mathbb{R}^{(n \times N) \times (n \times N)}$ with

$$W_Q > (K_1 \Pi - \Gamma)^T B^T P W^{-1} P B (K_1 \Pi - \Gamma) \tag{12}$$

let $Q > 0$ be the solution to the Lyapunov equation

$$\tilde{S}^T Q + Q\tilde{S} + W_Q = 0. \tag{13}$$

For any $\delta_i \in (0, 1), i = 1, ..., N$, let $c_\delta > 0$ be the largest positive scalar such that

$$\left| \begin{bmatrix} K_{i1} & K_{i1}\Pi_i - \Gamma_i \end{bmatrix} \begin{bmatrix} x_i \\ \hat{v}_i \end{bmatrix} \right| \leq (1 - \delta_i) u_{\max i} \tag{14}$$

for all

$$\begin{pmatrix} x \\ \hat{v} \end{pmatrix} \in \mathbf{X}_\delta := \left\{ \begin{pmatrix} x \\ \hat{v} \end{pmatrix} : \begin{pmatrix} x \\ \hat{v} \end{pmatrix}^T \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{pmatrix} x \\ \hat{v} \end{pmatrix} \leq c_\delta \right\}.$$

Then, there exists a $\rho^* > 0$ such that for any $\rho(\hat{e})$ with non-positive diagonal elements, it is locally Lipschitz in $\hat{e}$ and satisfies $\|\rho(\hat{e})\|_\infty \leq \rho^*$. The distributed CNF control law (8) with suitable positive number ε, solves the cooperative output regulation problem, provided that the initial states $x_i(0)$, $\hat{v}_i(0)$, and $v(0)$ satisfy

$$v(0) \in V_0 := \left\{ v(0) : \left| \Gamma_i e^{St} v(0) \right| \leq \delta_i u_{\max i}, t \geq 0, \forall i = 1, .., N \right\}$$

and $\begin{pmatrix} x(0)-\Pi\bar{v}(0) \\ \bar{v}(0)-\hat{v}(0) \end{pmatrix} \in \mathbf{X}_\delta$.

Proof: Consider the closed-loop system of node i under the distributed CNF control law (8), i.e.,

$$\begin{aligned}
\dot{x}_i &= A_i x_i + B_i \operatorname{sat}(u_i) + E_i v \\
\dot{v} &= Sv \\
u_i &= K_{i1} x_i + (\Gamma_i - K_{i1}\Pi_i)\hat{v}_i + \rho_i(\hat{e}_i) B_i^T P_i (x_i - \Pi_i \hat{v}_i) \\
\dot{\hat{v}}_i &= S\hat{v}_i + \varepsilon\left(\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{v}_j - \hat{v}_i) + a_{i0}(v - \hat{v}_i)\right) \\
e_i &= C_i x_i + F_i v \quad i = 1,2,\ldots,N.
\end{aligned} \tag{15}$$

According to the above denotations, we can obtain the following overall closed-loop system

$$\begin{aligned}
\dot{x} &= Ax + B\operatorname{sat}(u) + E\bar{v} \\
\dot{\bar{v}} &= (I_N \otimes S)\bar{v} \\
u &= K_1 x + (\Gamma - K_1\Pi)\hat{v} + \rho(\hat{e}) B^T P(x - \Pi\hat{v}) \\
\dot{\hat{v}} &= ((I_N \otimes S - \varepsilon(H\otimes I_q))\hat{v} + \varepsilon(H\otimes I_q)\bar{v} \\
e &= Cx + F\bar{v}.
\end{aligned} \tag{16}$$

Introduce a set of state transformations

$$\tilde{x} = x - \Pi\bar{v}, \quad \tilde{v} = \bar{v} - \hat{v}. \tag{17}$$

The closed-loop system (16) can be rewritten as

$$\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x} + B\operatorname{sat}(u_l + u_n) - B\Gamma\bar{v} \\
\dot{\tilde{v}} &= \tilde{S}\tilde{v}
\end{aligned} \tag{18}$$

with the linear part u_l and nonlinear part u_n being

$$\begin{aligned}
u_l &= K_1\tilde{x} + (K_1\Pi - \Gamma)\tilde{v} + \Gamma\bar{v} \\
u_n &= \rho(\hat{e}) B^T P(\tilde{x} + \Pi\tilde{v})
\end{aligned} \tag{19}$$

respectively. Then, (18) can be combined as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{bmatrix} = \begin{bmatrix} A + BK_1 & B(K_1\Pi - \Gamma) \\ 0 & \tilde{S} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \omega \tag{20}$$

where $\omega = \operatorname{sat}(u_l + u_n) - u_l$.

