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Event-triggered Sensor Scheduling for Remote State Estimation with Error-Detecting Code

Yuxing Zhong, *Student Member, IEEE*, Jiawei Tang, *Student Member, IEEE*, Nachuan Yang, *Student Member, IEEE*, Dawei Shi, *Senior Member, IEEE*, and Ling Shi, *Fellow, IEEE*

***Abstract*— This letter addresses the problem of remote state estimation subject to packet dropouts, focusing on the use of an event-triggered sensor scheduler to conserve communication resources. However, packet dropouts introduce significant challenges, as the remote estimator cannot distinguish between packet loss caused by poor channel conditions and the event trigger. To overcome this issue, we propose a novel formulation that incorporates error-detecting codes. We prove that the Gaussian property of the system state, commonly utilized in the literature, does not hold in this scenario. Instead, the system state follows an extended Gaussian mixture model (GMM). We present an exact minimum mean-squared error (MMSE) estimator and an approximate estimator, which significantly reduces algorithm complexity without sacrificing performance. Our simulation results show that the approximate estimator achieves nearly the same performance as the exact estimator while requiring much less computation time. Moreover, the proposed event trigger outperforms existing schedulers in terms of estimation accuracy.**

I. Introduction

The rapid development of sensor networks has brought both new opportunities and challenges to remote state estimation. Although sensor networks offer numerous benefits, sensor devices are often battery-powered, and communication bandwidth is limited, particularly in large-scale applications. As a result, there is a critical need to carefully schedule data transmissions to conserve communication resources. For instance, in artificial weir systems, sensors are deployed to monitor the fill levels of water reservoirs. To extend sensors' lifetime, scheduling the communication between sensors and remote estimator is crucial [1].

In comparison to offline schedulers, which lack the ability to adapt to real-time system dynamics, online event-triggered schedulers have been shown to be more flexible and efficient [2]. The triggering rules for such schedulers may involve thresholds based on innovation [3] or error covariance [4]. Wu et al. [5] proposed a deterministic event-triggered scheduling mechanism that leverages the extra information held by measurement transmission to improve scheduling efficiency. However, the estimator becomes challenging to design due to the destruction of the Gaussian property. To address this issue, Han et al. [6] proposed a stochastic event trigger that preserves the Gaussian property.

While the above traditional studies assume perfect communication between sensors and remote estimators, wireless communication suffers from packet dropouts. This makes it essential to explore scheduling methods for sensor measurements in packet-dropping networks. Unfortunately, it has been claimed by Kung et al. [7] that there does not exist any event-triggered scheduler that can preserve the Gaussian property in the presence of packet dropouts. As a result, the update rule becomes non-recursive, nonlinear, and highly complex. To tackle it, Xu et al. [8] derived the exact MMSE estimator, but the computation time required is exponential. They also proposed two approximate estimators to simplify the calculation. Huang et al. [9] explored the event-triggered scheduling problem in cognitive radio sensor networks, where the uncertainty from the event trigger and communication channels are decoupled. In this letter, we focus on the event-triggered sensor scheduling problem with error-detecting codes to detect packet dropouts. Due to the stochastic property of packet-dropping networks, it is impossible to decouple the event trigger and the packet loss entirely [9]. Therefore, the estimation process considered in this letter is much more complicated. On the other hand, since the error-detecting code is widely used in communications nowadays, proper use of it in our event-triggered sensor scheduling problem can significantly simplify the calculation and improve the estimation performance compared to that by Xu et al. [8].

The main contributions of this letter are listed as follows:

1) To cope with packet dropouts in the event-triggered sensor scheduling problem, we propose a novel formulation that incorporates error-detecting codes into transmitted packets. This approach is unique and, to the best of our knowledge, has not been explored previously. Notably, our developed results reduce to that of Han et al. [6] by assuming that no error will be detected.
2) We derive both the exact MMSE estimator and an approximate estimator to reduce the computational complexity. Our method aims to preserve the Gaussian property as much as possible, which significantly outperforms the existing trigger scheme by Wu et al. [5].

