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Mixed Gain/Phase Robustness Criterion for Structured Perturbations with an application to Power System Stability

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***Abstract*— A novel conception of phase for linear time-invariant multivariable systems was recently introduced. It enables robustness of such systems to be determined in terms of a phase-bounded set of perturbations via a so-called small phase theorem, in analogy to the well-known small gain theorem. However, it requires the system's frequency response to satisfy the relatively strong condition known as "sectoriality", which not all practical systems have. This paper aims to show that if the perturbation is assumed to have a block diagonal structure, a matrix-valued multiplier function can be calculated that can enable phase-based robustness margins to be defined in some cases when the original system is not sectorial. A real-world power systems example is presented to show how the small phase criterion using a multiplier can significantly reduce the conservatism of the small gain theorem, providing computationally straightforward methods to inform further nonlinear stability analysis of power systems.**

I. Introduction

The small gain theorem is a cornerstone result in the field of robust control. It generalizes the classical gain margin of LTI SISO systems to nonlinear and MIMO systems. Given a known system G, it provides a generalized "gain margin" that bounds a set of possible uncertainties or perturbations Δ producing a stable closed loop system, as illustrated in Fig. 1. In SISO analysis, the phase is as important as gain, but phase-based results for MIMO systems are relatively sparse.

Addressing this shortcoming, a novel conception of phase for LTI MIMO systems was introduced by [1], [2]. Any theory of phase for LTI MIMO systems rests upon a definition of matrix phase. Much earlier, [3] defined the *principal phases*, in terms of the unitary part of the polar decomposition of a matrix. In contrast, the recent paper [4] defined matrix phase in terms of the numerical range and the related sectorial decomposition. Using this definition of matrix phase, [1], [2] present a *small phase theorem* that generalizes the concept of SISO phase margin to MIMO systems while also generalizing the passivity theorem and the associated concepts of passivity and positive realness for LTI systems. Several extensions of this concept have already been made. [5] presents a version of the result for discrete-time systems, while [6] and [7] extend the concept of phase to nonlinear and linear time-periodic systems, respectively. [8] shows how the small gain and small phase theorems can be integrated on a frequency-wise basis to produce stability tests akin to classical Bode analysis for SISO systems.

An area of active research for applications of robust control is the small-signal stability of grid-connected power converters. Many power converters employ so-called "grid-following" control. For this type of control, the stability of the grid-connected converter can be analyzed as a robustness problem with the impedance of the grid connection playing the role of the uncertainty [9].

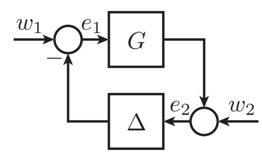

Fig. 1: Negative feedback interconnection of two systems.

Robust control techniques are a natural fit for this problem. For example, [10] and [11] apply $\mathcal{H}_\infty$ synthesis to produce controllers stabilizing grid-connected power electronics over wide ranges of operating conditions. [12] provides a fixed-structure $\mathcal{H}_\infty$ design procedure for a *network* of power converters with a decentralized stability criterion, but makes strict assumptions about the dynamics of the network itself. [13] compares the efficacy of several small-gain based stability criteria, which exhibit significant conservatism. In general, robust control results for grid-connected power converters are specific to a particular system structure, or are highly conservative.

This paper introduces a mixed gain/phase stability criterion. The phase portion of the criterion requires the plant frequency response satisfy a condition known as "sectoriality". We show that in the case of a structured perturbation, the criterion can be augmented with a multiplier function, allowing phase bounds to be calculated in some cases where the plant model is not sectorial. This might be thought of as a first step towards a phase analogue of the structured singular value concept for the small gain theorem [14]. [15] addresses conditions for the existence of related integral quadratic constraint (IQC) multipliers for robustness against gain-type perturbations, but does not address general block-wise perturbations, nor provide robustness bounds on the phase of a perturbation system. We then apply the mixed gain/phase criterion to estimate the stability boundary for a grid-connected converter system derived from an Australian wind farm, achieving lower conservatism than the small gain theorem alone.

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Section II collects relevant mathematical concepts, including the definition of matrix phase [4]. Section III summarizes standard results in robust control and introduces the small phase theorem [2]. Section IV contains this paper's main contributions, including the modified mixed gain/phase criterion and the application of multipliers for structured perturbations. In Section V, we present the grid-connected power converter example.

