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Distributed Finite-Time Supremum/Infimum Dynamic Consensus under Directed Network Topology

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***Abstract*— In this paper, we address the distributed supremum/infimum dynamic consensus problem in networked multi-agent systems. More in detail, by considering that each agent has access to a local exogenous time-varying signal, the objective is to have all the agents distributively track the global maximum supremum (or minimum infimum) of these exogenous signals. We propose a distributed protocol guaranteeing finite-time convergence under directed network topology. The sole requirements are the strong connectivity of the communication graph and the boundedness of the derivatives of the exogenous signals, with known bounds. The effectiveness of the proposed protocol is corroborated through numerical simulations in a precision farming case study.**

I. Introduction

Achieving consensus stands as a fundamental objective [1] in the domain of distributed networked multi-agent systems: each agent possesses local information and, through interactions with neighboring agents, the goal is to collectively converge toward a common solution. This capability finds great utility in a wide range of fields, such as collaborative decision-making, resource allocation, and distributed computing, among others. The seminal problem addressed in the context of consensus theory has been the static average consensus, in which the agents have to converge to the average of their initial values in a distributed fashion [2]. Building upon this, several variations and extensions have been explored in the literature. One notable extension is the dynamic average consensus problem, also known as average consensus tracking, where agents have access to local time-varying exogenous signals and they are tasked with dynamically tracking the average of these signals [3], [4], [5]. A further consensus problem that has received attention within the research community is the static maximum/minimum consensus problem. Here, the objective is to have the agents converge toward the maximum or minimum values of their initial states. For instance, the work in [6] proposes a protocol achieving asymptotic convergence with weakly connected and weakly-balanced directed graphs, and finite-time convergence with strongly connected interaction directed graphs. Asynchronous agents and bounded time delays are analyzed in [7] where finite-time convergence with directed graphs is reached, while jointly connected communication networks are considered in [8] where asymptotic convergence with double-integrator agents is proved. An application for shortest path planning based on static minimum consensus can also be found in [9]. As far as the *dynamic* maximum/minimum consensus problem, where the maximum/minimum of time-varying exogenous signals should be tracked, a few contributions exist in the literature. Among these, a protocol only achieving *bounded* error in case of time-varying exogenous signals is proposed in [10].

In this work, we focus on the supremum/infimum consensus problem, introduced in our previous papers [11], [12], where the objective is to reach consensus on the maximum supremum (or minimum infimum) of a given set of signals. These signals represent the initial state values in the static version, and time-varying exogenous references in the dynamic case. By noticing that the static version can be reformulated as a static maximum/minimum consensus problem, we narrow our focus to the dynamic scenario. In our earlier studies, we developed in [11] a protocol with finite-time convergence tailored for undirected communication graphs and based on known bounds of the exogenous signal derivatives. Subsequently, we relaxed the assumption of known bounds in [12] and devised an adaptive distributed protocol with finite-time convergence. Nonetheless, in both works, we operated under the assumption of undirected communication networks. It is worth noting that in real-world applications, scenarios involving directed communication graphs, or digraphs, are quite common. In such cases, the asymmetrical exchange of information among the agents introduces substantial complexities in the theoretical analysis compared to undirected graphs [13]. A contribution addressing the dynamic supremum consensus problem with digraphs can be found in [14]. However, it is important to note that only *asymptotic* convergence is guaranteed, while in practical applications, the necessity for finite-time convergence of the dynamic supremum consensus protocol may arise. This is particularly relevant, for instance, in tasks like anomaly detection or when estimating upper bounds for tuning subsequent algorithms.

Motivated by the above considerations, in this work we propose a distributed protocol solving the dynamic supremum or infimum consensus problem under directed network topologies and achieving *finite-time* convergence. The protocol requires strong graph connectivity and assumes bounded derivatives for the exogenous signals, with known bounds. Simulation results in a precision farming context inspired by the H2020 European project CANOPIES[1] are provided to validate the effectiveness of the protocol.

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[1] www.canopies-project.eu

II. Preliminaries and Problem Definition

A. Nonsmooth Theory

In this section, we recall some fundamental notions from nonsmooth theory that are necessary for our mathematical analysis. Consider a possibly discontinuous dynamical system with state vector $\boldsymbol{x} \in \mathbb{R}^n$

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ a measurable and essentially locally bounded function. When the the function $f(\cdot)$ is discontinuous, the solution is defined in the Filippov sense [15]. To simplify the notation, in the following, we omit the time dependency of the variables, unless necessary, e.g., we refer to $\boldsymbol{x}(t)$ simply as $\boldsymbol{x}$.

Definition 1 (Filippov Solution)**.** *A vector function $\boldsymbol{x}(\cdot)$ is a Filippov solution of (1) on a time interval $[t_0,\, t_1]$ if $\boldsymbol{x}(\cdot)$ is absolutely continuous on $[t_0,\, t_1]$ and for almost all $t \in [t_0,\, t_1]$ it holds $\dot{\boldsymbol{x}} \in K[f](\boldsymbol{x}, t)$, where $K[f](\boldsymbol{x}, t) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, with $2^{\mathbb{R}^n}$ the set of all subsets of $\mathbb{R}^n$, is a set-valued map defined as*

$$K[f](\boldsymbol{x}, t) = \bigcap_{\delta>0} \bigcap_{\mu\{H\}=0} \overline{\text{co}}\{f(B(\boldsymbol{x}, \delta) \setminus H, t)\}, \tag{2}$$

with $\bigcap_{\mu\{H\}=0}$ denoting the intersection over all sets H of Lebesgue measure zero, $B(\boldsymbol{x}, \delta)$ the ball of radius δ centered at $\boldsymbol{x}$, and $\overline{\text{co}}$ the convex closure.