Then, it is shown that the system (20) is asymptotically stable for all $\begin{pmatrix} x(0)-\Pi\bar{v}(0) \\ \bar{v}(0)-\hat{v}(0) \end{pmatrix} \in \mathbf{X}_\delta$. Consider a Lyapunov function candidate

$$V = \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}. \tag{21}$$

The time-derivative of (21) along the trajectories of the closed-loop system (20) is given by

$$\begin{aligned}
\dot{V} = &\begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}^T \begin{bmatrix} -W & PB(K\Pi - \Gamma) \\ (K\Pi - \Gamma)^T B^T P & -W_Q \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} \\
&+ 2\tilde{x}^T PB\omega
\end{aligned} \tag{22}$$

Next, the different scenarios of the input saturation are considered respectively in the remainder of this proof.

Case 1) Control inputs of all agents are unsaturated. That is, $|u_{li} + u_{ni}| \leq u_{\max i}$ holds for all $i = 1,\ldots,N$. In this case, $\omega_i = u_{ni}$ holds.

Case 2) Control inputs of all agents are beyond the upper limit. That is, for all $i = 1,\ldots,N, u_{li} + u_{ni} > u_{\max i}$ holds. Since $\omega_i = u_{\max i} - u_{li}, \omega_i < u_{ni}$. Noting that for all $\begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix} \in \mathbf{X}_\delta$, we have

$$\begin{aligned}
&\left| \begin{bmatrix} K_{i1} & K_{i1}\Pi_i - \Gamma_i \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \tilde{v}_i \end{bmatrix} + \Gamma_i v \right| \\
&\leq \left| \begin{bmatrix} K_{i1} & K_{i1}\Pi_i - \Gamma_i \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \tilde{v}_i \end{bmatrix} \right| + |\Gamma_i v| \leq u_{\max i}.
\end{aligned}$$

Thus, in this case, $0 \leq \omega_i < u_{ni}$ holds.

Case 3) Control inputs of all agents are beyond the lower limit. Similar to Case 2, $u_{ni} < \omega_i \leq 0$ holds.

Case 4) Control inputs of some agents are saturated, but the others are not. This case is a combination of the cases 1–3. In this case, ω_i can be expressed as $\omega_i = q_i u_{ni}$ with a suitable piecewise continuous function $q_i(t) \in [0,1]$. To conclude, we can define a diagonal matrix $q = \operatorname{diag}[q_1,\ldots,q_N]$ and

$$\omega = q\rho(\hat{e}) \begin{bmatrix} B^T P & B^T P\Pi \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}. \tag{23}$$

According to the above analysis, the derivative of V becomes

$$\begin{aligned}
\dot{V} &= 2\tilde{x}^T PBq\rho(\hat{e}) B^T P\tilde{x} \\
&\quad + \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}^T \begin{bmatrix} -W & \mathcal{M} \\ \mathcal{M}^T & -W_Q \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix} \\
&\leq \begin{bmatrix} \bar{x} \\ \tilde{v} \end{bmatrix}^T \begin{bmatrix} -W & 0 \\ 0 & -\bar{W}_Q \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{v} \end{bmatrix}
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\mathcal{M} &= PB\left(K\Pi - \Gamma + q\rho(\hat{e})B^T P\Pi\right) \\
\bar{x} &= \tilde{x} - W^{-1}\mathcal{M}\tilde{v}, \quad \bar{W}_Q = W_Q - \mathcal{M}^T W^{-1}\mathcal{M}.
\end{aligned}$$

By (12), we have $W_Q > (K\Pi - \Gamma)^T B^T P W^{-1} PB(K\Pi - \Gamma)$, and it is clear that there exists a $\rho^* > 0$ such that for any $\rho(\hat{e})$ with non-positive diagonal elements, it is locally Lipschitz in $\hat{e}$ and satisfies $\|\rho(\hat{e})\|_\infty \leq \rho^*$. Then, we have $\bar{W}_Q > 0$. Hence

$$\dot{V} \leq 0, \ \forall \begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix} \in \mathbf{X}_\delta$$

where X_δ is a invariant set of the overall closed-loop systems (20), and all trajectories starting from X_δ will remain inside and asymptotically converge to the origin. For the initial state claimed in Theorem 3.1, we have

$$\begin{aligned}
&\lim_{t\to\infty} \tilde{x}(t) = 0, \lim_{t\to\infty} x(t) = \Pi\bar{v}(t). \\
&\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (Cx(t) + F\bar{v}(t)) \\
&\qquad\qquad = \lim_{t\to\infty} (C\Pi + F)\bar{v}(t) = 0.
\end{aligned}$$