II. Mathematical Preliminaries [16] [4]

The set of real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$ respectively. The set of imaginary numbers is denoted $j\mathbb{R}$. We denote a *sector* of the complex plane as

$$\mathbb{S}(\underline{\theta}, \overline{\theta}) := \{\rho e^{j\theta} \mid \rho \geq 0 \text{ and } \theta \in [\underline{\theta}, \overline{\theta}]\}. \tag{1}$$

The set of complex column vectors is denoted $\mathbb{C}^n$. The Euclidean norm of a vector x is denoted $||x||$. The set of $n \times m$ complex matrices is denoted $\mathbb{C}^{n\times m}$. I_n and 0_n denote the n-dimensional identity and square zero matrix, respectively. The conjugate transpose of a matrix A is denoted A^*. The set of proper real rational functions with no poles in the closed RHP is denoted $\mathcal{RH}_\infty$, and matrices thereof as $\mathcal{RH}_\infty^{n\times m}$.

In the following, $A, B \in \mathbb{C}^{n\times n}$. The spectrum (set of eigenvalues) of A is denoted $\lambda(A)$ and its largest singular value by $\overline{\sigma}(A)$. A has a *Cartesian decomposition* $A = \mathcal{R}(A) + j\mathcal{I}(A)$, where $\mathcal{R}(A) = (A + A^*)/2$ and $\mathcal{I}(A) = (A - A^*)/2j$ are both Hermitian. A positive (semi-)definite Hermitian matrix A is denoted $A > 0$ ($A \geq 0$). The notation $A > B$ ($A \geq B$) means that $A - B$ is positive (semi-)definite.

The *numerical range* of A is defined as

$$W(A) := \{x^* A x \mid x \in \mathbb{C}^n,\ ||x|| = 1\}. \tag{2}$$

It has the following relevant properties:

(W1): $W(A)$ is a convex and compact set in $\mathbb{C}$.
(W2): $\lambda(A) \subset W(A)$.
(W3): $W(cA) = cW(A)$ for $c \in \mathbb{C}$.
(W4): $W(\mathcal{R}(A)) = \{x \in \mathbb{R} \mid x = \mathrm{Re}(z),\ z \in W(A)\}$.

The *angular numerical range* of A is defined as

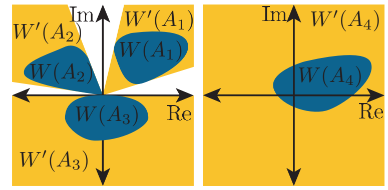
$$W'(A) := \{x^* A x \mid x \in \mathbb{C}^n,\ x \neq 0\}. \tag{3}$$

For any A, $W'(A) \cup \{0\} = \mathbb{S}(\underline{\theta}, \overline{\theta})$ for some $\underline{\theta}, \overline{\theta} \in \mathbb{R}$. It is the smallest sector containing $W(A)$. Define the *field angle* of A as $\delta(A) := \overline{\theta} - \underline{\theta}$. We say that a matrix A is *sectorial* if $0 \notin W(A)$, *quasi-sectorial* if $\delta(A) < \pi$, and *semi-sectorial* if $\delta(A) \leq \pi$. If A is non-semi-sectorial, then $W'(A) = \mathbb{C}$ and we define $\delta(A) = 2\pi$. Fig. 2 illustrates these cases.

Any quasi-sectorial A is congruent to a diagonal matrix:

$$A = T^* \begin{bmatrix} D & 0 \\ 0 & 0_{n-m} \end{bmatrix} T, \tag{4}$$

where $D \in \mathbb{C}^{m\times m}$ is unitary with $m \leq n$ and $T \in \mathbb{C}^{n\times n}$ is invertible. The matrix D is unique up to the ordering of its diagonal elements. By noting that (3) is invariant under congruence, it can be seen that $W'(A)$ is the set of strictly positive linear combinations of the elements of D, including 0 if $m < n$. Because $\delta(A) < \pi$, we may choose a unique pair $\overline{\phi}(A)$, $\underline{\phi}(A)$ so that $W'(A) \cup \{0\} = \mathbb{S}(\underline{\phi}(A), \overline{\phi}(A))$ and



(a) (b)

Fig. 2: A_1 is sectorial, A_2 is quasi-sectorial but not sectorial, A_3 is semi-sectorial but not quasi-sectorial, and A_4 is non-semi-sectorial.

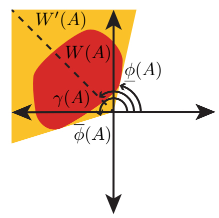


Fig. 3: Relationship between $\underline{\phi}(A)$, $\overline{\phi}(A)$ and $W(A)$.

the *phase center* $\gamma(A) = (\overline{\phi}(A) + \underline{\phi}(A))/2$ of A lies in $(-\pi, \pi]$, as illustrated in Fig. 3. The phases of the elements of D may be selected to lie in $[\underline{\phi}(A), \overline{\phi}(A)]$. Then, the m phases of D are called the *matrix phases* of A, and $\underline{\phi}(A)$ and $\overline{\phi}(A)$ are phases of A. This definition can be extended to semi-sectorial matrices, but we omit the details [2].