This definition allows to discard sets of measure zero, implying that Filippov solutions can be defined at points where the vector field is not defined. We now recall Clarke's generalized gradient, the chain rule, and the generalized Lyapunov theorem [16] for finite-time stability.

Definition 2 (Clarke's Generalized Gradient [17])**.** *Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke's generalized gradient at $(\boldsymbol{x}, t)$ is given by*

$$\partial V(\boldsymbol{x}, t) \triangleq \text{co}\left\{\lim_{k\to\infty} \nabla V(\boldsymbol{x}_k, t_k) : (\boldsymbol{x}_k, t_k) \to (\boldsymbol{x}, t), (\boldsymbol{x}_k, t_k) \notin \Omega_V\right\}, \tag{3}$$

with ∇V the gradient function, $\boldsymbol{x}_k \in \mathbb{R}^n$ a point of an infinite succession converging to $\boldsymbol{x}$, Ω_V a set of Lebesgue measure zero containing all points where $\nabla V(\boldsymbol{x}, t)$ is not defined.

Theorem 1 (Chain Rule [18])**.** *Let $\boldsymbol{x}(\cdot)$ be a Filippov solution to (1) on an interval containing t and $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz and regular function. Then, $V(\boldsymbol{x}, t)$ is absolutely continuous, $\frac{d}{dt}(V(\boldsymbol{x}, t))$ exists almost everywhere and*

$$\frac{d}{dt} V(\boldsymbol{x}, t) \in^{a.e.} \dot{\tilde{V}}(\boldsymbol{x}, t),$$

with $\dot{\tilde{V}}(\boldsymbol{x}, t)$ defined as

$$\dot{\tilde{V}}(\boldsymbol{x}, t) = \bigcap_{\boldsymbol{\xi} \in \partial V(\boldsymbol{x}, t)} \boldsymbol{\xi}^T \begin{pmatrix} K[f](\boldsymbol{x}, t) \\ 1 \end{pmatrix}. \tag{4}$$

Theorem 2 (Finite-Time Stability Theorem)**.** *Let $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^n$ be a Filippov solution to (1) and $V(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, be a time dependent regular function such that $V(\boldsymbol{x}, t) = 0$, $\forall \boldsymbol{x}(t) \in \mathcal{C}(t)$ and $V(\boldsymbol{x}, t) > 0$, $\forall \boldsymbol{x}(t) \notin \mathcal{C}(t)$, with $\mathcal{C}(t) \subset \mathbb{R}^n$ a compact set. Furthermore, let $\boldsymbol{x}(t)$ and $V(\boldsymbol{x}, t)$ be absolutely continuous on $[0, \infty)$ with $\frac{d}{dt}(V(\boldsymbol{x}, t)) \leq -\varepsilon < 0$ almost everywhere on $\{t : \boldsymbol{x}(t) \notin \mathcal{C}(t)\}$. Then, $V(\boldsymbol{x}, t)$ converges to 0 in finite-time and $\boldsymbol{x}(t)$ reaches the compact set $\mathcal{C}(t)$ in finite-time as well.*

Finally, we define the discontinuous function $\text{sign}(y)$, with $y \in \mathbb{R}$, and its corresponding set-valued function $\text{SIGN}(y)$

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0, \end{cases} \text{SIGN}(y) \in \begin{cases} 1 & \text{if } y > 0, \\ [-1, 1] & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases}$$

B. Network Modeling

Consider a network comprising n agents characterized by a directed topology. This topology is modeled by a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, n\}$ denotes the node set, and $\mathcal{E} \subset \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges. Specifically, the presence of an edge $(i, j) \in \mathcal{E}$ signifies that agent i can transmit information to agent j at time t. A graph is defined as strongly connected if there exists a directed path connecting every pair of nodes in the graph. The notation $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ denotes the neighborhood of agent i, while the extended neighborhood is denoted by $\overline{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}$. Additionally, we define the following set

$$\mathcal{N}_i^+(t) = \left\{ j \in \overline{\mathcal{N}}_i : x_j(t) = \max_{\ell \in \overline{\mathcal{N}}_i} \{x_\ell(t)\} \right\}, \tag{5}$$

collecting the agents with maximum state in the extended neighborhood of agent i, and indicate with i^+ the index of any agent belonging to $\mathcal{N}_i^+(t)$, i.e., $i^+ \in \mathcal{N}_i^+(t)$. Note that all agents in $\mathcal{N}_i^+(t)$ have the same state value.

C. Problem Formulation

Let us consider a networked multi-agent system comprising n agents, each characterized by first-order dynamics

$$\dot{x}_i(t) = u_i(t), \tag{6}$$

where $x_i(t) \in \mathbb{R}$ is the state variable of the agent i and $u_i(t) \in \mathbb{R}$ is the respective control input. Each agent i has access to a scalar exogenous reference signal $r_i(t)$, for which we make the following assumption.

