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A δ-persistently-exciting formation controller for non-holonomic systems over directed graphs

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***Abstract*— We study a formation control problem of nonholonomic vehicles which consists in making a group of them gather around a given rendezvous set-point and acquire a common orientation. This task may be regarded as part of a more complex maneuver, e.g., requiring the robots to advance in a scouting mission on a path composed of straight lines and occasional turns. The control approach relies on addressing separately the problems of stabilization (on the plane) and of orientation consensus. For the former we use individual controllers involving smooth time-varying terms and for the latter we use distributed consensus control, under the assumption that the robots form a directed graph that contains a spanning tree.**

***Index Terms*— Formation consensus, non-holonomic constraints, Lyapunov stability**

I. Introduction

Formation control of nonholonomic vehicles is a well-studied problem in multi-disciplinary literature, but one which has different interpretations and motivations. One pertains to *formation-consensus* in which case it is required for the networked-interconnected vehicles to reach a consensual, non-predefined posture. This is inherently a *set-point* stabilization problem (with the well-known technical difficulties that this implies for nonholonomic systems [1], [2]) that may involve full consensus in position and orientation [3]–[6] or partial, either in orientation [7] or in position [1], [8], [9]. The term formation control, however, may also refer to the case in which a group of robots are required to follow a leader, real or virtual, while keeping a formation [10]–[16]. We may generically refer to this scenario as that of leader-follower formation-*tracking* control.

In the case of formation-consensus the equilibrium is not pre-defined, but it depends on the topology of the network, as well as on the initial postures and the nonlinearities of the systems. In leader-follower-based formation-control problems the vehicles are meant to track a leader, so both problems are fundamentally different, are motivated by distinct practical applications and, from a control viewpoint, demand very different approaches. A well-known fact, *e.g.,* is that smooth controllers must rely on persistency of excitation [17] both in stabilization and tracking control tasks [2]. However, the two problems are not solvable simultaneously via the same controller [2].

Now, besides the difficulties imposed by the nonholonomic constraints, controlling *networked* autonomous vehicles imposes challenges related to the network's topology. Even though the literature on consensus is very rich and covers a wide range of graph topologies, this is so mostly for linear systems. Alternatively, in the context of formation and consensus of multiple vehicles, most works are restricted to undirected-graph topologies—see *e.g.,* [4], [6], [7], [9], [11], [14], [15], [18]–[20]. Articles covering directed-graph topologies are relatively scarce—see *e.g.,* [3], [21] for formation consensus and formation-tracking control, respectively. However, in both cases knowledge of the Laplacian is needed. Leader-follower-based schemes are often restricted to graphs that consist in (do not simply contain) a spanning tree [5], [12], [22], [23].

In this letter we study a problem of formation control that consists in all the robots converging to a static formation with a pre-imposed reference position on the Cartesian plane and acquiring a non-predefined orientation. This is a task that involves both stabilization and orientation consensus, so, as in [1], [7]–[9] we solve a problem of partial consensus. Our main contribution resides in that we consider that the systems are interconnected over a generic directed graph, *i.e.,* that it *contains* a spanning tree. Hence, from a network-topology viewpoint, the leader-follower scenarii are covered. Relative to [3], [21] we stress that our controller is fully distributed. Formation-tracking control is out of scope, but the problem that we address may be regarded as part of a more complex maneuver. For instance, as a byproduct of our main statement, we show that our controller applies also in simple tasks that require a group of vehicles to advance in formation; not following a target or a leader, as in [10], [13], [21], but advancing with constant forward speed, along paths composed of straight-lines—cf. [12] and predefined turns.

II. Problem formulation

Let us consider a group of N nonholonomic vehicles modeled by the kinematic equations

$$\dot{x}_i = \cos(\theta_i)v_i \qquad (1a)$$
$$\dot{y}_i = \sin(\theta_i)v_i \qquad (1b)$$
$$\dot{\theta}_i = \omega_i, \qquad i \le N, \qquad (1c)$$

where $(x_i, y_i) := p_i$ denote the Cartesian coordinates of the center of mass of the ith robot on the plane, θ_i denotes

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its orientation relative to the abscissae; these variables are measured and are to be controlled. The forward velocity v_i and the angular velocity ω_i constitute the control inputs.

It is assumed that each system has access to its own coordinates p_i and θ_i, as well as the orientation of neighbor vehicles, θ_j, where $j \in \mathcal{N}_i$ and $\mathcal{N}_i$ is the set of indexes corresponding to the neighbors of the ith vehicle. In that regard, we assume that through their interconnections the systems form a directed graph that contains a spanning tree. We stress that such condition is necessary for consensus and, therefore, cannot be relaxed. In *particular* it is satisfied in leader-follower configurations [5], [12], [22] in which the graph *consists* in a spanning tree and cycles are excluded. For these systems, we address the following problem.