The matrix phases of the factors of a matrix product bound the phases of the product's eigenvalues (just as the factors' singular values bound the magnitudes of the eigenvalues.) We restate Lemma 3 of [4], avoiding technical issues around multiples of 2π due to the choice of phase center.

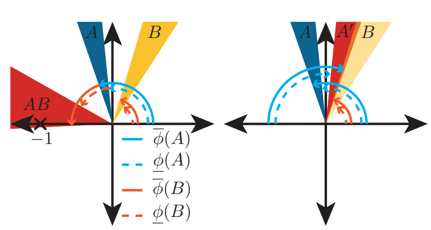
Lemma 1: [4] Suppose that $A \in \mathbb{C}^{n\times n}$ is quasi-sectorial, and $B \in \mathbb{C}^{n\times n}$ is semi-sectorial. Then:

$$\lambda(AB) \subset \mathbb{S}(\underline{\phi}(A) + \underline{\phi}(B), \overline{\phi}(A) + \overline{\phi}(B)). \tag{5}$$

This leads to a sufficient condition for $|I + AB| \neq 0$–or equivalently, for $-1 \notin \lambda(AB)$. For a quasi-sectorial A, we define the phase-bounded set of matrices:

$$\begin{aligned}\mathbf{P}(A) = \{B \in \mathbb{C}^{n\times n} \mid\ & B \text{ is semi-sectorial and} \\ & -1 \notin \mathbb{S}(\underline{\phi}(A) + \underline{\phi}(B), \overline{\phi}(A) + \overline{\phi}(B))\}.\end{aligned} \tag{6}$$

This membership test is illustrated in Fig. 4a. The shaded regions labeled A and B are respectively $\mathbb{S}(\underline{\phi}(A), \overline{\phi}(A))$ and $\mathbb{S}(\underline{\phi}(B), \overline{\phi}(B))$, and the sector shaded AB is $\mathbb{S}(\underline{\phi}(A) +$



(a) (b)

Fig. 4: Illustration of (a) (6) and (b) (7)

$\underline{\phi}(B), \overline{\phi}(A) + \overline{\phi}(B))$. This sector contains -1, so we may conclude $B \notin \mathbf{P}(A)$.

An equivalent test for the membership of B in $\mathbf{P}(A)$ is:

$$\mathbb{S}(\underline{\phi}(B), \overline{\phi}(B)) \cap \mathbb{S}(\pi - \overline{\phi}(A), \pi - \underline{\phi}(A)) = \{0\}. \quad (7)$$

This intersection-based test is illustrated in Fig. 4b. The sector labeled A' is $\mathbb{S}(\pi - \overline{\phi}(A), \pi - \underline{\phi}(A))$. As this intersects the sector labeled B, we again conclude that $B \notin \mathbf{P}(A)$.

From Lemma 1, it is clear that if $B \in \mathbf{P}(A)$, then $|I + AB| \neq 0$. We define the norm-bounded set of matrices:

$$\mathbf{N}(A) = \{B \in \mathbb{C}^{n\times n} : \overline{\sigma}(A)\overline{\sigma}(B) < 1\}. \quad (8)$$

If $B \in \mathbf{N}(A)$ then $|I + AB| \neq 0$.

III. Robust Stability of Feedback Interconnections

Consider the feedback interconnection of G and Δ shown in Fig. 1. If $G, \Delta \in \mathcal{RH}_\infty^{n\times n}$, the loop is *stable* if and only if the transfer function $(I + G\Delta)^{-1} \in \mathcal{RH}_\infty^{n\times n}$. We consider G a known "plant" and Δ an unknown "perturbation". The set of Δ forming a stable loop with G is denoted:

$$\mathcal{S}(G) := \{\Delta \in \mathcal{RH}_\infty^{n\times n} \mid (I + G\Delta)^{-1} \in \mathcal{RH}_\infty^{n\times n}\}. \quad (9)$$

The MIMO Nyquist criterion [17] provides a test for the membership of Δ in $\mathcal{S}(G)$. To quantify G's *robustness*, we need a measure of the "size" of the perturbation, which the Nyquist criterion does not provide. The famous small gain theorem provides such a measure, and a corresponding sufficient stability criterion.