The remainder of this letter is organized as follows. We

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Yuxing Zhong, Jiawei Tang, Nachuan Yang, and Ling Shi are with the Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong SAR (email: yzhongbc@connect.ust.hk; jtangas@connect.ust.hk; nyangaj@connect.ust.hk; eesling@ust.hk).

Dawei Shi is with the School of Automation, Beijing Institute of Technology, Haidian District, Beijing, China (email:daweishi@bit.edu.cn).

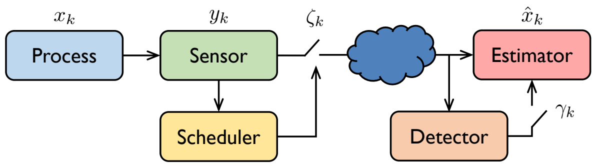

Fig. 1. System diagram.

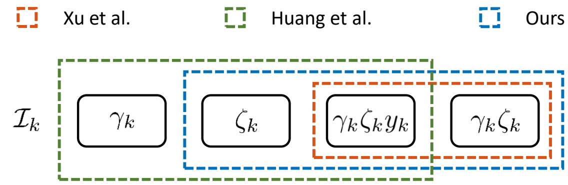

Fig. 2. Comparison with existing literature Xu et al. [8] and Huang et al. [9], where the problem formulation considered herein differs in the information set $\mathcal{I}_k$.

introduce the system model in Section II. We present the exact MMSE estimator and an approximate one in Section III and Section IV, respectively. We conduct simulations to evaluate the effectiveness of the event-triggered scheduler in Section V and end the letter with a conclusion in Section VI.

Notations: The notation $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and n dimensional column vectors, respectively. For a matrix X, X^T and $\det(X)$ represent its transpose and determinant, respectively. When a matrix X is positive semidefinite (resp. definite), we write $X \geq 0$ (resp. $X > 0$). We use $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ to denote the probability of an event and the expectation of a random variable (r.v.). The notation $f(\cdot)$ denotes a general probability density function (PDF) of an r.v. and $\mathcal{N}(\mu, \Sigma)$ denotes the PDF of a Gaussian distribution with mean μ and covariance Σ. The expression I denotes the identity matrix with compatible dimensions.

II. Problem Formulation

A. System Model

Consider a discrete-time linear time-invariant (LTI) system (Fig. 1) that can be described by

$$x_{k+1} = Ax_k + w_k, \quad y_k = Cx_k + v_k,$$

where $x_k \in \mathbb{R}^n$ is the system state, $y_k \in \mathbb{R}^m$ is the sensor measurement, and $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are independent and identically distributed (i.i.d.) white Gaussian noises with covariance matrices $Q \geq 0$ and $R > 0$, respectively. The initial state x_0 is Gaussian with mean zero and covariance $\Sigma_0 \geq 0$, and is uncorrelated with w_k and v_k for all k. We assume that $(A, \sqrt{Q})$ is stabilizable and (A, C) is detectable.

Remark 1: The i.i.d. assumption of noises is used to enable recursive updates in the standard Kalman filter, and the stability/detectability assumption is used to guarantee the filter's convergence without event triggers and packet dropouts [10]. Both are standard assumptions in the literature.

Motivated by the fact that some measurements benefit the state estimation more while others may not, we equip the sensor with an event-triggered scheduler. Let $\zeta_k \in \{0, 1\}$ denote the decision variable of the scheduler such that $\zeta_k = 1$ if the measurement is transmitted and $\zeta_k = 0$ otherwise. We consider the triggering rule by Han et al. [6], i.e.,

$$\zeta_k = \begin{cases} 0, & \lambda_k \leq \exp(-\frac{1}{2}y_k^T Y y_k), \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

where λ_k is a randomly generated variable uniformly distributed over $[0, 1]$, and $Y > 0$ is the triggering parameter. The essence of the designed trigger is that the scheduler will choose not to transmit the measurement with the following probability:

$$\mathbb{P}(\zeta_k = 0 | y_k) = \exp(-\frac{1}{2}y_k^T Y y_k). \tag{2}$$

Remark 2: The triggering parameter Y is selected to achieve a trade-off between the triggering rate and the estimation accuracy, i.e., the larger Y, the higher the triggering rate and thus, the better estimation performance.