Hence, the tracking error $\lim_{t\to\infty} e_i(t) = 0, i = 1,\ldots,N$. This completes the proof. ■

Remark 3: If the control input saturation is not considered, Theorem 3.1 will be reduced to the result of [5]. It can be proved from the Case 1 in the proof part. As a matter of

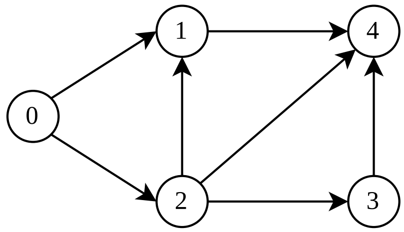


Fig. 1. Network topology for the example (Node 0 is the leader).

fact, adding an additional nonlinear part will not damage the solvability condition given in [5].

Remark 4: In this paper, we only consider SISO case, but it is not difficult to extend the results of this paper to multi-input multi-output (MIMO) case. In fact, for MIMO systems, the dynamic state feedback distributed CNF control law also has the same form as (8), except that $\rho_i(\hat{e})$ is a nonlinear function matrix. For more information on CNF controller design for MIMO systems, please see [20].

IV. Numerical Example

In what follows, we illustrate the theoretical results with the example used in [5] to show the improvement of transient performance under CNF control law. The interaction among agents is described as a directed graph shown in Fig. 1, including four agents and one leader. The arrows represent the informed agents. The multi-agent system is a group of double-integrator systems with sinusoidal disturbances of which the dynamics are

$$\begin{aligned}\dot{x}_{i1} &= x_{i2}\\ \dot{x}_{i2} &= \operatorname{sat}(u_i) + 0.5 * i * v_2 \quad i = 1,2,3,4\\ e_i &= x_{i1} - v_1\end{aligned}$$

and the exo-system is given by

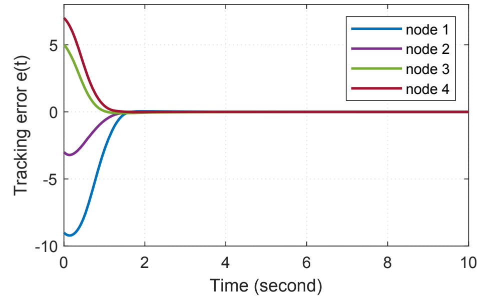
$$\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1.$$

In this example, the system matrices are given by

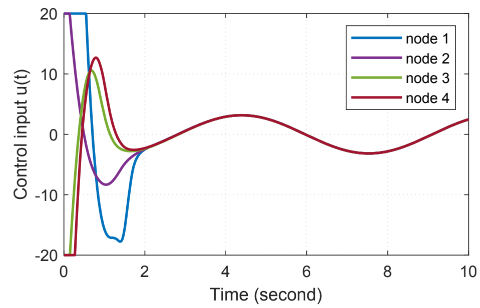
$$\begin{aligned}&A_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_i = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 * i \end{pmatrix}\\ &C_i = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D_i = 0 \quad F_i = \begin{pmatrix} -1 & 0 \end{pmatrix}\\ &S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i = 1,2,3,4.\end{aligned}$$

Thus, Assumptions 1 and 2 are satisfied, and Assumption 3 is also satisfied by choosing that $\Pi_i = I_2$ and $\Gamma_i = (-1, -0.5 * i)$ satisfy (5). By Theorem 3.1, a CNF dynamic control law of the form (8) can be synthesized to solve the cooperative output regulation problem. We further consider the input saturation by setting $u_{\max i}$ to 20. Set the linear feedback gain matrix $K_{i1} = (-8, -4)$, $\varepsilon = 5$, and $K_{i2} = \Gamma_i - K_{i1}\Pi_i = (7, 4 - 0.5 * i)$. Choose a positive definite matrix $W_i = \operatorname{diag}(1, 10)$. Then, by (9), we obtain