Formation-stabilization with orientation consensus: Let $p^* := [p_x^* \ p_y^*]^\top$ be a given reference constant set-point on the plane. Let $\delta_i := [\delta_{xi} \ \delta_{yi}]^\top$ be a given offset relative to p^*, so that, defining $\bar{p}_i := p_i - \delta_i - p^*$, we have that $\bar{p}_i = 0$ for all $i \leq N$ means that all the vehicles are positioned in a geometric formation with center at p^*. Then, the formation-stabilization-with-common-orientation control problem consists in designing decentralized control laws such that

$$\lim_{t\to\infty} |\bar{p}_i(t)| = 0, \quad \lim_{t\to\infty} \theta_i(t) = \theta_c, \tag{2a}$$

$$\lim_{t\to\infty} v_i(t) = 0, \quad \lim_{t\to\infty} \omega_i(t) = 0, \quad \forall i \in \{1, \cdots, N\}, \tag{2b}$$

where θ_c is a non-given constant that depends on the system's dynamics, initial conditions, and the networkś topology. This is a problem of partial consensus control—cf. [1], [7]–[9]; from a practical viewpoint, it is pertinent, *e.g.,* as the first part of a more complex maneuver [24] which may consist in making the robots gather and advance in formation, for instance in a scouting mission—see Section III-C.

III. Control approach and main results

A. Rationale

Formation control with consensual orientation, as stated above, may be solved by designing a formation-stabilization and a consensus controller independently. However, this presents two difficulties: the first stems from the fact that nonholonomic systems are known not to be stabilizable via smooth time-invariant feedback; the second is that of analyzing consensus over generic directed graph-networks for highly nonlinear systems, such as nonholonomic integrators. To circumvent the first difficulty we use the approach of so-called δ-persistently-exciting (δ-PE) controllers [12], which are designed to inject an external time-varying perturbation that is persistently exciting [17], as long as a certain function of the state is away from the origin. To achieve the consensus goal we use the control law

$$\omega_i = -k_w \sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i - \theta_j), \quad k_w > 0, \tag{3}$$

where $a_{ij} = 1$ if an interconnection from the node j to the node i exists and $a_{ij} = 0$ if otherwise. Note that since the interconnections are directed, in general, $a_{ij} \neq a_{ji}$. Disregarding the stabilization task, orientation consensus is achieved provided that the network's graph contains a directed spanning tree [25]. That is, (3) guarantees that $\theta_i \to \theta_j \to \theta_c$ for all $i, j \leq N$, where θ_c is constant.

Now, in relation to the stabilization objective we start by introducing the error coordinates

$$z_i := \begin{bmatrix} \phi(\theta_i)^\perp \\ \phi(\theta_i)^\top \end{bmatrix} \bar{p}_i, \qquad \begin{array}{l} \phi(\theta_i) = \begin{bmatrix}\cos(\theta_i) & \sin(\theta_i)\end{bmatrix}^\top \\ \phi(\theta_i)^\perp = \begin{bmatrix}-\sin(\theta_i) & \cos(\theta_i)\end{bmatrix}. \end{array} \tag{4}$$

Note that $\phi(\theta_i)^\perp \phi(\theta_i) \equiv 0$, so, after (4) and (1), we have

$$\dot{z}_i = \begin{bmatrix} \dot{z}_{1i} \\ \dot{z}_{2i} \end{bmatrix} = \begin{bmatrix} -z_{2i}\omega_i \\ z_{1i}\omega_i + v_i \end{bmatrix} \tag{5}$$

and since the matrix in (4) is globally invertible, we have that $z_i = 0$ if and only if $\bar{p}_i = 0$, so the formation-stabilization objective boils down to ensuring that $z_i \to 0$ holds for (5). To that end, we remark that Equations (5) may be assimilated to those of a stable (oscillatory) system with input v_i. Hence, for any $\omega_i \neq 0$, the control law

$$v_i = -k_{2i} z_{2i} + (k_{1i} - 1)\omega_i z_{1i}, \qquad k_{1i},\ k_{2i} > 0, \tag{6}$$

makes $z_i \to 0$. To see this more clearly note that the closed-loop system takes the form