Remark 3: When A is unstable, $\zeta_k = 1$ occurs almost surely after a long time. To avoid trivial problems, we assume A is stable in this letter. However, closed-loop schedulers [6][1] can serve a viable alternative method for unstable systems.

When $\zeta_k = 1$, the measurement packet is transmitted to the remote estimator over unreliable wireless communication channels. Nevertheless, such a problem is difficult to tackle, because the estimator cannot distinguish whether the packet loss is from the trigger or the channel. To enable the estimator to make such a distinction, we equip it with an error detector. The to-be-transmitted measurement, if any, is added with error-detecting codes[2] on the sensor side, with which the remote estimator can detect whether an error appears during the transmission. Let $\gamma_k \in \{0, 1\}$ be the binary variable such that $\gamma_k = 1$ if no error is detected and the received measurement y_k is used to improve the estimate of the system state whereas $\gamma_k = 0$ if an error is detected and the measurement is dropped by the remote estimator. We assume γ_k is an i.i.d. r.v.. Moreover, for notation consistency, we assume $\gamma_k = 0$ when $\zeta_k = 0$. With the error-detecting codes, the information set available to the estimator at time k is $\mathcal{I}_k = \{\gamma_i \zeta_i y_i\}_0^k \cup \{\gamma_i \zeta_i\}_0^k \cup \{\zeta_i\}_0^k$, where $\mathcal{I}_{-1} = \emptyset$. Based on the collected information, the estimator estimates the system state via

$$\begin{aligned} \hat{x}_k^- &\triangleq \mathbb{E}[x_k | \mathcal{I}_{k-1}], & e_k^- &\triangleq x_k - \hat{x}_k^-, & P_k^- &\triangleq \mathbb{E}[e_k^- e_k^{-T}], \\ \hat{x}_k &\triangleq \mathbb{E}[x_k | \mathcal{I}_k], & e_k &\triangleq x_k - \hat{x}_k, & P_k &\triangleq \mathbb{E}[e_k e_k^T], \end{aligned}$$

where $\hat{x}_k^-$ (resp. $\hat{x}_k$) is the priori (resp. posteriori) MMSE estimate of x_k, and e_k^- (resp. e_k) and P_k^- (resp. P_k) is the associated estimation error and error covariance, respectively.

It is worth noticing that our problem formulation differs from the works by Xu et al. [8] and Huang et al. [9] in the information set $\mathcal{I}_k$ on the estimator side (Fig. 2).

[1]The developed results in this letter can be directly extended to the closed-loop case by replacing the measurement y_k with the innovation z_k.

[2]Examples of error detection codes include parity check, checksum, and cyclic redundancy check, to name just a few.

B. Problem of Interest

As mentioned [7], deriving the MMSE estimate with an event-triggered scheduler subject to packet dropouts is challenging. In this letter, for the system in Fig. 1 with the event-triggered scheduler (1), we aim to compute the exact MMSE estimate $\hat{x}_k$ and the associated error covariance P_k. As the exact estimator is highly complicated and difficult to implement in practice, we also discuss how to simplify the calculation by approximations.

III. Exact MMSE Estimator

Before providing the exact MMSE estimator under the sensor scheduler (1), we first define a variable, i.e., $N_k \triangleq 2^{\sum_{\tau=1}^{k} \zeta_\tau \oplus \gamma_\tau}$ and an extended mixture model, where $\zeta_k \oplus \gamma_k = 0$ if $\zeta_k = \gamma_k$ and $\zeta_k \oplus \gamma_k = 1$ otherwise.