Assumption 1. *The reference signals $r_i(t)$ are absolutely continuous and it holds $|\psi| < \psi_r$ for all $\psi \in K[\dot{r}_i](t)$, $\forall i, t$, with ψ_r a positive known constant.*

We model the communication topology of the network as a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, for which the following assumption is made.

Assumption 2. *The communication graph $\mathcal{G}$ is strongly connected.*

By denoting the supremum (infimum) of a function $c : [0, \infty) \to \mathbb{R}$ as $c^+(t)\,(c^-(t))$, i.e.,

$$c^+(t) = \sup_{\tau \in [0,t]} \{c(\tau)\}, \left(c^-(t) = \inf_{\tau \in [0,t]} \{c(\tau)\} \right),$$

we define the maximum (minimum) of the supremum (infimum) of the exogenous signals $r_i(t)$ as $\overline{r}(t)$ ($\underline{r}(t)$), i.e.,

$$\overline{r}(t) = \max_{i\in\mathcal{V}}\{r_i^+(t)\}, \left(\underline{r}(t) = \min_{i\in\mathcal{V}}\{r_i^-(t)\}\right).$$

We can now formulate the distributed finite-time dynamic supremum (infimum) consensus problem.

Problem 1. *Consider a multi-agent system with n agents and let Assumptions 1-2 hold. The objective of the finite-time dynamic supremum (infimum) problem is to determine the control input $u_i(t)$ in* (6) *that ensures the finite-time tracking of the maximum (minimum) supremum (infimum) of the reference signals, i.e., such that there exists a finite-time $T > 0$ for which it holds, $\forall i \in \mathcal{V}$*

$$|x_i(t) - \overline{r}(t)| = 0, \quad (|x_i(t) - \underline{r}(t)| = 0) \quad t \geq T. \tag{7}$$

III. Proposed Protocol

In this section, we focus on the design and analysis of the control input $u_i(t)$ solving Problem 1. Due to space constraints, we will concentrate on the supremum case; however, it is worth noting that the infimum case can be approached through analogous reasoning. Let us denote with $\boldsymbol{x} = [x_1, ..., x_n]^T \in \mathbb{R}^n$ the stacked vector of the agents' states, and introduce the following variable $y_i \in \mathbb{R}$

$$y_i(\boldsymbol{x}, t) = (x_{i^+}(t) - x_i(t)) + \phi_i(\cdot)\,(r_i^+(t) - x_i(t)), \tag{8}$$

which is non-negative by construction and where $\phi_i(\cdot)$ is a selection function equal to

$$\phi_i(x_i(t), t) = \begin{cases} 0, & \text{if } x_i(t) \geq r_i^+(t), \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

We propose the following control input u_i, $\forall i \in \mathcal{V}$, to solve the finite-time dynamic supremum consensus problem

$$u_i(\boldsymbol{x}, t) = \alpha\,\text{sign}\,(y_i(\boldsymbol{x}, t))\,. \tag{10}$$

where α is a positive constant.

Remark 1. *To solve the dynamic infimum consensus consensus, the same form of the control law in* (10) *is preserved, and y_i is adapted as follows*

$$y_i(\boldsymbol{x}, t) = (x_{i^-}(t) - x_i(t)) + \phi_i(\cdot)\,(r_i^-(t) - x_i(t)), \tag{11}$$

where i^- represents any agent in the extended neighborhood of i with minimum state and $\phi_i(\cdot)$ is equal to 0 if $x_i(t) \leq r_i^-(t)$ and is 1 otherwise.

To prove the convergence of the proposed update law, we first provide some ancillary results. Let $\mathcal{I}^M(t)$ be the set of agents with maximum state in the network at time t, i.e.,

$$\mathcal{I}^M(t) = \{i \in \mathcal{V} \,|\, x_i(t) = \max_{\ell\in\mathcal{V}}\{x_\ell(t)\}. \tag{12}$$

The following lemma holds true.

Lemma 1. *Let Assumption 1 hold. Assume that the agents run the control input in* (10)*. Then, if $x_M(0) \leq \overline{r}(0)$, $\forall M \in \mathcal{I}_M(0)$ the following holds $x_i(t) \leq \overline{r}(t)$, $\forall i, t$.*

Proof. Consider an interval (t_1, t_2) such that $x_M(t) \geq \overline{r}(t)$, for all $t \in (t_1, t_2)$, where M is any agent in $\mathcal{I}^M(t)$. We first establish the following result

$$\dot{x}_M(t) = 0, \quad \forall t \in (t_1, t_2). \tag{13}$$

To this end, we observe that for all $t \in (t_1, t_2)$, it holds $x_M(t) \geq \overline{r}(t)$, by assumption, and $\overline{r}(t) \geq r_M^+(t)$, by definition of $\overline{r}(t)$. Therefore, it follows that $x_M(t) \geq r_M^+(t)$, for all $t \in (t_1, t_2)$, and, then, in view of (9), we obtain

$$\phi_M(x_M(t), t) = 0, \quad \forall t \in (t_1, t_2). \tag{14}$$

At this point, being $M \in \mathcal{I}^M(t)$, it holds by construction that

$$x_{M^+}(t) - x_M(t) = 0, \quad \forall t \in (t_1, t_2). \tag{15}$$

By plugging (14) and (15) in (8), it follows

$$y_M(\boldsymbol{x}, t) = 0, \quad \forall t \in (t_1, t_2). \tag{16}$$

At this point, we observe that (13) follows by replacing (16) in (10). Now that we have established (13), to prove our main result we show that