$$\begin{bmatrix} \dot{z}_{2i} \\ \dot{z}_{1i} \end{bmatrix} = \begin{bmatrix} -k_{2i} & k_{1i}\omega_i \\ -\omega_i & 0 \end{bmatrix} \begin{bmatrix} z_{2i} \\ z_{1i} \end{bmatrix}, \tag{7}$$

which has a structure familiar in adaptive control—cf. [12]. Hence, it may be showed that if $k_{2i} > 0$, $z_{2i} \to 0$ (see the proof of our main result below). Moreover, if ω_i depended only on t, it would be sufficient to make ω_i persistently exciting (PE), *i.e.*, to ensure that there exist μ and $T > 0$ such that

$$\int_t^{t+T} |\omega_i(s)| ds \geq \mu \quad \forall\, t \geq 0,$$

to ensure that $z_{1i} \to 0$. Making ω_i be PE, however, means that θ_i cannot converge to a constant value, which is in clear conflict with the orientation-consensus objective. In summary, persistency of excitation is needed to attain the formation control objectives, while it is required that ω_i converge to zero to achieve orientation consensus. To overcome this conundrum, we endow the angular-motion consensus control law (3) with an oscillatory perturbation that persists as long as $|z_{1i}| \neq 0$. That is, with a δ-persistently exciting function—cf. [12]. Hence, we redesign the consensus control law (3) to take the form

$$\omega_i = -k_w \sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i - \theta_j) + f_i(t)\beta_i(z_i), \quad k_w > 0. \tag{8}$$

In the previous expression, $f_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is designed to be twice continuously differentiable, bounded with bounded derivatives, and $\dot{f}_i$ is persistently exciting, that is, there exist T, $\mu > 0$ such that

$$\int_t^{t+T} |\dot{f}_i(s)| ds \geq \mu \quad \forall\, t \geq 0. \tag{9}$$

In addition, the function $\beta_i : \mathbb{R}^2 \to \mathbb{R}$ is chosen to be twice continuously differentiable and to satisfy $\beta_i(z_i) = 0$ if and only if $z_i = 0$. Loosely speaking, the purpose of the term $f_i(t)\beta_i(z_i)$ is to prevent ω_i from vanishing to zero (achieving consensus) as long as $z_i \neq 0$ (the stabilization task has not been achieved). Indeed, provided that $z_i(t)$ is bounded, $f_i(t)\beta_i(z_i)$ acts as a bounded disturbance that vanishes only if both objectives, consensus and stabilization are attained simultaneously. This rationale is formalized in following statement given next.

B. Formation stabilization

Proposition 1 (Formation stabilization): Consider a network of autonomous vehicles modeled as in (1), communicating over a directed-graph containing a directed spanning tree, in closed loop with the controller (6)-(8), where for each $i \leq N$, $f_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is twice continuously differentiable, bounded with bounded derivatives, and satisfies (9), whereas $\beta_i : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and satisfies $\beta_i(z_i) = 0$ if and only if $z_i = 0$. Then, the formation-stabilization-with-common-orientation goals in (2) are attained. □

Proof: Replacing the control laws (6)-(8) in the kinematics equations (5)-(1c) we obtain the closed-loop system

$$\dot{z}_{1i} = -\omega_i z_{2i} \tag{10a}$$

$$\dot{z}_{2i} = k_{1i}\omega_i z_{1i} - k_{2i} z_{2i} \tag{10b}$$

$$\dot{\theta}_i = -k_w \sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i - \theta_j) + f_i(t)\beta_i(z_i). \tag{10c}$$

To establish the statement of the proposition we start by showing that $z_i \in \mathcal{L}_\infty$, where $z_i := [z_{1i}\ z_{2i}]^\top$. To that end, we use the positive-definite, decrescent Lyapunov function

$$V_z(z) = \frac{1}{2}\sum_{i=1}^{N}[k_{1i}z_{1i}^2 + z_{2i}^2]. \tag{11}$$

Its total derivative along the trajectories of (10a)-(10b) yields

$$\dot{V}_z(z(t)) = -k_{2i}\sum_{i=1}^{N} z_{2i}(t)^2 \leq 0. \tag{12}$$

Integrating on both sides of $\dot{V}_z(z(t)) \leq 0$, from $t_o \geq 0$ to t, we obtain $[k_{1i}z_{1i}(t)^2 + z_{2i}(t)^2] \leq [k_{1i}z_{1i}(t_o)^2 + z_{2i}(t_o)^2]$ which implies that

$$|z_i(t)|^2 \leq \frac{\max\{k_{1i},\,1\}}{\min\{k_{1i},\,1\}}|z_i(t_o)|^2. \tag{13}$$

That is, z_i is uniformly globally bounded.