Definition 1: (Finite mixture models with negative components [11]) A finite mixture model is said to be with negative components if negative coefficients of components are allowed, i.e.,

$$f(x|\theta) = \sum_{i=1}^{N} \alpha_i f(x|\theta_i), \quad \sum_{i=1}^{N} \alpha_i = 1,$$

where N is the number of components, $\alpha_i \in \mathbb{R}$ is the coefficient of i-th component and θ is the distribution parameter.

Remark 4: A finite Gaussian mixture with negative components is an extension of GMM. It has been shown that the negative pattern can be regarded as a part of the positive pattern. Nevertheless, by using negative components, we speed up the computation.

We now present the exact distribution of x_k conditioned on $\mathcal{I}_{k-1}$ and $\mathcal{I}_k$, respectively.

Lemma 1: Consider the remote estimation in Fig. 1 with the event-triggered scheduler (1). The state x_k conditioned on $\mathcal{I}_{k-1}$ and $\mathcal{I}_k$ are an N_{k-1}-component and N_k-component Gaussian mixture with negative coefficients, respectively, i.e.,

$$f(x_k|\mathcal{I}_{k-1}) = \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^{-} \mathcal{N}_{x_k}(\mu_{k,i}^{-}, \Sigma_{k,i}^{-}),$$
$$f(x_k|\mathcal{I}_k) = \sum_{i=1}^{N_k} \alpha_{k,i} \mathcal{N}_{x_k}(\mu_{k,i}, \Sigma_{k,i}),$$

where $\alpha_{k,i}^{-}$, $\mu_{k,i}^{-}$, $\Sigma_{k,i}^{-}$ and $\alpha_{k,i}$, $\mu_{k,i}$, $\Sigma_{k,i}$ satisfy the following update rule.

Time Update:

$$\alpha_{k+1,i}^{-} = \alpha_{k,i},, \tag{3}$$
$$\mu_{k+1,i}^{-} = A\mu_{k,i}, \tag{4}$$
$$\Sigma_{k,i}^{-} = A\Sigma_{k,i}A^T + Q. \tag{5}$$

Measurement Update:

1) when $\zeta_k = \gamma_k$,

$$\alpha_{k,i} = \frac{\beta_{k,i}^{\gamma_k \zeta_k}}{\sum_{i=1}^{N_k} \beta_{k,i}^{\gamma_k \zeta_k}}, \tag{6}$$
$$\mu_{k,i} = (I - K_{k,i}C)\mu_{k,i}^{-} + \gamma_k K_{k,i} y_k, \tag{7}$$
$$\Sigma_{k,i} = \Sigma_{k,i}^{-} - K_{k,i}C\Sigma_{k,i}^{-}, \tag{8}$$

2) when $\zeta_k = 1$ and $\gamma_k = 0$, for $0 < i \le N_{k-1}$,

$$\alpha_{k,i} = \frac{\alpha_{k,i}^{-}}{1 - \sum_{i=1}^{N_{k-1}} \beta_{k,i}^{0}}, \tag{9}$$
$$\mu_{k,i} = \mu_{k,i}^{-}, \tag{10}$$
$$\Sigma_{k,i} = \Sigma_{k,i}^{-}, \tag{11}$$

otherwise, for $N_{k-1} < i \le N_k$,

$$\alpha_{k,i} = -\frac{\beta_{k,i}^{0}}{1 - \sum_{i=1}^{N_{k-1}} \beta_{k,i}^{0}}, \tag{12}$$
$$\mu_{k,i} = \mu_{k,i}^{-} - K_{k,i}C\mu_{k,i}^{-}, \tag{13}$$
$$\Sigma_{k,i} = \Sigma_{k,i}^{-} - K_{k,i}C\Sigma_{k,i}^{-}, \tag{14}$$