Theorem 1 (Small gain theorem for LTI systems [14]): Given $G, \Delta \in \mathcal{RH}_\infty^{n\times n}$, $(I + G\Delta)^{-1} \in \mathcal{RH}_\infty^{n\times n}$ if for all $\omega \in [0, \infty]$,

$$\overline{\sigma}\left(G(j\omega)\right)\overline{\sigma}\left(\Delta(j\omega)\right) < 1. \quad (10)$$

We define the following set of Δ satisfying the small gain theorem together with G:

$$\mathcal{N}(G) = \{\Delta \in \mathcal{RH}_\infty^{n\times n} : \forall\, \omega \in [0, \infty],\, \Delta(j\omega) \in \mathbf{N}(G(j\omega))\}. \quad (11)$$

A Bode-style magnitude plot can be used to test the membership of a particular perturbation in this set–if $\sigma(\Delta(j\omega)) < \sigma(G(j\omega))^{-1}$, then we conclude $\Delta \in \mathcal{N}(G)$.

An analogous small *phase* theorem, based on the phases of the systems' frequency responses was recently introduced by [2]. We restate it here in terms of complex plane sectors.

Theorem 2 (Small phase theorem [2]): Suppose $G, \Delta \in \mathcal{RH}_\infty^{n\times n}$ with $G(j\omega)$ quasi-sectorial and $\Delta(j\omega)$ semi-sectorial for all $\omega \in [0, \infty]$. $(I + G\Delta)^{-1} \in \mathcal{RH}_\infty^{n\times n}$ if

$$-1 \notin \mathbb{S}(\underline{\phi}(G(j\omega)) + \underline{\phi}(\Delta(j\omega)), \overline{\phi}(G(j\omega)) + \overline{\phi}(\Delta(j\omega))), \quad (12)$$

for all $\omega \in [0, \infty]$.

Given a known G with $G(j\omega)$ quasi-sectorial for $\omega \in [0, \infty]$, the set of all Δ satisfying the small phase theorem is:

$$\mathcal{P}(G) = \{\Delta \in \mathcal{RH}_\infty^{n\times n} : \forall\, \omega \in [0, \infty],\, \Delta(j\omega) \in \mathbf{P}(G(j\omega))\}. \quad (13)$$

Membership in $\mathcal{P}(G)$ can be tested by plotting the two sectors in (7) against frequency on a Bode phase plot. If the two regions do not intersect, we conclude $\Delta \in \mathcal{P}(G)$.

IV. Main Results

The results of this paper aim at dealing with defining phase-based robustness measures in the case when the frequency response of G is non-quasi-sectorial over some frequencies. We present a mixed gain-phase stability criterion that requires $G(j\omega)$ to be sectorial only when the gain criterion fails to hold. The criterion incorporates multiplier functions, which will be used to reduce the conservatism of the phase criterion. In particular, these multipliers will allow phase bounds to be found in some cases when $G(j\omega)$ is non-quasi-sectorial.

A. Small Gain/Phase Theorem

The small gain and phase criteria can be combined on a frequency-wise basis to produce a criterion that is less conservative than either taken alone. [8] provides such a result proved with IQCs, but it requires $G(j\omega)$ to be semi-sectorial for all $\omega \in [0, \infty]$. We provide a Nyquist-based proof for a mixed gain/phase criterion that only requires quasi-sectoriality of G for frequencies where the gain criterion fails, and allows for the introduction of a frequency-domain multiplier function.

Theorem 3: Given $G, \Delta \in \mathcal{RH}_\infty^{n\times n}$, and $M : j\mathbb{R} \to \mathbb{C}^{n\times n}$ with $M(j\omega)$ bounded and invertible for all $\omega \in [0, \infty]$, $(I + G\Delta)^{-1} \in \mathcal{RH}_\infty^{n\times n}$ if, for every $\omega \in [0, \infty]$, at least one of the following is true.

(A) (10) is satisfied.
(B) $G(j\omega)M(j\omega)$ is quasi-sectorial, $M(j\omega)^{-1}\Delta(j\omega)$ is semi-sectorial, and (12) is satisfied for $G(j\omega)M(j\omega)$ and $M(j\omega)^{-1}\Delta(j\omega)$.