Strictly speaking, the latter holds only on the interval of existence of $t \mapsto \omega(t)$. Say, for a certain $\infty > t_f > t_o$, any $t_o \geq 0$ and for all $t \in [t_o, t_o + t_f)$. In the sequel we show that $t_f = +\infty$, that $\omega := [\omega_1\ \cdots\ \omega_N]^\top$ is also uniformly bounded and, therefore, the arguments above hold for all $t \geq t_o \geq 0$.

To that end, we start by rewriting (8) in multi-variable compact form,

$$\omega = -L\theta + \alpha(t, z), \tag{14}$$

where $\alpha(t, z) := [\, f_1(t)\beta_1(z_1)\ \cdots\ f_N(t)\beta_N(z_N)\,]^\top$, $\theta := [\,\theta_1\ \cdots\ \theta_N\,]^\top$, and

$$L := [\ell_{ij}] \in \mathbb{R}^{N\times N}, \quad \ell_{ij} := \begin{cases} -a_{ij} & \forall\, i \neq j \\ \sum_{k\in\mathcal{N}_i} a_{ik}, & \forall\, i = j \end{cases} \tag{15}$$

is the Laplacian matrix. Then, following [26], we introduce the synchronization errors

$$e_\theta = [I_N - \mathbf{1}_N v_\ell^\top]\theta, \tag{16}$$

which correspond to the vector $e_\theta := [e_{\theta 1} \cdots\ e_{\theta N}]^\top$, where $e_{\theta i} := \theta_i - v_\ell^\top\theta$, v_ℓ is the left eigenvector associated to the unique zero eigenvalue of L, and $v_\ell^\top\theta$ is a weighted average of θ. That is, consensus is reached if $\theta_i \to v_\ell^\top\theta$ for all $i \leq N$ or, equivalently, $e_\theta \to 0$.

Remark 1: The errors e_θ satisfy

$$\dot{e}_\theta = -Le_\theta + [I_N - \mathbf{1}_N v_\ell^\top]\alpha(t, z). \tag{17}$$

To see this, we note that from (16),

$$\dot{e}_\theta = -L[I_N - \mathbf{1}_N v_\ell^\top]\theta + [I_N - \mathbf{1}_N v_\ell^\top]\alpha(t, z),$$

so (17) follows observing that $L\mathbf{1}_N = 0$. •

Next, let us assume that the graph contains a directed spanning tree then the statement of [26, Lemma 2] generates $P = P^\top \in \mathbb{R}^{N\times N}$ on solving, for any matrix $Q_L \in \mathbb{R}^{N\times N}, Q_L = Q_L^\top > 0$ and for $\sigma > 0$, the equation

$$PL + L^\top P = Q_L - \sigma[P\mathbf{1}_N v_\ell^\top + v_\ell \mathbf{1}_N^\top P], \tag{18}$$

where v_ℓ is the left eigenvector associated to the single zero eigenvalue of L. Then, consider the Lyapunov function $V_{e_\theta} = e_\theta^\top P e_\theta$. Its total derivative along the trajectories of (17) yields

$$\dot{V}_{e_\theta} = -e_\theta^\top[PL + L^\top P]e_\theta + 2e_\theta^\top P[I_N - \mathbf{1}_N v_\ell^\top]\alpha(t, z),$$

so, using (18), we obtain

$$\begin{aligned}\dot{V}_{e_\theta} = &- e_\theta^\top\left[Q_L - \sigma[P\mathbf{1}_N v_\ell^\top + v_\ell\mathbf{1}_N^\top P]\right]e_\theta \\ &+ 2e_\theta^\top P[I_N - \mathbf{1}_N v_\ell^\top]\alpha(t, z).\end{aligned}$$

However, since $v_\ell^\top\mathbf{1}_N = 1$, the second term on the right-hand side of the equation above equals to zero. Indeed, after (16), $P\mathbf{1}_N v_\ell^\top e_\theta = P\mathbf{1}_N v_\ell^\top[I_N - \mathbf{1}_N v_\ell^\top]\theta = P[\mathbf{1}_N v_\ell^\top - \mathbf{1}_N v_\ell^\top]\theta = 0_N$. Therefore,

$$\dot{V}_{e_\theta} = -e_\theta^\top Q_L e_\theta + 2e_\theta^\top P[I_N - \mathbf{1}_N v_\ell^\top]\alpha(t, z).$$

Now, let $\Delta := 2|P[I_N - \mathbf{1}_N v_\ell^\top]|$ where $|\cdot|$ denotes here the induced matrix norm. For any $t_f > 0$, let

$$\|\alpha\|_{t_f} := \sup_{t\in[t_o, t_o+t_f)} |\alpha(t, z(t))|.$$

Then, defining $q_m := \lambda_{\min}(Q_L)$, we have

$$\dot{V}_{e_\theta}(e_\theta(t)) \leq -q_m|e_\theta(t)|^2 + \Delta|e_\theta(t)|\|\alpha\|_{t_f} \tag{19}$$

for all $t \in [t_o, t_o + t_f)$. We show by contradiction that $t_f = +\infty$, so all of the above holds for all $t \geq t_o$.

