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Data-Driven Output Feedback Control for Unknown Switched Linear Systems

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Abstract— **This paper proposes a data-driven control method to stabilize unknown switched linear systems under arbitrary switching. We consider the case where the system state is not measurable and design an output feedback controller only using measured input-output data. First, the system with multiple outputs is transformed into a single-output system with observability preserved. Then, a data-based state-space representation that has the same input-output relationship as the original system is constructed using the input-output data of the single-output system, based on which the data-driven controller is designed. Sufficient conditions for asymptotic stability of the closed-loop system under arbitrary switching are established in terms of linear matrix inequalities (LMIs). Compared with the existing method, the proposed method decreases the dimension of the constructed data-based state-space representation, which may reduce the computational burden of the controller design. The effectiveness of the proposed controller is illustrated by an example.**

I. Introduction

Data-driven control (DDC) is a control strategy that learns controllers from data for unknown systems and is increasingly popular in a wide range of research fields. Depending on the different control design procedures, DDC is divided into indirect DDC and direct DDC. The former identifies the system model using data and designs controllers using the existing model-based methods. In contrast, the latter directly designs the controller using data by bypassing the system identification step, which may simplify the whole design process. Over the past two decades, numerous direct DDC methods have been proposed, such as unfalsified control [1], iterative feedback tuning [2], neural network control [3], and model-free adaptive control [4].

Willems' Fundamental Lemma has recently attracted much attention in DDC. It reveals that the input-output behavior of a linear time-invariant (LTI) system can be described entirely by using a persistently exciting (PE) input-output trajectory of the system [5]. Thanks to this lemma, a data-based representation of LTI systems can be constructed using an informative input-output sequence [6]. With the representation, many existing model-based control methods can be extended to data-based cases [7]–[10]. Basic control problems of unknown LTI systems, such as the state feedback control, output feedback control, and linear quadratic regulator, are solved by using data-dependent linear matrix inequalities (LMIs) in [7], [8]. A data-enabled predictive control algorithm is proposed to solve the optimal trajectory tracking problem [9]. In addition, Willems' Fundamental Lemma has also been used to solve problems of complex systems. Preliminary results have been reported for particular nonlinear systems. The stabilization problem of unknown nonlinear polynomial systems with noisy data is solved in [11]. The state feedback control problem for systems with quadratic nonlinearities is solved in [12].

On the other hand, as an important type of hybrid system, switched linear systems have attracted much attention in different fields [13]. The control problems of switched linear systems have been well-studied in the past two decades. However, most of the existing control design methods [14]–[16] are model-based and only apply to switched linear systems with identified models. How to design controllers for unknown switched linear systems is still an open problem. In recent years, a few direct DDC methods have been proposed for unknown switched linear systems. State feedback controllers are designed using input-state data for systems without and with disturbances in [17] and [18], respectively. However, the system state may not be measurable in practice and thus is not available for the state feedback controller design. Moreover, the switching signals in [17], [18] are time-dependent and require a large enough dwell time.

In view of the abovementioned issue, this paper will propose an output feedback control by only using input and output data to stabilize unknown switched linear systems under arbitrary switching. First, a non-switched LTI system with multiple outputs is transformed into a new single-output (SO) system with observability preserved. Based on this SO system, a new system with the same input-output relationship as the original one is built. Then, a data-based representation is constructed for the new system using the input and output data. This method is extended to represent each subsystem of the switched linear system by using data. Finally, output feedback control laws are designed for each subsystem to stabilize the switched linear system under arbitrary switching.

The remainder of this paper is organized as follows. The switched linear system to be studied is given in Section II. Section III proposes a data-based representation for the system. Then, a data-driven output feedback controller is designed in Section IV. Section V gives a numerical example to illustrate the effectiveness of the proposed methods. Finally, Section VI gives some concluding remarks.

Notation: Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural

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numbers, integers, real numbers, and complex numbers, respectively. I_n is the identity matrix with dimensions $n \times n$. $0_{n\times m}$ is a zero matrix with dimensions $n\times m$. Given a matrix M, let $M^\top$ denote its transpose, M^{-1} denote its inverse if it is nonsingular, and $M > 0$ means it is positive definite. $\mathsf{diag}(M_1, \ldots, M_n)$ denotes the block diagonal matrix with matrices M_i, $i = 1, \ldots, n$. For a signal $z(i) : \mathbb{Z} \to \mathbb{R}^n$, define $z_{[i,j]}$ as $z_{[i,j]} = [z(i)^\top, z(i+1)^\top, \ldots, z(j)^\top]^\top$, where $i, j \in \mathbb{Z}$ and $i < j$. $Z_{i|L|j}$ denotes the Hankel matrix of order L, associated with $z_{[i,j]}$, which is given by

$$Z_{i|L|j} = \begin{bmatrix} z(i) & z(i+1) & \cdots & z(j-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ z(i+L-1) & z(i+L) & \cdots & z(j) \end{bmatrix},$$

where $i, j \in \mathbb{Z}$ and $L \in \mathbb{N}$ satisfying $L \leq j - i + 1$. For simplicity, $Z_{i|j}$ is used to denote the case with $L = 1$.