Proof: We will proceed by contraposition. Suppose that $(I + G\Delta)^{-1}$ is unstable. Then by the generalized Nyquist criterion [17], an eigenlocus of $G(j\omega)\Delta(j\omega)$ encircles or intersects -1 at least once as ω increases from 0 to ∞. If an eigenlocus encircles or intersects -1, then it must intersect the negative real axis at a point ≤ -1. Therefore, for some $\omega_0 \in \mathbb{R}$ and $k \geq 1$, we may say $-k \in \lambda(G(j\omega_0)\Delta(j\omega_0))$. By the properties of singular values, this implies that (A) is not true at ω_0. Because $M(j\omega)$ is bounded and invertible, we also have $-k \in \lambda(G(j\omega_0)M(j\omega_0)M(j\omega_0)^{-1}\Delta(j\omega_0))$. Applying contraposition to Lemma 1 with $A = G(j\omega_0)M(j\omega_0)$ and $B = M(j\omega_0)^{-1}\Delta(j\omega_0)$, we see that (B) is also false at ω_0. This implies that if the loop is unstable, (A) and (B) are simultaneously false for at least one ω_0, completing the proof by contraposition. ∎

It should be remarked that $M(j\omega)$ need not be the frequency response of a stable, minimum phase system – the proof above allows any matrix-valued function that is bounded and invertible at every frequency.

Given G and M, the set of all Δ that satisfy the premises of Theorem 3 is:

$$\begin{aligned}\mathcal{M}(G,M) = \{\Delta \in \mathcal{RH}_\infty^{n\times n} \mid \forall\, \omega \in [0,\infty],\\ \Delta(j\omega) \in \mathbf{N}(G(j\omega)) \text{ or}\\ M(j\omega)^{-1}\Delta(j\omega) \in \mathbf{P}(G(j\omega)M(j\omega))\}.\end{aligned} \tag{14}$$

In the case where $M(j\omega) = I$, it is simple to see that $\mathcal{N}(G) \cup \mathcal{P}(G) \subset \mathcal{M}(G, I) \subset \mathcal{S}(G)$.

B. Structured Perturbations

Often, a mixed phase/gain criterion may provide little benefit over the small gain theorem because G's frequency response is non-quasi-sectorial at important frequencies. We show that if a block diagonal structure for Δ is assumed, a phase criterion for the perturbation can be found in some cases when the frequency response of G is non-quasi-sectorial.

Define the set of block diagonal matrices in $\mathbb{C}^{n\times n}$:

$$\mathbf{\Delta} = \{\text{blkdiag}(X_1,\dots,X_b) \mid X_i \in \mathbb{C}^{n_i\times n_i}\}. \tag{15}$$

for some number of square blocks b with dimensions $n_1,\dots,n_b$ so that $\sum n_i = n$. In the remainder of the paper, we suppose Δ has the matching structure:

$$\Delta = \text{blkdiag}(\Delta_1,\dots,\Delta_b),\ \Delta_i \in \mathcal{RH}_\infty^{n_i\times n_i}. \tag{16}$$

We will now show how a multiplier can potentially reduce the conservatism of the phase criterion under a structured Δ. The cost of introducing a multiplier M is that the phase criterion applies not to $\Delta(j\omega)$, but to $M(j\omega)^{-1}\Delta(j\omega)$. Generally there is *no* relationship between the matrix phase of two matrices and their product [2], so we must consider multipliers so that the phases of $\Delta(j\omega)$ and $M(j\omega)^{-1}\Delta(j\omega)$ have a tractable relationship.

Consider the following set of invertible matrices and corresponding multiplier functions:

$$\mathbf{M} := \{\text{blkdiag}(c_1 I_{n_1},\dots,c_b I_{n_b}) \mid c_i \in \mathbb{C}\setminus\{0\}\}, \tag{17}$$

$$M(j\omega) := \text{blkdiag}(m_1(j\omega)I_{n_1},\dots,m_b(j\omega)I_{n_b}). \tag{18}$$

where each $m_i(j\omega) : j\mathbb{R} \to \mathbb{C}\setminus\{0\}$. A multiplier of this type commutes with $\Delta(s)$ and preserves its block-diagonal structure:

$$\begin{aligned}M(j\omega)^{-1}\Delta(j\omega) = \Delta(j\omega)M(j\omega)^{-1}\\ = \text{blkdiag}(\Delta_1(j\omega)/m_1(j\omega),\dots,\Delta_b(j\omega)/m_b(j\omega)).\end{aligned} \tag{19}$$

Multipliers of this form are closely related to those used in the so-called $\mathcal{D}$-scaling procedure for estimating the structured singular value. However, we consider a single sided transformation $G \to GM$, rather than the similarity transformation $G \to D^{-1}MD$ used in $\mathcal{D}$-scaling. The phases of each block $m_i(j\omega)^{-1}\Delta_i(j\omega)$ are the phases of $\Delta_i(j\omega)$ minus the phase of $m_i(j\omega)$, and by (4), the phases of $M(j\omega)^{-1}\Delta(j\omega)$ are comprised of the blocks' phases up to adjustments for phase center. The phase criterion imposed on $M(j\omega)^{-1}\Delta(j\omega)$ by (14) may then be interpreted as a criterion on the phases of each block $\Delta_i(j\omega)$ adjusted by the phase of $m_i(j\omega)$.