Assume that $|\omega(t)| \to \infty$ as $t \to t_f < \infty$. Then, since $\theta(t) = \int_{t_o}^{t}\omega(t)dt$ we have $|\theta(t)| \to \infty$ as $t \to t_f$. On the other hand, $V_{e_\theta}(e_\theta(t)) := \theta(t)^\top[I_N - \mathbf{1}_N v_\ell^\top]^\top[I_N - \mathbf{1}_N v_\ell^\top]\theta(t)$, so $V_{e_\theta}(e_\theta(t)) \to \infty$ as $t \to t_f$. Next, defining $\nu(s) := V_{e_\theta}(e_\theta(s))$, after (19), we have

$$\dot{\nu}(t) \leq \frac{\Delta^2}{2q_m}\|\alpha\|_{t_f}^2. \tag{20}$$

Integrating on both sides of (20), from t_o to t_f, we obtain

$$\lim_{t\to t_f} \nu(t) \leq \frac{\Delta^2}{2q_m}\|\alpha\|_{t_f}^2[t_f - t_o] + \nu(t_o).$$

By assumption $t_f < \infty$, so the right-hand side of the inequality above is finite, whereas

$$\lim_{t\to t_f} \nu(t) = \lim_{t\to t_f} V_{e_\theta}(e_\theta(t)) = +\infty,$$

which is a contradiction. We conclude that $t_f = +\infty$ so (12) holds along trajectories for all $t \geq t_o$ and $z_i(t)$ is uniformly globally bounded. Also, in view of the latter, the continuity of $\alpha(t,\cdot)$ and the uniform boundedness of $\alpha(\cdot, z)$, $t \mapsto \alpha(t, z(t))$ is uniformly bounded, that is, $\|\alpha\|_\infty^2 < \infty$. From this and integrating on both sides of (19) along the trajectories, from t_o to ∞, we obtain that e_θ is uniformly globally bounded.

We show next that ω is also uniformly globally bounded. For this, we note that the Laplacian matrix L admits the decomposition $L = UMU^{-1}$ where U is a matrix whose columns are the right eigenvectors of L, *i.e.*, $U = [\mathbf{1}_N \ U_1]$, $U^{-1} = [v_\ell \ U_1^{\dagger\top}]^\top$, and M contains the Jordan blocks corresponding to the eigenvalues of L. The first eigenvalue of M being zero and simple, we also have $L = U_1\bar{M}U_1^\dagger$, where $\bar{M}$ contains all the Jordan blocks of L corresponding to the non-zero eigenvalues. It follows that $L\theta = U_1\bar{M}U_1^\dagger\theta$, so $|L\theta| \leq \lambda_{\max}(L)|U_1U_1^\dagger\theta| = \lambda_{\max}(L)|[I_N - \mathbf{1}_N v_\ell^\top]\theta| = \lambda_{\max}(L)|e_\theta|$. We conclude that $|L\theta(t)|$ is uniformly globally bounded. From this, and the boundedness of f_i and $\alpha(t, z(t))$, it follows that ω in (14) is also uniformly globally bounded, and so is $\dot{\theta} = \omega$. Consequently, $\dot{z}_i$ is uniformly globally bounded as it is a function of z_i and ω_i—see Eq. (7).

In addition, for further development, we remark that $\dot{\omega}$ is also globally uniformly bounded, since so are all the terms on the right hand side of

$$\dot{\omega}_i = -k_w \sum_{j\in\mathcal{N}_i} a_{ij}(\omega_i - \omega_j) + f_i(t)\frac{\partial\beta_i(z_i)}{\partial z_i}\dot{z}_i + \dot{f}_i(t)\beta_i(z_i). \tag{21}$$

Next, we establish the convergence of z and $L\theta$. This part of the proof relies on a recursive application of Barbălat's Lemma.