where

$$K_{k,i} = \Sigma_{k,i}^{-}C^T[C\Sigma_{k,i}^{-}C^T + R + (1 - \gamma_k\zeta_k)Y^{-1}]^{-1},$$
$$\beta_{k,i}^{0} = \frac{\alpha_{k,i}^{-} \exp(-\frac{c_{k,i}}{2})}{\sqrt{\det[I + Y(C\Sigma_{k,i}^{-}C^T + R)]}},$$
$$\beta_{k,i}^{1} = \alpha_{k,i}^{-}\mathcal{N}_{y_k}(C\mu_{k,i}^{-}, C\Sigma_{k,i}^{-}C^T + R),$$
$$c_{k,i} = (\mu_{k,i}^{-})^T C^T (C\Sigma_{k,i}^{-}C^T + R + Y^{-1})^{-1} C\mu_{k,i}^{-}.$$

The initial condition is

$$\hat{x}_0^{-} = 0, \quad P_0^{-} = \Sigma_0, \quad \alpha_0^{-} = 1. \tag{15}$$

Proof: See the Appendix. ■

Theorem 1: For the remote estimation in Fig. 1 with the event-triggered scheduler (1), $\hat{x}_k^{-}$, $\hat{x}_k$ and P_k^{-}, P_k satisfy the following update rule:

$$\hat{x}_k^{-} = \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^{-}\mu_{k,i}^{-}, \quad \hat{x}_k = \sum_{i=1}^{N_k} \alpha_{k,i}\mu_{k,i},$$
$$P_k^{-} = \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^{-} \left[\Sigma_{k,i}^{-} + (\hat{x}_k - \mu_{k,i}^{-})(\hat{x}_k - \mu_{k,i}^{-})^T\right],$$
$$P_k = \sum_{i=1}^{N_k} \alpha_{k,i} \left[\Sigma_{k,i} + (\hat{x}_k - \mu_{k,i})(\hat{x}_k - \mu_{k,i})^T\right].$$

Proof: The update rule follows immediately from Lemma 1 and the calculation of the Gaussian sum filter [10], and thus is omitted here. ■

Let T be the simulation length. Define the triggering rate:

$$r \triangleq \limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}[\zeta_k].$$

Discussion 1: Similar to [6, Theorem 3], we can obtain $r = 1 - \det(I + \Pi Y)^{-\frac{1}{2}}$, where $\Sigma = A\Sigma A^T + Q$ and $\Pi = C\Sigma C^T + R$. Therefore, by minimizing $\lim_{k\to\infty} \mathbb{E}[P_k]$ subject to a triggering rate constraint, or vice versa, we can balance the trade-off between the triggering rate and the estimation performance.

IV. Approximate MMSE Estimator

Theorem 1 provides an exact MMSE estimator for the studied problem. However, the computation is non-recursive, nonlinear and suffers from the curse of dimensionality. Handling such an estimator involves formidable exponentially time and space complexity, which motivates our design of approximate MMSE estimators.

In this section, we first state how the Gaussian property will corrupt during the estimation process, and then provide a recursive approximate estimator with minor approximations.

Lemma 2: For the remote estimation problem in Fig. 1 with the event-triggered scheduler (1), if $f(x_k|\mathcal{I}_{k-1})$ is Gaussian distributed, i.e., $f(x_k|\mathcal{I}_{k-1}) = \mathcal{N}_{x_k}(\hat{x}_k^-, P_k^-)$, then $f(x_k|\mathcal{I}_k)$ and $f(x_{k+1}|\mathcal{I}_k)$ are still Gaussians when $\zeta_k = \gamma_k$ and become GMMs with two components otherwise.

Proof: The proof is similar to that of Lemma 1 and thus is omitted here. ∎

Since the Gaussian property only corrupts when $\zeta_k = 1$ and $\gamma_k = 0$, a considerate approximation is to assume $f(x_k|\mathcal{I}_{k-1})$ is Gaussian at each time k, i.e.,

$$f(x_k|\mathcal{I}_{k-1}) \approx \mathcal{N}_{x_k}(\hat{x}_k^-, P_k^-). \tag{16}$$

This is a widely used approach in nonlinear filtering and the same approximation is also adopted by Wu et al. [5], Qian et al. [12] and Ribeiro et al. [13].