$$x_M(t) \leq \overline{r}(t), \quad \forall t. \tag{17}$$

To this end, let us assume by contradiction that there exists an instant $t_2 > 0$ such that $x_M(t_2) - \overline{r}(t_2) > 0$. Then, by continuity of $x_M(t) - \overline{r}(t)$ and since $x_M(0) - \overline{r}(0) \leq 0$, there exists an instant $t_1 \in [0, t_2)$ such that $x_M(t) - \overline{r}(t) \geq 0$ for all $t \in [t_1, t_2]$, where in particular, $x_M(t_1) - \overline{r}(t_1) = 0$ and $x_M(t) - \overline{r}(t) > 0$ for all $t \in (t_1, t_2]$. Then, it holds 13. Moreover, since by Assumption 1 it holds $\dot{\overline{r}}(t) < \psi_r$ almost everywhere in (t_1, t_2), it follows that

$$\underbrace{\dot{x}_M(t)}_{0} - \underbrace{\dot{\overline{r}}(t)}_{\leq\psi_r} \leq 0 \quad \forall t \in (t_1, t_2) \setminus \mathcal{T},$$

for some set of measure zero $\mathcal{T}$, where the last inequality holds since $\psi_r \geq 0$. Then, being $x_M(t) - \overline{r}(t)$ absolutely continuous, it follows that

$$x_M(t_2) - \overline{r}(t_2) = \underbrace{x_M(t_1) - \overline{r}(t_1)}_{=0} + \int_{t_1}^{t_2} \underbrace{\dot{x}_M(t) - \dot{\overline{r}}(t)}_{\leq 0}\,dt \leq 0, \tag{18}$$

which yields $x_M(t_2) - \overline{r}(t_2) \leq 0$, thus contradicting the assumption that $x_M(t_2) - \overline{r}(t_2) > 0$. By the arbitrariness of t_2, the result in (17) follows and concludes our proof. $\square$

We additionally define the set $\mathcal{I}^m(t)$ of agents holding the minimum state in the network at t, i.e.,

$$\mathcal{I}^m(t) = \{i \in \mathcal{V} \,|\, x_i(t) = \min_{\ell\in\mathcal{V}}\{x_\ell(t)\}\}, \tag{19}$$

and the respective subset of the agents having zero variable y_i, i.e.,

$$\mathcal{I}^z(t) = \{m \in \mathcal{I}^m(t) \,|\, y_m(\boldsymbol{x}, t) = 0\}, \tag{20}$$

Based on these sets, we can prove the following results.

Proposition 1. *Let $m \in \mathcal{I}^z(t)$. Then, it follows that $x_m(t) \geq r_m^+(t)$ and*

$$\mathcal{N}_m(t) \subseteq \mathcal{I}^m(t). \tag{21}$$

Proof. Let us first observe that, being $m \in \mathcal{I}^z(t)$, it holds $y_m(\boldsymbol{x}, t) = 0$. Since $y_m(\cdot)$ is the sum of non negative terms, it follows from (8) that the two terms $x_{m^+}(t) - x_m(t)$ and $\phi_m(\cdot)(r_m^+(t) - x_m(t))$ are zero. From $x_{m^+}(t) - x_m(t) = 0$, we have that $x_{m^+}(t) \in \mathcal{I}^m(t)$, by which the result in (21) follows. The inequality $x_m(t) \geq r_m^+(t)$ follows immediately by $\phi_m(\cdot)(r_m^+(t) - x_m(t)) = 0$. □

Proposition 2. *Let Assumption 2 hold. Consider that the agents run the control input in* (10)*. Moreover, suppose that* $\mathcal{I}^m(t) \equiv \mathcal{I}^z(t)$ *holds. Then, all the agents belong to* $\mathcal{I}^z(t)$*, i.e.,* $\mathcal{V} \equiv \mathcal{I}^z(t)$.

Proof. This proposition easily follows by considering that, since $\mathcal{I}^m(t) \equiv \mathcal{I}^z(t)$ and being the graph strongly connected, then starting from any agent $m \in \mathcal{I}^z(t)$ we can recursively apply (21) until all agents are reached, leading to $\mathcal{V} \equiv \mathcal{I}^z(t)$. □

Lemma 2. *Let Assumption 2 hold. Moreover, assume that* $x_M(0) \leq r_M(0)$, $\forall M \in \mathcal{I}_M(0)$*. Then, it holds*

$$x_m(t) = \overline{r}(t), m \in \mathcal{I}^m(t) \iff \mathcal{I}^m(t) \equiv \mathcal{I}^z(t). \tag{22}$$

Proof. Suppose that $x_m(t) = \overline{r}(t)$ for $m \in \mathcal{I}^m(t)$. In order to prove that $\mathcal{I}^m(t) \equiv \mathcal{I}^z(t)$, we show that $y_m(\boldsymbol{x}, t) = 0$ holds for all $m \in \mathcal{I}^m(t)$. To this end, as per (8), we show that for such agents the terms $x_m^+(t) - x_m(t)$ and $\phi_m(\cdot)(r_m^+(t) - x_m(t))$ are zero. Since by assumption it holds $x_m(t) = \overline{r}(t)$ for $m \in \mathcal{I}^m(t)$ and since, by definition, $\overline{r}(t) \geq r_i^+(t)$ for all $i \in \mathcal{V}$, it follows that $x_m(t) \geq r_m^+(t)$, by which, from (9), we have $\phi_i(\cdot)(r_m^+(t) - x_m(t)) = 0$. Let us now observe that, in view of Lemma 1, it holds $x_i(t) \leq \overline{r}(t)$ for all $i \in \mathcal{V}$, leading to

$$\overline{r}(t) = x_m(t) \leq x_i(t) \leq \overline{r}(t) \quad \forall i \in \mathcal{V},$$

by which it follows that $\mathcal{V} \equiv \mathcal{I}^m(t)$, hence, $x_{m^+}(t) - x_m(t) = 0$ for all $m \in \mathcal{I}^m(t)$.