First, we integrate on both sides of (12) from t_o to ∞. Since z is globally uniformly bounded we obtain that $z_2 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Since, moreover, $\dot{z}_2 \in \mathcal{L}_\infty$ we obtain, by [17, Lemma 3.2.5] that $z_2 \to 0$ asymptotically. To establish the convergence of z_1 we proceed by establishing, first, the convergence of $\omega_i z_{1i}$ for any $i \leq N$. This follows from the fact that both terms on the right hand side of $k_{1i}\omega_i z_{1i} = k_{2i}z_{2i} - \dot{z}_{2i}$—cf. (10b), converge to zero. That $z_{2i} \to 0$ was established above, while the convergence of $\dot{z}_{2i}$ follows from Barbălat's Lemma. Indeed, on one hand, $\ddot{z}_{2i}$ is uniformly globally bounded since so are all terms on the right hand side of

$$\begin{aligned}\ddot{z}_{2i} &= k_{1i}\dot{\omega}_i z_{1i} + k_{1i}\omega_i\dot{z}_{1i} - k_{2i}\dot{z}_{2i} \\ &= k_{1i}\Big[-k_w\sum_{j\in\mathcal{N}_i} a_{ij}(\omega_i - \omega_j) + f_i(t)\frac{\partial\beta_i(z_i)}{\partial z_i}\dot{z}_i\Big]z_{1i} \\ &\quad + k_{1i}\dot{f}_i(t)\beta_i(z_i)z_{1i} - k_{1i}\omega_i^2 z_{2i} - k_{1i}k_{2i}\omega_i z_{1i} + k_{2i}^2 z_{2i}.\end{aligned} \tag{22}$$

On the other hand,

$$\lim_{t\to\infty}\int_{t_o}^t \dot{z}_{2i}(\tau)d\tau = \lim_{t\to\infty} z_{2i}(t) - z_{2i}(t_o) = -z_{2i}(t_o).$$

Now, since $|\omega_i z_{1i}| \to 0$ and both ω_i and z_{1i} are uniformly globally bounded, either $z_{1i} \to 0$ or $\omega_i \to 0$. In the first case, the proof is completed. Hence, let us assume that $\omega_i \to 0$ and turn our attention to the term $k_{1i}\dot{f}_i(t)\beta_i(z_i)z_{1i}$ on the right-hand side of (22). By assumption $\beta_i(z_i) = 0$ if and only if $z_i = 0$ and $\dot{f}_i$ is persistently exciting. Therefore, since $t \mapsto \beta_i(z_i(t))$ is uniformly globally bounded we see that $z_{1i} \to 0$ if and only if so does

$$\begin{aligned}k_{1i}\dot{f}_i(t)\beta_i(z_i)z_{1i} &= \ddot{z}_{2i} + k_{1i}\omega_i^2 z_{2i} + k_{1i}k_{2i}\omega_i z_{1i} - k_{2i}^2 z_{2i} \\ &\quad -k_{1i}\Big[-k_w\sum_{j\in\mathcal{N}_i} a_{ij}(\omega_i - \omega_j) + f_i(t)\frac{\partial\beta_i(z_i)}{\partial z_i}\dot{z}_i\Big]z_{1i}.\end{aligned}$$

We proceed to establish the convergence of all the terms on the right-hand side of the previous expression, individually. As a matter of fact, this has already been done for the second, third, and fourth terms, since both z_{2i} and $\omega_i z_{1i}$ converge to zero. Furthermore, the convergence of the last term on the right-hand side follows from that of $\dot{z}_i$ and the uniform boundedness of $f_i(\cdot)$ and $\beta_i(z_i(\cdot))$. In addition, by assumption $\omega_i \to 0$, so it is only left to show that $\ddot{z}_2 \to 0$. To that end, we use Barbălat's Lemma once more. First, we observe that, for any $t_o \geq 0$

$$\lim_{t\to\infty}\int_{t_o}^t \ddot{z}_{2i} = \lim_{t\to\infty}\dot{z}_{2i}(t) - \dot{z}_{2i}(t_o) = -\dot{z}_{2i}(t_o). \tag{23}$$

On the other hand, we observe that $t \mapsto \ddot{z}_{2i}(t)$ is uniformly continuous since $z_{2i}^{(3)}$ is uniformly globally bounded. Indeed, so are all the terms bounded on the right-hand side of $z_{2i}^{(3)}$. This completes the proof. ■

C. Application to formation scouting

After Proposition 1, the controller (6)-(8) guarantees that a group of vehicles meet at a specified rendezvous set-point and acquire a consensual orientation. This task may be considered as the starting of a gather-and-scout maneuver in which it is also required to advance in formation along a path composed of straight lines and occasional turns. Besides the practical motivations of such a mission, it is worth noticing that a formation cannot be rigidly maintained over arbitrary paths, but it can if they mainly consist in straight lines [27], [28].