Remark 5: Even though it is impossible for an event-triggered scheduler to preserve the Gaussian property in the presence of packet dropouts, Lemma 2 shows that the Gaussian property is destroyed only in the case where $\zeta_k = 1$ and $\gamma_k = 0$. It means that the approximation is more accurate compared with the existing trigger [5] that cannot preserve Gaussian even in other cases.

Attributed to the approximation (16), we now obtain a simple linear recursive update of the estimate and error covariance, which significantly simplifies the calculation.

Theorem 2: Consider the remote estimation in Fig. 1 with the triggering mechanism (1). Under approximation (16), the estimator takes the following recursive form:

Time Update:

$$\begin{aligned}\hat{x}_{k+1}^- &= A\hat{x}_k,\\ P_{k+1}^- &= AP_kA^T + Q,\end{aligned}$$

Measurement Update:

1) for $\zeta_k = \gamma_k$,

$$\begin{aligned}\hat{x}_k &= (I - K_kC)\hat{x}_k^- + \gamma_k K_k y_k,\\ P_k &= P_k^- - K_kCP_k^-,\end{aligned}$$

2) for $\zeta_k = 1$ and $\gamma_k = 0$,

$$\begin{aligned}\hat{x}_k &= \hat{x}_k^- - (1-\varrho_k)K_kC\hat{x}_k^-,\\ P_k &= P_k^- - (1-\varrho_k)K_kCP_k^-\\ &\quad + \varrho_k(1-\varrho_k)K_kC\hat{x}_k^-(\hat{x}_k^-)^TC^TK_k^T,\end{aligned}$$

where

$$K_k = P_k^-C^T[CP_k^-C^T + R + (1-\gamma_k\zeta_k)Y^{-1}]^{-1},$$

$$\varrho_k = \left\{1 - \frac{\exp(-\frac{c_k}{2})}{\sqrt{\det[I + Y(CP_k^-C^T + R)]}}\right\}^{-1},$$

$$c_k = (\hat{x}_k^-)^TC^T(CP_k^-C^T + R + Y^{-1})^{-1}C\hat{x}_k^-.$$

Proof: By Lemma 1 and moment matching [14], the proof is straightforward thus omitted here. ∎

Unlike Theorem 1 that requires exponentially increasing complexity, the approximate estimator provided in Theorem 2 has a simpler recursive form and can be easily implemented in practice.

V. Simulation

In this section, we demonstrate the effectiveness of the event trigger and the approximation algorithm via simulations. We use the following setup throughout this section:

1) The sequence $\{\gamma_k\}_0^k$ is assumed to follow an i.i.d. Bernoulli distribution with $\mathbb{P}(\gamma_k = 1) = \gamma \in (0, 1)$.
2) The empirical mean error is defined as

$$\mathcal{E} \triangleq \frac{1}{T}\sum_{k=0}^{T-1}(x_k - \hat{x}_k)^T(x_k - \hat{x}_k).$$

A. Approximation Assessment

Fig. 3 presents a performance comparison between the exact MMSE estimator and the approximate one on the following system:

$$A = \begin{bmatrix}0.8 & 1\\ 0 & 0.95\end{bmatrix},\ C = \begin{bmatrix}0.5 & 0.3\\ 0 & 1.4\end{bmatrix},\ Q = R = \Sigma_0 = I.$$

Notably, the approximate estimator achieves nearly the same empirical error as the exact one while requiring significantly less computation time. This finding suggests that the approximation approach in Section IV is small and effective in practical applications.