II. Problem formulation

Consider a discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \tag{1a}$$

$$y(k) = Cx(k), \tag{1b}$$

where $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, and $y(k) \in \mathbb{R}^p$ are the control input, state, and output, respectively; A_i, B_i, $\forall i \in \mathcal{S} = \{1, \ldots, s\}$ and C are the related matrices with appropriate dimensions. The switching signal $\sigma(k) \in \mathcal{S}$ is a piecewise constant function of time k. We assume that the state $x(k)$ is unmeasurable. Without loss of generality, the following assumptions are made for the system (1).

Assumption 1: The system matrices A_i, B_i, $\forall i \in \mathcal{S}$, and C are unknown. The number of the subsystems s and the dimensions n, m and p are known.

Assumption 2: The common output matrix C has full row rank.

Assumption 3: Each subsystem i, $i \in \mathcal{S}$ is controllable and observable.

Assumption 4: The system matrix A_i for each subsystem i, $\forall i \in \mathcal{S}$, has n distinct eigenvalues.

Remark 1: Assumption 1 is commonly used in the study of DDC problem [7]. Assumption 2-4 can be checked using input-output data (e.g., Markov parameters [19]) of the subsystem i. Take a single-input system of (1) for an example. By applying a special input signal with $u(k) = 1$ at $k = 0$ and $u(k) = 0$ otherwise, to each subsystem i under the zero initial condition, a length $2n$ output sequence $y_{i,[0,2n-1]}$ can be collected. Denote $H_{i,1|n|2n-1}$ as the Hankel matrix of order n, associate with $y_{i,[1,2n-1]}$. From [19], if $\text{rank}(H_{i,1|n|2n-1}) = n$, then the collected input-output data has a minimal n-dimensional state-space realization. Since the original n-dimensional subsystem i is similar to this minimal realization, it is controllable and observable, i.e., Assumption 3 is satisfied. Further, if $\text{rank}([y_i(1), \ldots, y_i(n)]) = p$, then the subsystem output matrix has full row rank, i.e., Assumption 2. For Assumption 4, if there exists an $\eta_i \in \mathbb{R}^{1\times p}$ such that $\text{rank}(\Psi_i H_{i,1|n|2n-1}) = n$ with $\Psi_i = \mathsf{diag}(\eta_i, \ldots, \eta_i)$, then A_i has n distinct eigenvalues. How to find such an η_i will be discussed in Remark 2 and 5.

This paper is aimed to design an output feedback controller to stabilize the unknown switched linear system (1) under arbitrary switching. Traditionally, one can use a set of input-output data satisfying the PE condition to identify the model of the unknown system (1). Then, the output feedback controller can be designed using the model-based methods [15]. To simplify the design procedure, we bypass the system identification step and use the input-output data to design an output feedback controller directly. To end this section, we review the PE condition for future reference.

Definition 1 ([5]): The sequence $z_{[i,j]}$ is said to be PE of order L if the Hankel matrix $Z_{i|L|j}$ has full row rank.

III. Input-Ouput Data-Based Representation

This section proposes a data-based representation of the system (1). Usually, the existing works, even for non-switched LTI systems, use input-state-output data to build such a representation. However, in practice, the system states may not be measurable. In view of this issue, we will first build a data-based representation for LTI systems by only using input-output data and then extend it to switched linear systems.

Consider an LTI system as follows:

$$x(k+1) = Ax(k) + Bu(k), \tag{2a}$$

$$y(k) = Cx(k), \tag{2b}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Moreover, the system (2) follows the same assumptions as those for the subsystem i of the system (1).

A. A New State-Space Model

This subsection builds a new state-space model for the system (2) only using its input and output data. Compared with the original system model (2), this new model has the same input and output as (2), but a different state $\hat{x}$ that is created by the past input-output data of (2).

In [7], the p-dimensional output of the system (2) is directly used to construct $\hat{x}$, which may lead to a high dimension of $\hat{x}$. To solve this issue, this paper will change the multiple-output (MO) system into an SO system with observability preserved. The one-dimensional output of the new system is actually a linear combination of the p-dimensional system output. Then, the output of the SO system is used to construct $\hat{x}$.

Before proceeding, we introduce the following lemma.

Lemma 1: Consider the system (2) with $p \geq 2$ outputs. If the pair (A, C) is observable and A has n distinct eigenvalues, then there exists at least a row vector $\eta \in \mathbb{R}^{1\times p}$ such that the pair (A, ηC) is observable.