This task may be simply achieved by introducing *individual* reference models for each vehicle, given by $\dot{p}_i^* := \phi(\theta_i)v^*$ with $p_i^*(0) = p_j^*(0)$ for all $i,\, j \leq N$, and constant speed v^*. Then, we redefine the control law (6) as

$$v_i = -k_{2i}z_{2i} + (k_{1i}-1)\omega_i z_{1i} + v^*, \qquad k_{1i},\ k_{2i} > 0. \quad (24)$$

For the vehicles to agree on a consensual orientation, we add to the network an orientation leader with dynamics $\dot{\theta}^* = \omega^*$, where ω^* is set to zero (to move on a straight line) or to a constant (to take a turn). More precisely, for ν turns in a path, let $M := 2\nu + 1$ and, for each $k \in \{1, 2, \ldots, M+1\}$, let $\{t_k\}$ be an increasing finite sequence of times, such that $\omega^*(t) = 0$ for all $t \in [t_k, t_{k+1})$ with k odd, and $\omega^*(t) \neq 0$ for all $t \in [t_k, t_{k+1})$, with k even. In addition, let $(t_{j+1} - t_j) \gg (t_{j+2} - t_{j+1})$ for any odd number $j \in \{1, 2, \ldots, M-2\}$, and $t_{M+1} = +\infty$, so $\omega^*(t) = 0$ for all $t \geq t_M$.

Since an orientation-leader node is added, we also modify the orientation-consensus controller so that, on the consensus manifold $\{\theta_i = \theta^*\}$, the individual reference models generate a consensual trajectory, solution of $\dot{p}^* := \phi(\theta^*)v^*$, that is, $p_i^*(t) = p_j^*(t)$ for all $t \geq 0$ and $i,\, j \leq N$. As a result, they advance in the same direction while keeping the formation. The orientation-consensus controller, which is given in the following statement, is decentralized and modifies the generic directed graph into one that typically appears in leader-follower configurations—cf. [21].

Proposition 2 (formation scouting): Let v^* be a given constant velocity and, for ω^* as defined above, let $\dot{\theta}^* = \omega^*$ and $\dot{p}_i^* := \phi(\theta_i)v^*$. In addition, let

$$\omega_i = -k_w\left[\sum_{j\in\mathcal{N}_i} a_{ij}(\theta_i - \theta_j) + b_i(\theta_i - \theta^*)\right] + f_i(t)\beta_i(z_i), \quad (25)$$

where β_i and f_i are as in Proposition 1, $b_i > 0$ for at least one node from which any other node can be reached while $b_i \geq 0$ for any other node, and a_{ij} are such that the Laplacian matrix L defined in (15) contains a unique zero eigenvalue and all others have strictly positive real parts (the graph contains a spanning tree). Then, for the closed-loop system, $\lim_{t\to\infty} |\bar{p}_i(t)| = 0$ and $\lim_{t\to\infty} \theta_i(t) = \lim_{t\to\infty} \theta^*(t)$. □

Proof: The closed-loop equations correspond to Eqs. (10a)-(10b) and, in terms of the synchronization errors, $e_\theta = \theta - \mathbf{1}_N\theta^*$, $\dot{e}_\theta = -k_w[L+B]e_\theta + \alpha(t,z) - \mathbf{1}_N\omega^*$, where $B = \text{diag}[b_i]$. Under the above-stated assumptions, $[L+B]$ is a non-singular M-matrix, so for any $Q = Q^\top > 0$, there exists a matrix $P = P^\top > 0$ that solves the Lyapunov equation $-P[L+B] - [L+B]^\top P = -Q$ (for details, see Theorem 4.25 in [29]). By construction, $\omega^*(t) = 0$ for all $t \geq t_M$, so the proof follows as in that of Proposition 1. ■

IV. Simulation Results

We simulated two experiments involving a group of eight vehicles interconnected as in the directed graph showed in Fig. 1, which contains a directed spanning tree. The first experiment consists in making all robots meeting around the origin, so $p^* = (0,0)^\top$, at the vertices of an octagon and acquiring a common non pre-imposed orientation θ_c, so we