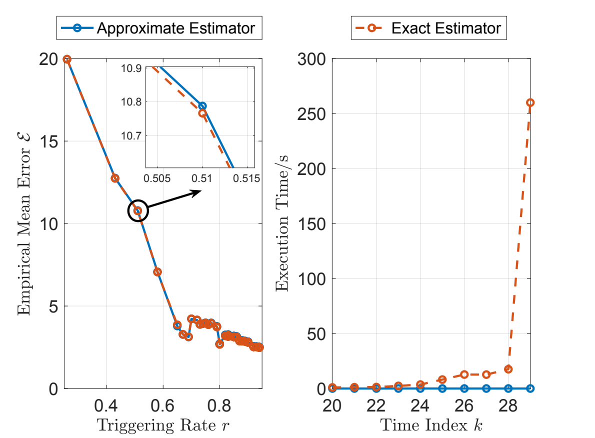

Fig. 3. The empirical mean error and the execution time of the exact estimator and the approximate estimator.

B. Policy Assessment

Since other existing event triggers [4], [5] cannot be directly applied to packet-dropping networks, in this part, we compare the designed event-triggered scheduler with the one by Xu et al. [8]. Fig. 4 demonstrates the comparison on an artificial weir system [1]:

$$A = \begin{bmatrix} 0.9 & 0 & 0 \\ 0.43 & 0.8 & 0 \\ 0.15 & 0.35 & 0.7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$Q = R = \Sigma_0 = 5I.$$

It should be noted that our proposed scheduler outperforms the scheduler by Xu et al. [8]. Particularly, the improvement is significant for lower values of γ. This can be attributed to our scheduler's better estimates when $\zeta_k = 0$ and $\gamma_k = 0$.

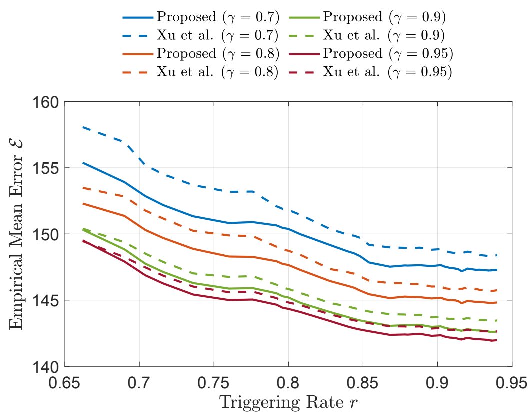

Fig. 4. The empirical mean error $\mathcal{E}$ versus the triggering rate r, where the different triggering rates are obtained by adjusting the value of Y.

VI. Conclusion

In this letter, we considered an event-based sensor scheduler for remote state estimation problems with error-detecting codes. We derived the exact MMSE estimator with exponential complexity and provided an approximate estimator to reduce the computational burden. Simulations showed that the approximate estimator can achieve nearly the same empirical error while requiring much lower computation time. Additionally, the designed event trigger significantly outperforms existing schedulers in terms of estimation accuracy. Future work includes event-triggered schedulers in other channel models, i.e., packet-delaying or fading models.

Appendix

Proof of Lemma 1: We prove the lemma by induction. Since $\mathcal{I}_{-1} = \emptyset$, x_0 is Gaussian thus GMM and (15) holds. For measurement update, we assume that $f(x_k|\mathcal{I}_{k-1})$ is GMM with N_{k-1} components. Since all noises are i.i.d. and $y_k = Cx_k + v_k$, $f(x_k, y_k|\mathcal{I}_{k-1})$ is GMM as well, i.e.,

$$f(x_k, y_k|\mathcal{I}_{k-1}) = \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^{-} \mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i}),$$

where $\eta_{k,i}$ and $\Phi_{k,i}$ is the mean the covariance of the Gaussian component that is partitioned as

$$\eta_{k,i} = \begin{bmatrix} \eta_{k,i}^{x} \\ \eta_{k,i}^{y} \end{bmatrix} = \begin{bmatrix} \mu_{k,i}^{-} \\ C\mu_{k,i}^{-} \end{bmatrix},$$

$$\Phi_{k,i} = \begin{bmatrix} \Phi_{k,i}^{xx} & \Phi_{k,i}^{xy} \\ (\Phi_{k,i}^{xy})^T & \Phi_{k,i}^{yy} \end{bmatrix} = \begin{bmatrix} \Sigma_{k,i}^{-} & \Sigma_{k,i}^{-}C^T \\ C\Sigma_{k,i}^{-} & C\Sigma_{k,i}^{-}C^T + R \end{bmatrix}.$$