Proof: Let $\lambda_\iota \in \mathbb{C}$ and $\omega_\iota \in \mathbb{C}^n$, $\forall \iota \in \mathcal{N} = \{1, \ldots, n\}$ be the eigenvalues and the corresponding right non-zero eigenvectors of matrix A, respectively. Then, we have $A\omega_\iota = \lambda_\iota\omega_\iota$. Since all the eigenvalues λ_ι, $\forall \iota \in \mathcal{N}$, are distinct, any of the corresponding non-zero eigenvectors can be expressed as $a_\iota\omega_\iota$, $\forall \iota \in \mathcal{N}$, $\forall a_\iota \in \mathbb{R}$ and $a_\iota \neq 0$.

Define $W = [a_1\omega_1, \ldots, a_n\omega_n]$ and $CW = [Ca_1\omega_1, \ldots, Ca_n\omega_n] = [\mu_1^\top, \ldots, \mu_p^\top]^\top$ with $\mu_j, j \in \mathcal{P} = \{1, \ldots, p\}$ being the j-th row of CW. Since the pair (A, C) is observable, based on the Hautus lemma [20], we have $Ca_\iota\omega_\iota \neq 0_{p\times 1}$, $\forall\iota \in \mathcal{N}$, which indicates that there exists at least one non-zero entry in each column of CW, i.e., $Ca_\iota\omega_\iota$, $\forall\iota \in \mathcal{N}$.

Let $\eta = [\eta_1, \ldots, \eta_p]$ with η_j, $j \in \mathcal{P}$ being the j-th entry of η. Next, we claim that there exists at least one non-zero η, making $\eta CW = \sum_{j=1}^{p} \eta_j\mu_j$ with no zero entries. Set $\eta_1 = 1$ and choose η_2 such that entries of $\mu_1 + \eta_2\mu_2$ are non-zero if the corresponding ones of μ_1 or μ_2 are non-zero. It is obvious that such an η_2 always exists. Similarly, choose η_k, $3 \leq k \leq p$ such that the entries of $\sum_{j=1}^{k} \eta_j\mu_j$ are non-zero if the corresponding ones of $\sum_{j=1}^{k-1} \eta_j\mu_j$ or μ_k are non-zero. With the selected η, entries of ηCW are non-zero if the corresponding ones of any μ_j, $j \in \mathcal{P}$ are non-zero. Since there exists at least one non-zero entry in each column of CW, we conclude that all entries of ηCW are non-zero. Further, based on the Hautus lemma [20], the pair (A, ηC) is observable. ■

Remark 2: If matrices A and C are known, then η can be selected by ensuring that $\eta C\omega_\iota \neq 0$, $\forall\iota \in \mathcal{N}$. When A and C are unknown, η can be calculated based on data. Similar to Remark 1, we can obtain a Hankel matrix $H_{1|n|2n-1}$ by applying the special input $u(k) = 1$ at $k = 0$ and $u(k) = 0$ otherwise to the system (2) and collecting the output data. Then, η can be chosen such that $\text{rank}(\Psi H_{1|n|2n-1}) = n$, where $\Psi = \text{diag}(\eta, \ldots, \eta)$. Further, this problem can be cast as an unconstrained optimization problem to maximize $J(\eta) = \text{rank}(\Psi H_{1|n|2n-1})$. However, $J(\eta)$ is non-convex, and there is no theoretical guarantee of finding the globally optimal solution. Alternatively, we can employ heuristic optimization methods [21], such as the genetic algorithm and particle swarm optimization algorithm, to search for such an η.

From Lemma 1, there always exists a row vector $\eta \in \mathbb{R}^{1\times p}$ such that the observable matrix $\bar{M}_o = [(\eta C)^\top, (\eta CA)^\top, \ldots, (\eta CA^{n-1})^\top]^\top$ has full rank if the pair (A, C) is observable and matrix A has n distinct eigenvalues.

Let $\bar{x}(k) = \bar{M}_o x(k)$. Then, (2) with the new output $\bar{y}(k) = \eta y(k)$ can be transformed into an observability form [22]

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k), \tag{3a}$$

$$\bar{y}(k) = \bar{C}_\eta\bar{x}(k), \tag{3b}$$

where $\bar{A} = \left[\begin{array}{c|c} 0_{(n-1)\times 1} & I_{n-1} \\ \hline -\alpha_n & -\alpha_{n-1} \cdots -\alpha_1 \end{array}\right]$, $\bar{B} = [b_1, \ldots, b_n]^\top$, and $\bar{C}_\eta = [1, 0_{1\times(n-1)}]$ with $\alpha_i \in \mathbb{R}$ and $b_i \in \mathbb{R}^m$, $i \in \mathcal{N}$. Furthermore, (3) can be written in an ARX form [23]

$$\begin{aligned}\bar{y}(k+n) =& \beta_n^\top u(k) + \beta_{n-1}^\top u(k+1) + \cdots + \beta_1^\top u(k+n-1) \\ & - \alpha_n\bar{y}(k) - \cdots - \alpha_1\bar{y}(k+n-1),\end{aligned} \tag{4}$$

$$\text{where } \beta_i = \sum_{\iota=0}^{i-1} \alpha_\iota b_{i-\iota}, \forall i \in \mathcal{N} \text{ and } \alpha_0 = 1. \tag{5}$$