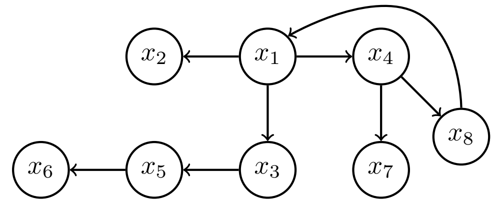

Fig. 1. Connection digraph for the eight communicating agents

use the controller (6)-(8). The initial positions on the plane are depicted in Fig. 2 with a '$\circ$' while the initial orientations are set to $\theta(0) := [\pi/2\ 0\ \pi/8\ -\pi/8\ \pi/3\ \pi/5\ -\pi/7\ \pi/6]$. The control gains were set to the same values for all robots, in a way to produce a little oscillatory transient; we used $k_{1i} = 3.5$, $k_{2i} = 30$, and $k_w = 1$. In addition, we used $f_i(t) = \cos(15t) - \sin(35t)$ and $\beta_i(z_i) = 22z_{1i}$.

The paths followed by the robots on the plane are also showed in Fig. 2 below, and their final consensual orientation ($\theta_c \approx 35\,[\text{deg}]$) is illustrated with pointing arrows.

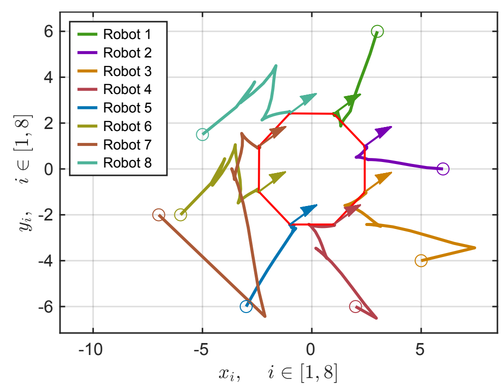

Fig. 2. Vehicles on the phase plane converging to a rendezvous point

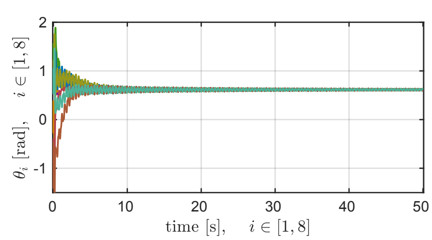

Fig. 3. Systems' orientations achieving consensus

In Fig. 3 we show the variation of orientations. The consensus errors in Cartesian positions rapidly become very small in norm while the orientations take a longer to settle due to the persistent, but vanishing perturbation $f_i(t)\beta_i(z_i)$ in (8), needed to achieve set-point stabilization.

In a second simulated experiment, the robots gather in an octagonal formation and advance in formation along the consensual direction, only that in this case a virtual orientation reference $\dot{\theta}^* = \omega^*$ is accessed only by node 1 so $b_1 = 1$ and $b_k = 0, k \in \{2, \cdots, 8\}$. The squad is required to advance at a constant speed $v^* = 0.05$[m/s] along straight paths and turning at an angular velocity of $\omega^* = 0.1$[rad/s] for pre-planned short periods of time. For this, we used the controller (24)-(25). The paths followed by the vehicles are showed in Fig. 4.

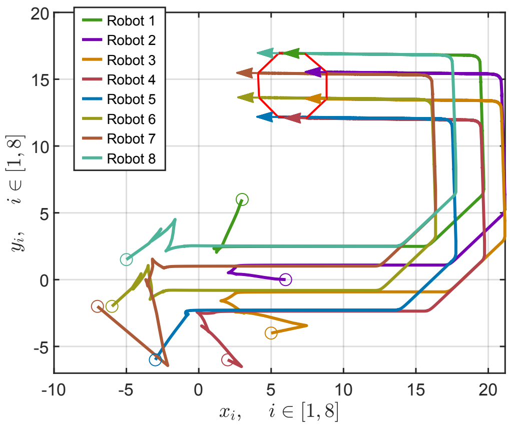

Fig. 4. Systems' paths on the phase plane converging to a formation and moving with constant speed

V. Conclusions

The decentralized controller for formation-stabilization and orientation consensus in Section III-B addresses, individually for each robot, the problem of formation stabilization and, in a decentralized multi-agent fashion, orientation-consensus control. With such controller, moreover, a simple task of formation-scouting may be achieved. Our contributions, however, are limited as they do not cover the case of full-consensus formation control, *i.e.,* including position-consensus, nor forward velocity consensus. Specifically for the latter, the use of second-order models appears imperative. Designing a fully distributed controller to address these problems over a directed graph containing a spanning-tree for nonholonomic systems is to the best of our knowledge an open problem. The Lyapunov-based analysis that we provide may serve as basis to extend our results to these scenarii and, in turn analyze robustness with respect to input disturbances.

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