$$P_i = \begin{pmatrix} 10.3750 & 0.0625 \\ 0.0625 & 1.2656 \end{pmatrix}.$$

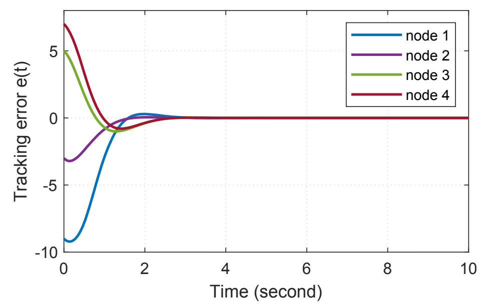


(a) Tracking error

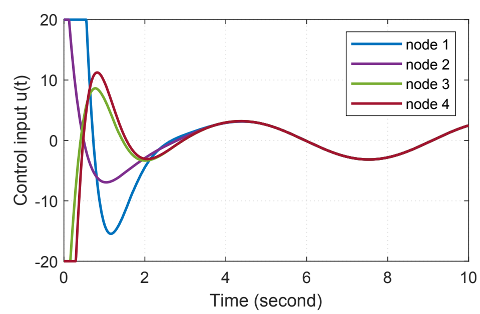


(b) Control input

Fig. 2. Tracking error and control input of each agent under CNF control law



(a) Tracking error



(b) Control input

Fig. 3. Tracking error and control input of each agent under linear feedback control law

The nonlinear functions $\rho_i(\hat{e}_i)$ are designed as

$$\begin{aligned}&\rho_1(\hat{e}_1) = -5e^{-20\gamma_1|\hat{e}_1(t)|}, \ \rho_2(\hat{e}_2) = -2e^{-12\gamma_2|\hat{e}_2(t)|},\\ &\rho_3(\hat{e}_3) = -3e^{-4\gamma_3|\hat{e}_3(t)|}, \ \rho_4(\hat{e}_4) = -3e^{-4\gamma_4|\hat{e}_4(t)|}.\end{aligned}$$

The simulation results are shown in Figs. 2 and 3. Fig. 2(a) shows the evolution of state difference between four agents and the leader under the CNF control law. It is shown that the overall closed-loop system is asymptotically stable and the tracking error of each agent i converges to zero. It can be seen that compared with the case using only linear control law in Fig. 3(a), CNF control law makes the tracking errors have barely overshoot and improves the transient performance effectively. Fig. 2(b) and Fig. 3(b) show the control input of each agent under the two control laws.

Remark 5: Specially, for double-integrator dynamics, CNF control law can provide guaranteed transient performance through numerous computations. However, for other high-order linear systems, it will be much more difficult due to the freedom of K_1, W and ρ.

V. Conclusion

In this paper, the cooperative output regulation problem of multi-agent systems with input saturation is investigated. A distributed dynamic control law based on the CNF control technique is designed. By appropriately selecting the nonlinear function, the transient performance of each agent can be significantly improved. The effectiveness of the theoretical results is demonstrated by a numerical example. Future work will focus on the case where the information graph is time-varying.

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[2] S. Knorn, Z. Chen, and R. H. Middleton, "Overview: Collective control of multiagent systems," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 4, pp. 334–347, 2015.

[3] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[4] X. Wang, Y. Hong, J. Huang, and Z.-P. Jiang, "A distributed control approach to a robust output regulation problem for multi-agent linear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2891–2895, 2010.

[5] Y. Su and J. Huang, "Cooperative output regulation of linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 1062–1066, 2012.

[6] M. Lu and L. Liu, "Cooperative output regulation of linear multi-agent systems by a novel distributed dynamic compensator," *IEEE Transactions on Automatic Control*, vol. 62, no. 12, pp. 6481–6488, 2017.

[7] Z. Lin, "Control design in the presence of actuator saturation: from individual systems to multi-agent systems," *Science China Information Sciences*, vol. 62, pp. 1–3, 2019.

[8] Y. Li, J. Xiang, and W. Wei, "Consensus problems for linear time-invariant multi-agent systems with saturation constraints," *IET Control Theory & Applications*, vol. 5, no. 6, pp. 823–829, 2011.

[9] W. Ren and R. W. Beard, "Consensus algorithms for double-integrator dynamics," *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*, pp. 77–104, 2008.

[10] Z. Meng, Z. Zhao, and Z. Lin, "On global leader-following consensus of identical linear dynamic systems subject to actuator saturation," *Systems & Control Letters*, vol. 62, no. 2, pp. 132–142, 2013.

[11] H. Chu, B. Yi, G. Zhang, and W. Zhang, "Performance improvement of consensus tracking for linear multiagent systems with input saturation: A gain scheduled approach," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 50, no. 3, pp. 734–746, 2017.