Define the state $\hat{x}(k) \in \mathbb{R}^{\hat{n}}$ with $\hat{n} = nm + n$ as

$$\begin{aligned}\hat{x}(k) = [&u(k-n)^\top, \ldots, u(k-1)^\top, \\ &\bar{y}(k-n), \ldots, \bar{y}(k-1)]^\top.\end{aligned} \tag{6}$$

Then, (4) can be transformed into a new state-space form

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k), \tag{7a}$$

$$\bar{y}(k) = \hat{C}_\eta\hat{x}(k), \tag{7b}$$

where $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times m}$, $\hat{C}_\eta \in \mathbb{R}^{1\times\hat{n}}$ are defined as

$$\hat{A} = \left[\begin{array}{cc|c} 0_{(n-1)m\times m} & I_{(n-1)m} & \\ 0_{m\times m} & 0_{m\times(n-1)m} & 0_{nm\times n} \\ \hline 0_{(n-1)\times m} & 0_{(n-1)\times(n-1)m} & \\ \beta_n^\top & \beta_{n-1}^\top \cdots \beta_1^\top & \bar{A} \end{array}\right], \tag{8a}$$

$$\hat{B} = [0_{m\times(n-1)m}, I_m, 0_{m\times n}]^\top, \tag{8b}$$

$$\hat{C}_\eta = [\beta_n^\top, \ldots, \beta_1^\top, -\alpha_n, \ldots, -\alpha_1]. \tag{8c}$$

Subsequently, we will establish the relation between the original system output $y(k)$ and the state $\hat{x}(k)$ of (7). To do so, the relationship between $\bar{x}(k)$ in (3) and $\hat{x}(k)$ in (6) is first derived. Substituting (7b) into $\bar{y}(k) = \bar{x}_1(k)$, i.e., (3b), gives

$$\bar{x}_1(k) = \hat{C}_\eta\hat{x}(k). \tag{9}$$

Combining (9), (7a), $\beta_1 = b_1$ in (5) with $\bar{x}_1(k+1) = \bar{x}_2(k) + b_1^\top u(k)$, i.e., the first row of (3a), gives

$$\begin{aligned}\bar{x}_2(k) &= \bar{x}_1(k+1) - b_1^\top u(k) = \hat{C}_\eta\hat{x}(k+1) - b_1^\top u(k), \\ &= \hat{C}_\eta(\hat{A}\hat{x}(k) + \hat{B}u(k)) - b_1^\top u(k) = \hat{C}_\eta\hat{A}\hat{x}(k).\end{aligned} \tag{10}$$

Repeating the process (10) derives $\bar{x}_i(k)$, $i = 3, \ldots, n$ as

$$\bar{x}_i(k) = \hat{C}_\eta\hat{A}^{i-1}\hat{x}(k). \tag{11}$$

From (9)-(11), the relation between $\bar{x}(k)$ and $\hat{x}(k)$ is given by

$$\bar{x}(k) = \Phi\hat{x}(k) \text{ with } \Phi = [(\hat{C}_\eta)^\top, \ldots, (\hat{C}_\eta\hat{A}^{n-1})^\top]^\top. \tag{12}$$

Substituting (12) into $\bar{x}(k) = \bar{M}_o x(k)$ gives the relation between $x(k)$ and $\hat{x}(k)$ as

$$x(k) = \bar{M}_o^{-1}\Phi\hat{x}(k). \tag{13}$$

Furthermore, substituting (13) into (2b) gives

$$y(k) = C\bar{M}_o^{-1}\Phi\hat{x}(k). \tag{14}$$

By combining (7a) with (14), a new state-space model of the system (2) is obtained as follows:

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k), \tag{15a}$$

$$y(k) = \hat{C}\hat{x}(k) \text{ with } \hat{C} = C\bar{M}_o^{-1}\Phi. \tag{15b}$$

Remark 3: It should be noted that the new system (15) constructed based on the SO system (3) has the same input-output relationship as the original system (2). However, different from [7], the state $\hat{x}$ of (15) is constructed by the input-output data of the SO system (3), rather than those of the original MO system (2). Thus, the order of the new system (15) can be reduced significantly, which may decrease the computational burden of the controller design.

Remark 4: It can be proved that the system (15) is completely controllable by directly calculating the controllability matrix and its rank. By substituting (8a) and (8c) into Φ in (12), it can be proved that Φ has full row rank, which together with the full row rank of C and the non-singularity of $\bar{M}_o$ gives that $\hat{C}$ has full row rank based on (15b), i.e., $\text{rank}(\hat{C}) = p$. These properties will be used for the controller design later.