Conversely, suppose that $\mathcal{I}^m(t) \equiv \mathcal{I}^z(t)$. Then, by Proposition 2, it holds $\mathcal{V} \equiv \mathcal{I}^z(t)$. In this view, let us observe that, by virtue of Proposition 1, it holds $x_i(t) \geq r_i^+(t)$ for all $i \in \mathcal{V}$. Moreover, since $x_m(t) = x_i(t)$ for all i, it follows that

$$x_m(t) \geq \max_{i \in \mathcal{V}} \{r_i^+(t)\} = \overline{r}(t), \quad \forall m \in \mathcal{I}^m(t),$$

which combined with the result of Lemma 1, leads to the condition $x_m(t) = \overline{r}(t)$ for all $m \in \mathcal{I}^m(t)$. □

At this point, we can provide theoretical guarantees that the control input in (10) solves Problem 1.

Theorem 3. *Consider a multi-agent system of* n *agents interconnected by a directed graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ *and let Assumptions 1-2 hold. Suppose that the agents run* (10) *with gain* α *satisfying*

$$\alpha \geq n\psi_r + \varepsilon \tag{23}$$

with $\varepsilon > 0$*, and assume that* $x_M(0) \leq r_M(0)$, $\forall M \in \mathcal{I}^M(0)$*. Then, the agents track the maximum supremum* $\overline{r}(t)$ *in finite-time and an upper bound* T *to the convergence time is*

$$T = \frac{1}{\varepsilon} \left(\overline{r}(0) - x_m(0)\right), \quad m \in \mathcal{I}^m(0). \tag{24}$$

Proof. Let $|\mathcal{I}^m(t)|$ be the cardinality of the set $\mathcal{I}^m(t)$, we introduce the following auxiliary function $h(\boldsymbol{x}, t)$

$$h(\boldsymbol{x}, t) = \overline{r}(t) - \frac{1}{|\mathcal{I}^m(t)|} \sum_{m \in \mathcal{I}^m(t)} x_m(t),$$

which is non-negative by virtue of Lemma 1. We consider the following Lyapunov candidate

$$V(\boldsymbol{x}, t) = |h(\boldsymbol{x}, t)|. \tag{25}$$

Notably, $V(\boldsymbol{x}, t) = 0$ if and only if $x_m(t) = \overline{r}(t)$. Additionally, since from Lemma 1 it holds $x_i(t) \leq \overline{r}(t)$ and by construction it holds $x_m(t) \leq x_i(t)$, $\forall i \in \mathcal{V}$, we have that $V(\boldsymbol{x}, t) = 0$ if and only if $x_i(t) = \overline{r}(t)$ for all i. Note that the cardinality of the set $\mathcal{I}^m(t)$ is a piecewise constant function and the discontinuity time instants belong to a set of measure zero [19]. Consequently, these isolated instants can be ignored in the nonsmooth analysis and $\mathcal{I}^m(t)$ can be studied as a set with constant cardinality.

Let us first compute the generalized derivative $\dot{\tilde{V}}(\boldsymbol{x}, t)$, as defined in Theorem 1. To this end, the Clarke's generalized gradient $\partial V(\boldsymbol{x}, t)$, defined in (3), is given by

$$\partial V(\boldsymbol{x}, t) \subseteq \left[\ \partial_x V^T(\boldsymbol{x}, t),\, \partial_t V(\boldsymbol{x}, t)\ \right]^T,$$

where

$$\begin{aligned} \partial_x V(\boldsymbol{x}, t) &\subseteq -\operatorname{SIGN}(h(\boldsymbol{x}, t)) \frac{1}{|\mathcal{I}^m(t)|} s^m(t), \\ \partial_t V(\boldsymbol{x}, t) &\subseteq \operatorname{SIGN}(h(\boldsymbol{x}, t)) K[\dot{\overline{r}}](t), \end{aligned} \tag{26}$$

with $s^m(t) \in \mathbb{R}^n$ a selection vector with component i equal to 1 if $i \in \mathcal{I}^m(t)$, 0 otherwise. By applying Theorem 1, the generalized derivative $\dot{\tilde{V}}(\boldsymbol{x}, t)$ is obtained as

$$\dot{\tilde{V}}(\boldsymbol{x}, t) = \bigcap_{[\boldsymbol{\eta}^T\ \psi]^T \in \partial V} \boldsymbol{\eta}^T K[\dot{\boldsymbol{x}}](\boldsymbol{x}, t) + \psi. \tag{27}$$

where the set-valued map $K[\dot{\boldsymbol{x}}](\boldsymbol{x}, t)$ can be computed as [20]

$$K[\dot{\boldsymbol{x}}](\boldsymbol{x}, t) \subseteq \Big[K[\dot{x}_1](\boldsymbol{x}, t),\, K[\dot{x}_2](\boldsymbol{x}, t),\, \ldots, K[\dot{x}_n](\boldsymbol{x}, t)\Big]^T, \tag{28}$$

where, since it holds $y_i(\boldsymbol{x}, t) \geq 0, \forall i$ by construction, we have from Lemma 1 in [21]

$$K[\dot{x}_i](\boldsymbol{x}, t) = \alpha \operatorname{SIGN}^+(y_i(\boldsymbol{x}, t)), \quad \forall i \in \mathcal{V}.$$