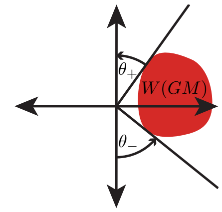


Fig. 5: Illustration of $W(GM)$ when $\mathcal{R}(GM) \geq \epsilon I$

How should we select multipliers from this structured set? Recalling (14), at each frequency $\omega \in [0,\infty]$, we require $G(j\omega)M(j\omega)$ to be quasi-sectorial if possible. Additionally, applying the criterion (7) to $G(j\omega)M(j\omega)$ and $M(j\omega)^{-1}\Delta(j\omega)$ we see from Fig. 4b that minimizing $\delta(G(j\omega)M(j\omega))$ maximizes the angular size of the sector that the phases of $\Delta(j\omega)$ can lie in. These considerations motivate the following optimization problem.

Problem 1: Given some $G \in \mathcal{RH}_\infty^{n\times n}$, find $M : j\mathbb{R} \to \mathbf{M}$ so that for each $\omega \in [0,\infty]$, $\delta(G(j\omega)M(j\omega))$ is minimized. We address this problem by calculating optimal values for the multiplier pointwise in frequency.

Theorem 4: Suppose $\epsilon > 0$ and $G \in \mathbb{C}^{n\times n}$ are given. There exists $M \in \mathbf{M}$ so that GM is quasi-sectorial if and only if the LMI constraints:

$$\mathcal{R}(GM) \geq 0, \tag{20}$$

$$MM^* \geq \epsilon I, \tag{21}$$

$$\tau\mathcal{R}(GM) \geq -\mathcal{I}(GM), \tag{22}$$

$$\tau\mathcal{R}(GM) \geq \mathcal{I}(GM), \tag{23}$$

are feasible for some $\tau \in \mathbb{R}$. Additionally, if M minimizes the τ required for feasibility, then that M minimizes $\delta(GM)$, and therefore solves Problem 1.

Proof: Recall that if GM is semi-sectorial then $W(GM)$ lies in a closed half-plane. Then by (W3) there always exists a $c \in \mathbb{C}\setminus\{0\}$ so that all $z \in W(cGM)$ satisfy $\text{Re}(z) \geq 0$. Then, by (W4), the existence of M so that GM is sectorial is equivalent to (20), while (21) enforces M's invertibility.

Define the following angles (see Fig. 5):

$$\theta_+(GM) = \pi/2 - \overline{\phi}(GM), \tag{24}$$

$$\theta_-(GM) = \pi/2 + \underline{\phi}(GM). \tag{25}$$

Define $\psi(GM) = \min(\theta_+(GM), \theta_-(GM))$. Because $\delta(GM) = \pi - (\theta_+(GM) + \theta_-(GM))$ and $\delta(e^{j\alpha}GM) = \delta(GM)$ for any $\alpha \in \mathbb{R}$, any solution maximizing $\psi(GM)$ will also minimize $\delta(GM)$. To express the maximization of $\psi(GM)$ as an LMI condition, we note $\psi(GM)$ is the largest possible angle $\alpha \in [0,\pi)$ that simultaneously satisfies:

$$\mathcal{R}(e^{j\alpha}GM) \geq 0, \tag{26}$$

$$\mathcal{R}(e^{-j\alpha}GM) \geq 0. \tag{27}$$

If $\alpha \neq 0$, then applying $e^{j\alpha} = \cos\alpha + j\sin\alpha$ yields

$$\cot\alpha\,\mathcal{R}(GM) \geq -\mathcal{I}(GM), \tag{28}$$

$$\cot\alpha\,\mathcal{R}(GM) \geq \mathcal{I}(GM). \tag{29}$$

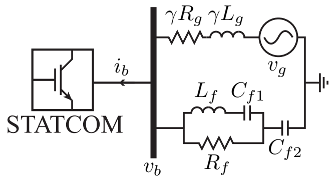


Fig. 6: One-line diagram of Stockyard Hill, eastern substation

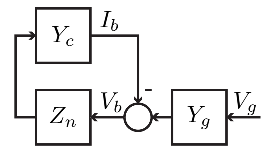


Fig. 7: Grid/converter interaction feedback loop

TABLE I: System parameters. A per-unit base of $S_b = 100$ MVA, $V_b = 132$ kV and $\omega_b = 100\pi$ rad/s is used.