B. Data-Based Representation

The equivalent system (15) is constructed by using input-output data of the system (2), but its system matrices are still unknown. To solve this issue, this subsection will build a data-based representation of (2) with the help of (15).

Let $(u_{[0,T]},\ y_{[0,T]})$ be an input-output trajectory of the system (2). With the definition of $\hat{x}(k)$ in (6), we define $U_{n|T}$, $Y_{n|T}$, and $\hat{X}_{n+\kappa|T+\kappa}$, $\kappa = 0, 1$ as the input, output and state Hankel matrices of order 1, associated with $u_{[n,T]}$, $y_{[n,T]}$, and $\hat{x}_{[n+\kappa,T+\kappa]}$, respectively.

Assume that the input sequence $u_{[n,T]}$ is PE of order $\hat{n}+1$. From Willems' Fundamental Lemma [5], [7], we have

$$\text{rank}\big([U_{n|T}^\top, \hat{X}_{n|T}^\top]^\top\big) = \hat{n} + m. \tag{16}$$

Let $u(k) = Ky(k)$ be the output feedback controller of the system (15) with the control gain $K \in \mathbb{R}^{m\times p}$. Then, the closed-loop system (15) with the controller becomes

$$\hat{x}(k+1) = (\hat{A} + \hat{B}K\hat{C})\hat{x}(k), \tag{17a}$$
$$y(k) = \hat{C}\hat{x}(k). \tag{17b}$$

With the condition (16), an equivalent data-based representation of (17) is proposed in the following theorem.

Theorem 1: Suppose the condition (16) holds. The data-based representation of the system (15) with the output feedback controller $u(k) = Ky(k)$ is given by

$$\hat{x}(k+1) = \hat{X}_{n+1|T+1}G\hat{x}(k), \tag{18a}$$
$$y(k) = Y_{n|T}G\hat{x}(k), \tag{18b}$$

with $G \in \mathbb{R}^{(T-n+1)\times\hat{n}}$ satisfying

$$[(K\hat{C})^\top, I_{\hat{n}}]^\top = [U_{n|T}^\top, \hat{X}_{n|T}^\top]^\top G. \tag{19}$$

Proof: The proof of (18a) is similar to that of Theorem 2 in [7] and thus is omitted. The proof of (18b) is given below. From (15b) and (19), we have $I_{\hat{n}} = \hat{X}_{n|T}G$ and

$$Y_{n|T} = \hat{C}\hat{X}_{n|T}. \tag{20}$$

Right-multiplying both side of (20) with G yields $\hat{C} = Y_{n|T}G$ and substituting the result into (15b) gives (18b). ■

C. Extension to Switched Linear Systems

This subsection extends the obtained results in Subsection III-A and III-B to the switched linear system (1). For simplicity, we only consider the case that the input-output data of each subsystem can be collected independently.

We first introduce the following lemma to show that there always exists a common η for all subsystems in (1) if each pair (A_i, C) is observable and A_i has n distinct eigenvalues.

Lemma 2: Consider the switched linear system (1) with $p \geq 2$ outputs. If the pair $(A_i,\ C)$, $\forall i \in \mathcal{S}$ is observable and matrix A_i has n distinct eigenvalues, then there exists at least a common row vector $\eta \in \mathbb{R}^{1\times p}$ such that the pair $(A_i,\ \eta C)$, $\forall i \in \mathcal{S}$, is observable.

Proof: The proof is similar to that of Lemma 1, and thus is omitted. ■

Remark 5: The process of finding such a common η for the switched linear system (1) is similar to that shown in Remark 2, and thus is omitted.

Similar to (2), the system (1) can be converted to

$$\hat{x}(k+1) = \hat{A}_{\sigma(k)}\hat{x}(k) + \hat{B}_{\sigma(k)}u(k), \tag{21a}$$
$$y(k) = \hat{C}_{\sigma(k)}\hat{x}(k), \tag{21b}$$

where $\hat{A}_i \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B}_i \in \mathbb{R}^{\hat{n}\times m}$, $\hat{C}_i \in \mathbb{R}^{p\times\hat{n}}$, $\forall i \in \mathcal{S}$, and $\hat{x}(k)$ is constructed by using the same steps as those of (6). Similar to Remark 4, for any $i \in \mathcal{S}$, the pair $(\hat{A}_i,\ \hat{B}_i)$ is controllable, and the output matrix $\hat{C}_i$ has full row rank.

Let $(u_{i,[0,T]},\ y_{i,[0,T]})$ be the input-output trajectory of the i-th, $i \in \mathcal{S}$ subsystem of (1). Define $U_{i,n|T}$, $Y_{i,n|T}$, and $\hat{X}_{i,n+\kappa|T+\kappa}$, $\forall i \in \mathcal{S}$ and $\kappa = 0, 1$ as the input, output, and state Hankel matrices of order 1, respectively.