Our objective is to demonstrate that

$$\dot{V}(\boldsymbol{x}, t) < -\varepsilon < 0, \quad \text{when } V(\boldsymbol{x}, t) \neq 0, \tag{29}$$

$$\dot{V}(\boldsymbol{x}, t) = 0, \quad \text{when } V(\boldsymbol{x}, t) = 0. \tag{30}$$

We first focus on proving (29) and analyze a generic element of the set-valued derivative (27)

$$\begin{aligned}\boldsymbol{\eta}^T K[\dot{\boldsymbol{x}}](\boldsymbol{x},t)+\psi \\ \subseteq \mathrm{SIGN}(h(\cdot))\left(-\frac{1}{|\mathcal{I}^m(t)|}\sum_{m\in\mathcal{I}^m(t)} K[\dot{x}_m](\boldsymbol{x},t)+K[\dot{\bar{r}}](t)\right) \\ = \mathrm{SIGN}(h(\cdot))\left(-\frac{\alpha}{|\mathcal{I}^m(t)|}\sum_{m\in\mathcal{I}^m(t)} \mathrm{SIGN}^+(y_m(\boldsymbol{x},t))+K[\dot{\bar{r}}](t)\right).\end{aligned} \tag{31}$$

By the considerations above, it follows that if $V(\boldsymbol{x},t)\neq 0$ it holds $x_m(t) \neq \bar{r}(t)$, i.e., $h(\boldsymbol{x},t) > 0$ and $\mathrm{SIGN}(h(\boldsymbol{x},t))=\{1\}$. As far as the set-valued functions $\mathrm{SIGN}^+(y_m(\cdot))$ are concerned, they can be computed as

$$\begin{aligned}\mathrm{SIGN}(y_m(\cdot)) &= \{1\}, & \forall m\in\mathcal{I}^m(t)\setminus\mathcal{I}^z(t),\\ \mathrm{SIGN}^+(y_m(\cdot)) &= [0,1], & \forall m\in\mathcal{I}^z(t).\end{aligned}$$

By Lemma 2 we have that $\mathcal{I}^m(t)\cap\mathcal{I}^z(t)\neq\emptyset$, implying that there must exist at least one agent $k\in\mathcal{I}^m(t)\setminus\mathcal{I}^z(t)\neq\emptyset$. Hence, by recalling that $\dot{V}(\boldsymbol{x},t)\in^{a.e.}\dot{\tilde{V}}(\boldsymbol{x},t)$, an upper bound on the generalized time-derivative of V can be obtained as follows

$$\begin{aligned}\dot{V}(\boldsymbol{x},t) &\leq -\frac{1}{|\mathcal{I}^m(t)|}\alpha+\psi_r \\ &\leq -\frac{1}{n}\alpha+\psi_r\leq-\varepsilon<0\end{aligned} \tag{32}$$

where we exploited that $|\psi|\leq\psi_r, \forall\,\psi\in K[\dot{\bar{r}}]$ as stated in Assumption 1 and that the gain α is selected as $\alpha\geq n\psi_r+\varepsilon$.

We now focus on proving (30). To this end, let us observe that, by Lemma 2 when $h(\boldsymbol{x},t)=0$ it holds $\mathcal{I}^m(t)\equiv\mathcal{I}^z(t)$. Hence, we have $y_m(\cdot)=0$ for all $m\in\mathcal{I}^m(t)$, leading to

$$\mathrm{SIGN}^+(y_m(\boldsymbol{x},t)) = [0,1], \text{ and } \mathrm{SIGN}(h(\boldsymbol{x},t)) = [-1,1].$$

Moreover, since for α satisfying (23), it holds $\psi<\psi_r<\alpha/n$ for all $\psi\in K[\dot{\bar{r}}](t)$, we can conclude from (27), that

$$\dot{\tilde{V}}(\boldsymbol{x},t)=\{0\}, \tag{33}$$

from which the inequality (29) follows. By combining (29) and (30), we prove that $V(\boldsymbol{x},t)$ vanishes in finite-time and then remains zero. To conclude our proof, we derive the bound in (24). Since it holds $\dot{V}(x_m(t),t)<-\varepsilon$ until $V(\boldsymbol{x},t)=0$, we can write the following

$$\begin{aligned}V(\boldsymbol{x}(t),t) &= V(\boldsymbol{x}(0),0)+\int_0^t \underbrace{\dot{V}(\boldsymbol{x}(\tau),\tau)}_{<-\varepsilon}\,d\tau \\ &< V(\boldsymbol{x}(0),0)-\varepsilon t,\end{aligned} \tag{34}$$