Variable	Value	Variable	Value	Variable	Value
L	0.2 p.u.	k_{pPLL}	420	L_f	1.329 p.u.
R	0.02 p.u.	k_{iPLL}	45000	R_f	9.183 p.u.
C_{dc}	1.6 p.u.	k_{pac}	0.175	C_{f1}	0.752 p.u.
R_{dc}	0 p.u.	k_{iac}	220	C_{f2}	0.251 p.u.
k_{pi}	3.5	k_{pdc}	0.3	L_g	0.981 p.u.
k_{ii}	80	k_{idc}	7	R_g	0.196 p.u.

As $\cot\alpha$ is decreasing on $\alpha \in (0, \pi)$ with range $\mathbb{R}$ and minimising $\cot\alpha$ maximises $\psi(GM)$ as required. (22-23) follow by setting $\tau = \cot\alpha$. ■

This optimization problem is a generalized eigenvalue problem for which solvers exist, such as `gevp` from MATLAB's LMI Toolbox. Numerical issues can arise for singular G, in which case a bisection algorithm is more appropriate.

V. Example

During its commissioning, the Stockyard Hill Wind Farm in Victoria, Australia suffered a small-signal stability issue related to the interaction of a static synchronous compensator (STATCOM) and a harmonic filter. Fig. 6 illustrates the relevant portion of the system.

The STATCOM is a power converter intended to regulate the system voltage. Its connection to the power grid is represented by an RL Thevenin equivalent circuit shown in the top branch. The bottom branch is the harmonic filter circuit. When designing a converter's controller, it is important to understand the range of grid impedances under which the system will be stable. We will show how the mixed small gain/phase criterion predicts the boundary of stability with less conservatism compared to the small gain theorem.

To this end, we vary the grid strength parameter $\gamma \geq 0$, which scales the grid impedance R_g, L_g in Fig. 6. $\gamma = 0$ represents a perfectly stiff grid, where the filter is shorted out by the grid Thevenin voltage, and the system is guaranteed to be stable. As γ increases, representing a weaker grid or longer transmission line, the grid impedance increases and the filter becomes more weakly damped. We aim to estimate the smallest γ for which the system becomes unstable.

We divide the system into a *converter* subsystem representing the STATCOM with bus voltage v_b and current i_b as input and output respectively, and a *network* subsystem combining the grid and filter models, with i_b and v_b as input and output respectively. The linearized subsystem models are derived in *dq-frame* variables [18], yielding 2×2 MIMO transfer functions. The STATCOM transfer function $Y_c(s)$ is taken from (9) in [19] using parameters given in the first two columns of Table I, and:

$$I_b(s) = Y_c(s)V_b(s), \tag{30}$$

where $Y_c(s) \in \mathcal{RH}_\infty^{2\times 2}$.

The network model has a per-phase impedance $Z_{n,1\phi}(s) \in \mathcal{RH}_\infty$. Its dynamics rewritten in dq frame variables remain LTI [20]. Define the unitary matrix $U_J = \left[\begin{smallmatrix} 1 & 1 \\ -j & j \end{smallmatrix}\right]/\sqrt{2}$. Then the dq-frame transfer matrix of the network impedance is

$$Z_n(s) = U_J \begin{bmatrix} Z_{n,1\phi}(s+j\omega_b) & 0 \\ 0 & Z_{n,1\phi}(s-j\omega_b) \end{bmatrix} U_J^*. \tag{31}$$

[20] shows that $Z_n \in \mathcal{RH}_\infty^{2\times 2}$. Then:

$$V_b(s) = -Z_n(s)I_b(s) + Z_n(s)Y_g(s)V_g(s). \tag{32}$$

(30) and (32) comprise a feedback loop, illustrated in Fig. 7. We identify $G = Y_c$ as the plant, and $\Delta = Z_n$ as the perturbation. Y_c is non-quasi-sectorial at all frequencies of interest, so we must use a multiplier. We perform a change of variables $\tilde{I}_b(s) = U_J I_b(s)$ and $\tilde{V}_b(s) = U_J V_b(s)$, yielding equivalent subsystems $\tilde{G}(s) = U_J^* Y_c(s) U_J$ and $\tilde{\Delta}(s) = U_J^* Z_n(s) U_J$. $\tilde{\Delta}$ has a diagonal structure (15) with $b = 2$ and $n_1 = n_2 = 1$. We then solve the LMI problem of Theorem 4 for $\tilde{G}(j\omega)$ to find a diagonal multiplier $\tilde{M}(j\omega)$, which can be transformed to the structure of (31) by $M(j\omega) = U_J \tilde{M}(j\omega) U_J^*$. It can be shown that $M(j\omega)$ still solves Problem 1 for G with respect to the perturbation structure in (31).