For each subsystem i, $i \in \mathcal{S}$ of the system (21), we assume that the input sequence $u_{i,[n,T]}$ is PE of order $\hat{n}+1$. From Willems' Fundamental Lemma [5], [7], it follows that

$$\text{rank}\big([U_{i,n|T}^\top,\ \hat{X}_{i,n|T}^\top]^\top\big) = \hat{n} + m,\ \forall i \in \mathcal{S}. \tag{22}$$

Let $u(k) = K_i y(k)$ with $K_i \in \mathbb{R}^{m\times p}$, $\forall i \in \mathcal{S}$, be the output feedback controller for the system (21). Then, the closed-loop switched linear system is given by

$$\hat{x}(k+1) = (\hat{A}_{\sigma(k)} + \hat{B}_{\sigma(k)}K_{\sigma(k)}\hat{C}_{\sigma(k)})\hat{x}(k), \tag{23a}$$
$$y(k) = \hat{C}_{\sigma(k)}\hat{x}(k). \tag{23b}$$

Similar to (18), the data-based representation of the closed-loop system (23) can be derived as

$$\hat{x}(k+1) = \hat{X}_{\sigma(k),n+1|T+1}G_{\sigma(k)}\hat{x}(k), \tag{24a}$$
$$y(k) = Y_{\sigma(k),n|T}G_{\sigma(k)}\hat{x}(k), \tag{24b}$$

with $G_i \in \mathbb{R}^{(T-n+1)\times\hat{n}}$, $\forall i \in \mathcal{S}$, satisfying

$$[(K_i\hat{C}_i)^\top, I_{\hat{n}}]^\top = [U_{i,n|T}^\top, \hat{X}_{i,n|T}^\top]^\top G_i. \tag{25}$$

IV. Data-driven output feedback controller

This section proposes a data-driven output feedback control scheme to stabilize the switched linear system (1) under arbitrary switching. Since the input-output behavior of the closed-loop system (1) with the output feedback controller is the same as that of the data-based representation (24), we use (24) to derive the data-driven control scheme.

Inspired by [15], a multiple Lyapunov function method is employed to design the control scheme for the switched linear system (24). We select the following piecewise quadratic function as a Lyapunov function candidate for (24)

$$V(\hat{x}(k)) = \sum_{i=1}^{s} \xi_i(k)V_i(\hat{x}(k)), \tag{26}$$

where $V_i(\hat{x}(k)) = \hat{x}(k)^\top P_i\hat{x}(k)$ with the positive definite matrix $P_i \in \mathbb{R}^{\hat{n}\times\hat{n}}$, and $\xi_i(k) = 1$ if $\sigma(k) = i$, otherwise $\xi_i(k) = 0$. Then, we introduce a lemma that provides a necessary and sufficient condition such that the closed-loop system (24) admits a Lyapunov function in the form of (26).

Lemma 3 ([14]): Consider the closed-loop system (24). There exists a Lyapunov function in the form of (26) such that (24) is asymptotically stable under arbitrary switching if and only if there exist matrices $P_i = P_i^\top$, $\forall i \in \mathcal{S}$, satisfying

$$\begin{bmatrix} P_i & \tilde{A}_i^\top P_j \\ P_j\tilde{A}_i & P_j \end{bmatrix} > 0, \forall (i,j) \in \mathcal{S}\times\mathcal{S}, \tag{27}$$

where $\tilde{A}_i = \hat{A}_i + \hat{B}_iK_i\hat{C}_i = \hat{X}_{i,n+1|T+1}G_i$.

Based on Lemma 3, the proposed controller for the system (1) can be designed by solving LMIs in the following theorem.

Theorem 2: Consider the switched linear system (1) under Assumption 1-4. For any $i \in \mathcal{S}$ and given the group of input-output data ($u_{i,[0,T]}$, $y_{i,[0,T]}$), suppose $u_{i,[n,T]}$ is PE of order $\hat{n}+1$. If there exist matrices Q_i, S_i, and non-singular matrices R_i with appropriate dimensions such that

$$\begin{bmatrix} \hat{X}_{j,n|T}Q_j & \hat{X}_{i,n+1|T+1}Q_i \\ Q_i^\top \hat{X}_{i,n+1|T+1}^\top & \hat{X}_{i,n|T}Q_i \end{bmatrix} > 0, \tag{28a}$$

$$S_i Y_{i,n|T} - U_{i,n|T}Q_i\hat{X}_{i,n|T} = 0, \tag{28b}$$

$$R_i Y_{i,n|T} - Y_{i,n|T}Q_i\hat{X}_{i,n|T} = 0 \tag{28c}$$

hold for all $i, j \in \mathcal{S}$, then the output feedback controller $u(k) = K_i y(k)$ with $K_i = S_i R_i^{-1}$, $\forall i \in \mathcal{S}$ asymptotically stabilizes the switched system (1) under arbitrary switching.

Proof: Based on Lemma 3, if LMIs (27) are feasible, the closed-loop system (24) is asymptotically stable. Therefore, it is sufficient to prove that LMIs (25) and (27) can be guaranteed by LMIs (28a)-(28c).