from which the bound in (24) is obtained by solving

$$V(\boldsymbol{x}(0),0)-\varepsilon T=0.$$

This concludes the proof. □

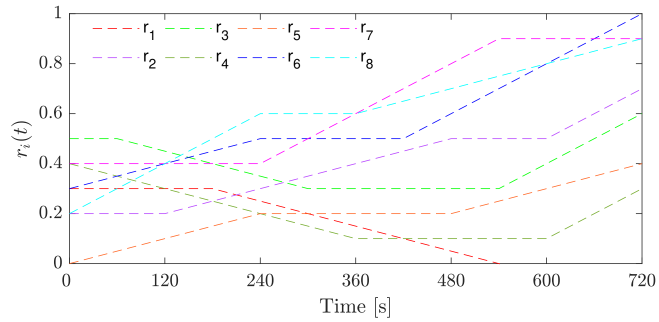

Fig. 1: Temporal evolution of the reference signals $r_i(t)$.

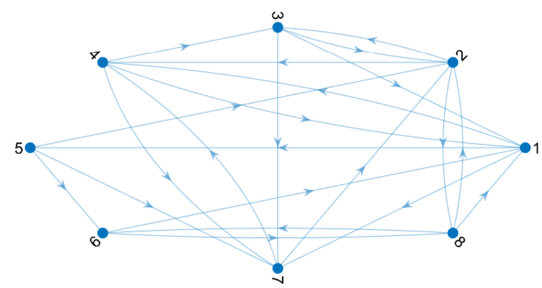

Fig. 2: Directed graph $\mathcal{G}$ encoding the strongly connected communication topology.

IV. Simulation Results

In this section, we validate the proposed distributed protocol and the results on finite time convergence stated in Theorem 3. Motivated by the needs of the H2020 European project CANOPIES, we consider a precision agricultural context in which a network of robots makes use of the maximum supremum information to plan their interventions. More in detail, we consider a team of robots composed of both *farming* and *logistic* units. The farming robots are tasked with agronomic operations, such as harvesting, and fill in their onboard boxes with collected items. Meanwhile, the logistic robots are in charge of promptly unloading the boxes filled by the farming robots, enabling them to resume their agronomic duties without delay. In this setting, each farming robot i senses a box-filling signal $r_i(t)$, ranging from 0, i.e., empty box, to 1, i.e., completely full box. The objective is to enable the robots to collectively track the maximum supremum of the box-filling signals, ensuring that the logistic robots can intervene when required.

We consider a network of $n=8$ agents, each of which perceives a box-filling signal $r_i(t)$, for all $i\in\mathcal{V}$, modeled as a piecewise linear signal, with values in $[0,1]$. The temporal evolution of the reference signals on a time frame of 720 s (12 minutes) is depicted in Figure 1, and the following initial values are considered

$$\boldsymbol{r}(0)=[0.3,\ 0.2,\ 0.5,\ 0.4,\ 0.0,\ 0.3,\ 0.4,\ 0.2]^T \tag{35}$$

In particular, we consider derivative bound $\psi_r=0.1$. The agents' states are initialized as follows

$$\boldsymbol{x}(0)=[0.2,\ 0.1,\ 0.3,\ 0.3,\ 0.0,\ 0.1,\ 0.2,\ 0.1]^T, \tag{36}$$

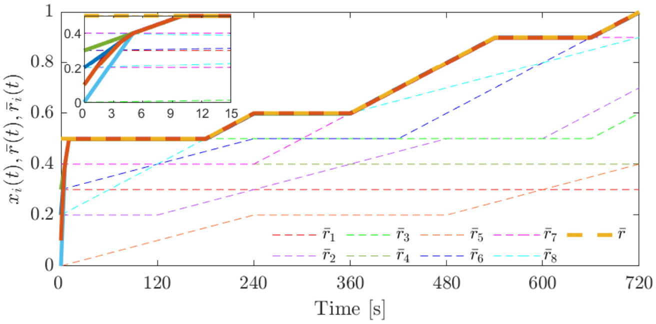

Fig. 3: Temporal evolution of the agents' states $x_i(t)$ (solid lines), the supremum reference signals $\overline{r}_i(t)$ (fine dotted lines) and the maximum supremum $\overline{r}(t)$ (thick dotted line).

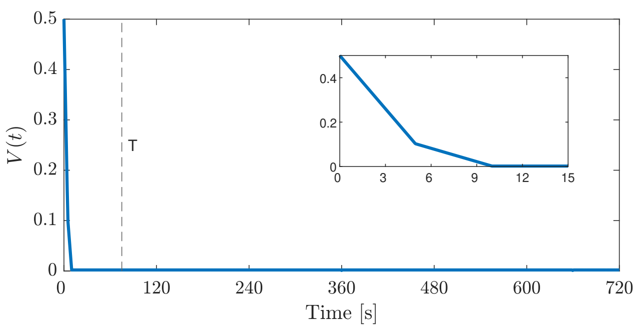

Fig. 4: Temporal evolution of the Lyapunov function $V(\boldsymbol{x}, t)$.

fulfilling the requirements of Theorem 3. We set $\varepsilon = 0.4$ and $\alpha = n\,\psi_r + \varepsilon = 1.2$, by virtue of Theorem 3. The described setting leads, from (24), to the upper bound on convergence time $T = 75$ s. Concerning the communication topology, Figure 2 depicts the directed graph $\mathcal{G}$, which, as per Assumption 2, is strongly connected.