[12] H. Chu and W. Zhang, "Adaptive consensus tracking for linear multi-agent systems with input saturation," *Transactions of the Institute of Measurement and Control*, vol. 38, no. 12, pp. 1434–1441, 2016.

[13] L. Shi, Y. Li, and Z. Lin, "Semi-global leader-following output consensus of heterogeneous multi-agent systems with input saturation," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 16, pp. 4916–4930, 2018.

[14] Q. Wang, C. Yu, and H. Gao, "Semiglobal synchronization of multiple generic linear agents with input saturation," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 18, pp. 3239–3254, 2014.

[15] Z. Zhao, Y. Hong, and Z. Lin, "Semi-global output consensus of a group of linear systems in the presence of external disturbances and actuator saturation: an output regulation approach," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 7, pp. 1353–1375, 2016.

[16] Z. Lin, R. Mantri, and A. Saberi, "Semi-global output regulation for linear systems subject to input saturation-a low-and-high gain design," in *Proceedings of 1995 American Control Conference-ACC'95*, vol. 5. IEEE, 1995, pp. 3214–3218.

[17] Z. Lin, M. Pachter, and S. Banda, "Toward improvement of tracking performance nonlinear feedback for linear systems," *International Journal of Control*, vol. 70, no. 1, pp. 1–11, 1998.

[18] M. C. Turner, I. Postlethwaite, and D. J. Walker, "Non-linear tracking control for multivariable constrained input linear systems," *International Journal of Control*, vol. 73, no. 12, pp. 1160–1172, 2000.

[19] B. M. Chen, T. H. Lee, K. Peng, and V. Venkataramanan, "Composite nonlinear feedback control for linear systems with input saturation: theory and an application," *IEEE Transactions on Automatic Control*, vol. 48, no. 3, pp. 427–439, 2003.

[20] Y. He, B. M. Chen, and C. Wu, "Composite nonlinear control with state and measurement feedback for general multivariable systems with input saturation," *Systems & Control Letters*, vol. 54, no. 5, pp. 455–469, 2005.

[21] B. Zhang and W. Lan, "Improving transient performance for output regulation problem of linear systems with input saturation," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 10, pp. 1087–1098, 2013.

[22] C. Wang, X. Yu, and W. Lan, "Semi-global output regulation for linear systems with input saturation by composite nonlinear feedback control," *International Journal of Control*, vol. 87, no. 10, pp. 1985–1997, 2014.

[23] C. Hu, R. Wang, F. Yan, and N. Chen, "Robust composite nonlinear feedback path-following control for underactuated surface vessels with desired-heading amendment," *IEEE Transactions on Industrial Electronics*, vol. 63, no. 10, pp. 6386–6394, 2016.

[24] R. Wang, C. Hu, F. Yan, and M. Chadli, "Composite nonlinear feedback control for path following of four-wheel independently actuated autonomous ground vehicles," *IEEE Transactions on Intelligent Transportation Systems*, vol. 17, no. 7, pp. 2063–2074, 2016.

[25] T. Lu and W. Lan, "Composite nonlinear feedback control for strict-feedback nonlinear systems with input saturation," *International Journal of Control*, vol. 92, no. 9, pp. 2170–2177, 2019.

[26] E. Jafari and T. Binazadeh, "Observer-based improved composite nonlinear feedback control for output tracking of time-varying references in descriptor systems with actuator saturation," *ISA Transactions*, vol. 91, pp. 1–10, 2019.

[27] Z. Hou and I. Fantoni, "Interactive leader–follower consensus of multiple quadrotors based on composite nonlinear feedback control," *IEEE Transactions on Control Systems Technology*, vol. 26, no. 5, pp. 1732–1743, 2017.

[28] C. Lei, W. Sun, and J. T. Yeow, "A distributed output regulation problem for multi-agent linear systems with application to leader-follower robot's formation control," in *2016 35th Chinese Control Conference (CCC)*. IEEE, 2016, pp. 614–619.

[29] C. Godsil and G. F. Royle, *Algebraic Graph Theory*. Springer Science & Business Media, 2001, vol. 207.

[30] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[31] J. Huang, *Nonlinear Output Regulation: Theory and Applications*. SIAM, 2004.

[32] W. Lan and B. M. Chen, "On selection of nonlinear gain in composite nonlinear feedback control for a class of linear systems," in *2007 46th IEEE Conference on Decision and Control*. IEEE, 2007, pp. 1198–1203.