First, we prove that LMIs (28a) are equivalent to (27) by choosing $P_i = (\hat{X}_{i,n|T}Q_i)^{-1}$, $\forall i \in \mathcal{S}$. Let $\Gamma_i = P_i^{-1}$. Applying Schur complement to (27) twice yields that

$$\begin{bmatrix} \Gamma_j & \Gamma_i\tilde{A}_i^\top \\ \tilde{A}_i\Gamma_i & \Gamma_i \end{bmatrix} > 0 \tag{29}$$

holds for all $i, j \in \mathcal{S}$.

Replacing $\tilde{A}_i$ in (29) with $\hat{X}_{i,n+1|T+1}G_i$ gives

$$\begin{bmatrix} \Gamma_j & \Gamma_i G_i^\top \hat{X}_{i,n+1|T+1}^\top \\ \hat{X}_{i,n+1|T+1}G_i\Gamma_i & \Gamma_i \end{bmatrix} > 0. \tag{30}$$

Let $Q_i \in \mathbb{R}^{(T-n+1)\times\hat{n}}$ satisfy

$$Q_i = G_i\Gamma_i,\ \forall i \in \mathcal{S}. \tag{31}$$

Combining (31) with (25) implies

$$K_i\hat{C}_i\Gamma_i = U_{i,n|T}Q_i, \tag{32}$$

$$\Gamma_i = \hat{X}_{i,n|T}Q_i,\ \forall i \in \mathcal{S}. \tag{33}$$

Then, substituting (31) and (33) into (30) gives (28a).

Subsequently, we further give the derivation of the LMIs (28b) and (28c). These two LMIs are used to guarantee that control gains $K_i = S_i R_i^{-1}$, $\forall i \in \mathcal{S}$ satisfy the condition (25).

From Remark 4, $\hat{C}_i$, $\forall i \in \mathcal{S}$ has full row rank. Thus, there exist nonsingular matrices $R_i \in \mathbb{R}^{p\times p}$, $i = 1, \ldots, s$, such that

$$R_i\hat{C}_i = \hat{C}_i\Gamma_i,\ \forall i \in \mathcal{S}. \tag{34}$$

Substituting $K_i = S_i R_i^{-1}$ and (34) into (32) gives

$$S_i\hat{C}_i = U_{i,n|T}Q_i,\ \forall i \in \mathcal{S}. \tag{35}$$

On the other hand, substituting (33) into (34) gives

$$R_i\hat{C}_i = \hat{C}_i\hat{X}_{i,n|T}Q_i,\ \forall i \in \mathcal{S}. \tag{36}$$

Since $Y_{i,n|T} = \hat{C}_i X_{i,n|T}$, the LMIs (28b) and (28c) can be derived by right-multiplying both sides of (35) and (36) with $\hat{X}_{i,n|T}$, respectively.

In conclusion, if the LMIs (28a)-(28c) are feasible, the LMIs (25) and (27) can be guaranteed by choosing $P_i = (\hat{X}_{i,n|T}Q_i)^{-1}$, and $K_i = S_i R_i^{-1}$, $\forall i \in \mathcal{S}$. Based on Lemma 3, the closed-loop system (24) is asymptotically stable, i.e., the designed output feedback controller asymptotically stabilizes the switched system (1) under arbitrary switching. ■

Remark 6: According to the definition of $\hat{x}$ in (6), if the output matrices are different, there will be a state jump at each switching instant, which may cause difficulties in the stability analysis. To avoid such an issue, this paper requires the output matrices in (1) to be identical, i.e., $C_1 = \cdots = C_n = C$. Of course, subsystems in (1) may have different output matrices in practice, and how to extend the obtained results to this general case will be studied in future work.

Remark 7: As discussed in Remark 3, one of the major advantages of our method is the dimension reduction of the state in (24), and thus may reduce the computational burden of the controller design. This advantage becomes particularly significant when using input-output data to design controllers for complex systems such as hybrid systems (e.g., unmanned aerial vehicles [24]) and large-scale systems with interconnected MIMO subsystems (e.g., multi-agent systems [25]). The application of our method to these practical scenarios will be a subject of study in future work.

V. Numerical Example

This section gives a numerical example to show the effectiveness of our methods. Consider the switched linear system (1) with four subsystems and system matrices as

$$A_i = \begin{bmatrix} 0.36 & 0 & 0 & 0 & \theta_i^1 \\ 0 & \theta_i^2 & 0.78 & 0 & 0 \\ 0 & 0.76 & 0.39 & 0 & 0.3 \\ 0 & 0 & 0 & \theta_i^3 & 0.3 \\ \theta_i^4 & 0 & 0.5 & 0.24 & 0.5 \end{bmatrix}, B_i = \begin{bmatrix} \theta_i^5 & 0.2 \\ 0.37 & 0.6 \\ 0.88 & 0.5 \\ 0.63 & \theta_i^6 \\ 0.86 & 0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.55 & 0.50 & 0 & 0 & 0.34 \\ 0 & 0 & 0.74 & 0.40 & 0 \end{bmatrix}, i = 1, \ldots, 4,$$

where θ_i^ι, $i = 1, \ldots, 4$, $\iota = 1, \ldots, 6$ is given in Table. I.