Figure 3 reports the temporal evolution of the agent's states $x_i(t)$ (solid lines), the supremum of the reference signals $\overline{r}_i(t)$ (fine dotted lines) and the maximum supremum $\overline{r}(t)$ (thick orange dotted line). The plot in the top left corner of the figure highlights the results obtained during the first 15 s of the simulation, showing that the agents reach consensus on the supremum signal $\overline{r}(t)$ (thick dotted orange line) at time $t \approx 10$ s. From this time on, tracking of the signal is achieved. Interestingly, the agents do not lose track of the supremum also at instants at which the reference signal representing the supremum changes. An example of this occurs at time $t \approx 360$ s, when the supremum signal switches from $\overline{r}_8(t)$ (turquoise thin dotted line) to $\overline{r}_2(t)$ (purple thin dotted line).

Finally, in Fig. 4 the temporal evolution of the Lyapunov function $V(\boldsymbol{x}, t)$ is depicted. The plot in the top right corner shows the transient of the simulation. We can observe that $V(\boldsymbol{x}, t)$ starts from the value $V(\boldsymbol{x}(0), 0) = |\overline{r}(0) - x_m(0)| = 0.5$, then vanishes in $t \approx 10$ s and remains zero.

V. Conclusions

We proposed a distributed protocol for achieving finite-time tracking of the maximum supremum (or the minimum infimum) of exogenous time-varying signals. Each agent has access to a time-varying reference signal, with bounded derivative, and the communication topology is described by directed graphs. We validated the distributed protocol in a precision agriculture scenario where the exogenous signals model the filling status of boxes filled by farming robots and their monitoring enables the intervention of logistic robots to empty such boxes. As future work, we aim to relax the assumption about the knowledge on the derivatives' bound and define an adaptive distributed protocol.

References

[1] J. Qin, Q. Ma, Y. Shi, and L. Wang, "Recent advances in consensus of multi-agent systems: A brief survey," *IEEE Trans. Ind. Electr.*, vol. 64, no. 6, pp. 4972–4983, 2016.

[2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1520–1533, 2004.

[3] D. P. Spanos, R. Olfati-Saber, and R. M. Murray, "Dynamic consensus on mobile networks," in *IFAC World Congress*, 2005, pp. 1–6.

[4] H. Moradian and S. S. Kia, "On robustness analysis of a dynamic average consensus algorithm to communication delay," *IEEE Trans. Control. Netw. Syst.*, vol. 6, no. 2, pp. 633–641, 2018.

[5] S. S. Kia, B. Van Scoy, J. Cortes, R. A. Freeman, K. M. Lynch, and S. Martinez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms," *IEEE Control Systems Magazine*, vol. 39, no. 3, pp. 40–72, 2019.

[6] J. Cortés, "Distributed algorithms for reaching consensus on general functions," *Automatica*, vol. 44, no. 3, pp. 726 – 737, 2008.

[7] S. Giannini, A. Petitti, D. Di Paola, and A. Rizzo, "Asynchronous max-consensus protocol with time delays: Convergence results and applications," *IEEE Trans. Circuits Syst. I: Regul. Pap.*, vol. 63, no. 2, pp. 256–264, 2016.

[8] Y. Zhang and S. Li, "Second-order min-consensus on switching topology," *Automatica*, vol. 96, pp. 293 – 297, 2018.

[9] ——, "Distributed biased min-consensus with applications to shortest path planning," *IEEE Trans. Automat. Contr.*, vol. 62, no. 10, pp. 5429–5436, 2017.

[10] D. Deplano, M. Franceschelli, and A. Giua, "Dynamic min and max consensus and size estimation of anonymous multi-agent networks," *IEEE Trans. Automat. Contr.*, pp. 1–1, 2021.

[11] M. Lippi, A. Furchì, A. Marino, and A. Gasparri, "Finite-time distributed protocol for tracking the upper (lower) bound for a set of time-varying reference signals," in *Mediterranean Conf. Control and Autom.*, 2022, pp. 908–913.

[12] ——, "An adaptive distributed protocol for finite-time infimum or supremum dynamic consensus," *IEEE Control Syst. Lett.*, vol. 7, pp. 401–406, 2023.

[13] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *Americ. Control Conf.*, 2005, pp. 1859–1864.

[14] A. Sen, S. R. Sahoo, and B. Singh, "Global max-tracking of multiple time-varying signals using a distributed protocol," *IEEE Control Syst. Lett.*, vol. 6, pp. 2888–2893, 2022.

[15] A. F. Filippov, *Differential equations with discontinuous righthand sides: control systems*. Springer Science & Business Media, 2013.

[16] M. Santilli, M. Franceschelli, and A. Gasparri, "Dynamic resilient containment control in multirobot systems," *IEEE Trans. Robot.*, vol. 38, no. 1, pp. 57–70, 2022.

[17] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

[18] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 9, 1994.

[19] M. Franceschelli, A. Giua, and A. Pisano, "Finite-time consensus on the median value with robustness properties," *IEEE Trans. Automat. Contr.*, vol. 62, no. 4, pp. 1652–1667, 2016.

[20] B. Paden and S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. 34, no. 1, 1987.

[21] M. Lippi, A. Furchì, A. Marino, and A. Gasparri, "An adaptive distributed protocol for finite-time infimum or supremum dynamic consensus," *IEEE Control Syst. Lett.*, vol. 7, pp. 401–406, 2022.