The frequency responses $Y_c(j\omega)$ and $M(j\omega)$ were calculated for a frequency range $[1, 200]$ Hz. $Z_n(j\omega)$ was calculated using the parameters in the right-hand column of Table I for three values of γ. Singular value and phase plots are shown in Fig. 8 for $Y_c(j\omega)M(j\omega)$ and $M(j\omega)^{-1}Z_n(j\omega)$ to test the membership $Z_n \in \mathcal{M}(Y_c, M)$. The gain criterion of (14) is satisfied at frequencies where $\overline{\sigma}(Z_n(j\omega)) < \overline{\sigma}(Y_c(j\omega))^{-1}$, and the phase criterion is satisfied for frequencies where the sectors defined in (7) do not intersect, where $A = Y_c(j\omega)M(j\omega)$ and $B = M(j\omega)^{-1}Z_n(j\omega)$. Nonlinear simulations were performed for each case and the q-axis converter current on startup is plotted in Fig. 9.

For $\gamma < 0.36$, stability is guaranteed by the gain criterion alone. For $\gamma = 0.74$, the gain criterion fails for frequencies up to 40 Hz and a small region around 135 Hz, but the phase criterion is satisfied at these frequencies. Therefore, the combined gain/phase criterion with multiplier provides a substantially less conservative estimate of the set of permissible grid impedances. For $\gamma = 1$, Fig. 9 reveals the onset of instability. The gain and phase criterion fail

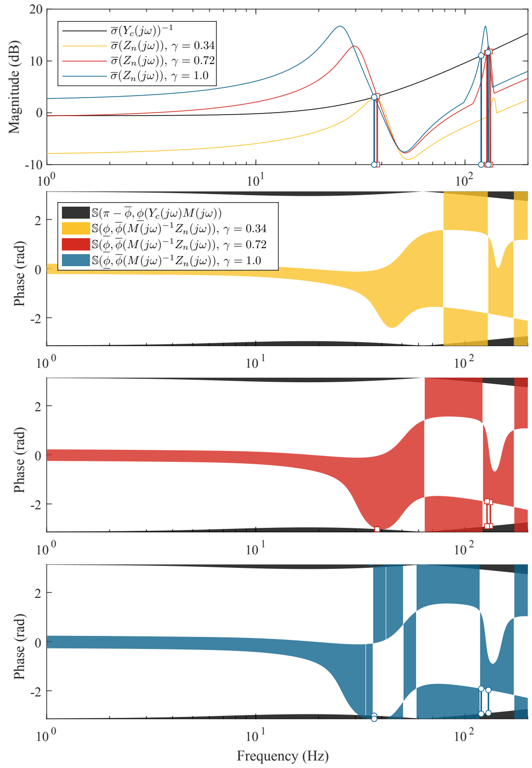


Fig. 8: Singular value and phase range plot for Y_c and Z_n. Gain crossover frequencies marked.

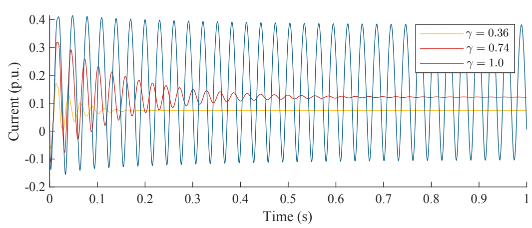


Fig. 9: Transient q axis current of i_b

simultaneously for a region from 32-38 Hz, in which the oscillatory frequency of 33.4 Hz falls.

VI. Conclusion

The recent definition of multivariable system phase is highly promising. To bridge the gap between this novel theory and practice, this paper provides a mixed gain/phase stability test that only requires sectoriality at frequencies where the gain criterion fails and augments it with a multiplier for structured perturbations, broadening the range of cases where the phase criterion applies. The practical usefulness of these methods for robustness analysis is demonstrated on a scenario based on a real-life incident of small-signal instability in an Australian wind farm. Substantial improvements in conservatism are achieved over the small gain theorem, which has practical implications in terms of informing nonlinear analysis and design for power systems. The authors hope that this paper's contribution will be the first step toward a general theory of phase for structured perturbations in analogy to the structured singular value.

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