TABLE I

Parameters in the switched linear system.

i	θ_i^1	θ_i^2	θ_i^3	θ_i^4	θ_i^5	θ_i^6
1	0.35	0.25	0.38	0.35	0.50	0.44
2	0.45	0.10	0.30	0.45	0.32	0.12
3	0.15	0.15	0.48	0.15	0.29	0.34
4	0.25	0.15	0.28	0.25	0.19	0.24

It is obvious that the four subsystems are unstable without control. We assume that system matrices are unknown and are used to build the test system in the simulation. Assume that the system state is unmeasurable. Based on Remark 1, by applying two special inputs $u_1(k) = [1, 0]^\top$ at $k = 0$ and $u_1(k) = 0_{2\times1}$ otherwise and $u_2(k) = [0, 1]^\top$ at $k = 0$ and $u_2(k) = 0_{2\times1}$ otherwise to these four subsystems and collecting the output data, respectively, we can verify that the four subsystems are controllable and observable, and their output matrices have full row rank. Further, based on Remark 2 and 5, a common row vector $\eta = [0.5, 0.5]$ makes the four pairs (A_i,ηC), $i = 1, \ldots, 4$ observable. Then, based on Section III-A, a new state variable can be constructed by using the past 5-step input-output data of the system and the selected η as $\hat{x}(k) = [u(k-5)^\top, \cdots, u(k-1)^\top, \eta y(k-5), \cdots, \eta y(k-1)]^\top$.

By simulation, we collect a set of input-output sequences of the switched linear system, denoted by $(u_{i,[0,55]}, y_{i,[0,55]})$, $i = 1, \ldots, 4$, in which the control input is randomly generated between -0.1 and 0.1. The input sequence $u_{i,[5,55]}$ is PE of order 16, i.e., $\hat{n} + 1$. Referring to Subsection III-C, the input, state, and output Hankel matrices of order 1 are denoted as $U_{i,5|55}$, $\hat{X}_{i,5+\kappa|55+\kappa}$ and $Y_{i,5|55}$, $i = 1, \ldots, 4$, $\kappa = 0, 1$, respectively. Then, solving the LMI problem in Theorem 2 can obtain the controller $u(k) = K_i y(k)$, $i = 1, \ldots, 4$.

The switching signal and output results of the closed-loop system with the initial condition $x_0 = [-0.32, 0.95, -0.56, -0.31, 0.79]^\top$ are given in Fig. 1. The switching signal is generated randomly. In the last two sub-graphs of Fig. 1, the blue lines are the system outputs with the proposed switched data-driven control (SDDC) scheme; and the red dashed lines are those with the switched model-based control (SMBC) scheme proposed in [14]. It can be observed that the SDDC scheme can reach the control target and achieve a similar control performance to the SMBC scheme.

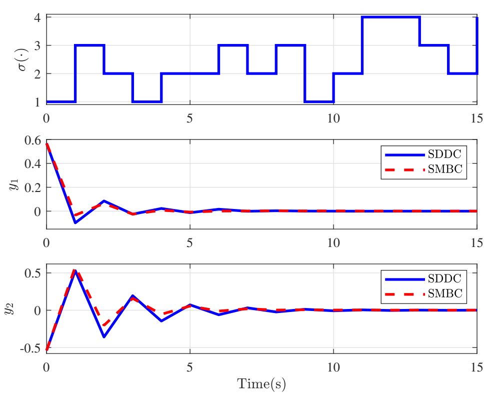

Fig. 1. The switching signal and output trajectories of the switched linear system.

A DDC method for non-switched LTI systems is proposed in [7] by using the original system output to construct the new system state $\hat{x}$. Using our method, the dimension of $\hat{x}$ is 15 compared with 20 using [7]; thus, our method may have a lower computation burden. Moreover, it is worth pointing out that subsystems with the state $\hat{x}(k)$ directly constructed by the system output are uncontrollable. Therefore, we cannot design the controllers by using our method or any other methods based on Willems' Fundamental Lemma.

VI. Conclusion

This paper has studied the problem of designing data-driven output feedback controllers for unknown switched linear systems under arbitrary switching. First, inspired by Willems' Fundamental Lemma, a data-based representation of the closed-loop system with an output feedback controller has been constructed using only input-output data. Based on this representation, the control problem has been formulated as a data-dependent LMI problem. Then, an output feedback controller has been designed using the feasible solution to the LMI problem. A numerical example has been given to show the correctness and effectiveness of the proposed controller.

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