We consider the following three cases:

1) When $\zeta_k = 0$, the joint conditional PDF of x_k and y_k is

$$\begin{aligned} &f(x_k, y_k|\mathcal{I}_k) = f(x_k, y_k|\zeta_k = 0, \mathcal{I}_{k-1}) \\ =&\frac{\mathbb{P}(\zeta_k = 0|x_k, y_k, \mathcal{I}_{k-1}) f(x_k, y_k|\mathcal{I}_{k-1})}{\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})} \\ =&\frac{\mathbb{P}(\zeta_k = 0|y_k) f(x_k, y_k|\mathcal{I}_{k-1})}{\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})} \\ =&\frac{\exp(-\frac{1}{2}y_k^T Y y_k) \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^{-} \mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i})}{\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})}, \end{aligned} \tag{17}$$

where the first equality is obtained from Bayes' theorem, the second equality follows the fact that ζ_k merely depends on y_k, and the third equality follows directly from (2) and the assumption.

Let

$$\theta_{k,i} = y_k^T Y y_k + \left(\begin{bmatrix} x_k \\ y_k \end{bmatrix} - \eta_{k,i} \right)^T \Phi_{k,i}^{-1} \left(\begin{bmatrix} x_k \\ y_k \end{bmatrix} - \eta_{k,i} \right). \tag{18}$$

By manipulating (18) and applying [6, Lemma 1], we obtain

$$\theta_{k,i} = \left(\begin{bmatrix} x_k \\ y_k \end{bmatrix} - \bar{\eta}_{k,i} \right)^T \bar{\Phi}_{k,i}^{-1} \left(\begin{bmatrix} x_k \\ y_k \end{bmatrix} - \bar{\eta}_{k,i} \right) + c_{k,i}, \tag{19}$$

where

$$\bar{\eta}_{k,i} = \begin{bmatrix} \bar{\eta}_{k,i}^{x} \\ \bar{\eta}_{k,i}^{y} \end{bmatrix}, \quad \bar{\Phi}_{k,i} = \begin{bmatrix} \bar{\Phi}_{k,i}^{xx} & \bar{\Phi}_{k,i}^{xy} \\ (\bar{\Phi}_{k,i}^{xy})^T & \bar{\Phi}_{k,i}^{yy} \end{bmatrix},$$
$$c_{k,i} = (\mu_{k,i}^{-})^T C^T (C\Sigma_{k,i}^{-}C^T + R + Y^{-1})^{-1} C\mu_{k,i}^{-},$$

with

$$\begin{aligned} \bar{\eta}_{k,i}^{x} &= \mu_{k,i}^{-} - \Sigma_{k,i}^{-}C^T (C\Sigma_{k,i}^{-}C^T + R + Y^{-1})^{-1} C\mu_{k,i}^{-}, \\ \bar{\eta}_{k,i}^{y} &= [I + (C\Sigma_{k,i}^{-}C^T + R)Y]^{-1} C\mu_{k,i}^{x}, \\ \bar{\Phi}_{k,i}^{xx} &= \Sigma_{k,i}^{-} - \Sigma_{k,i}^{-}C^T (C\Sigma_{k,i}^{-}C^T + R + Y^{-1})^{-1} C\Sigma_{k,i}^{-}, \\ \bar{\Phi}_{k,i}^{xy} &= \Sigma_{k,i}^{-}C^T [I + Y(C\Sigma_{k,i}^{-}C^T + R)]^{-1}, \\ \bar{\Phi}_{k,i}^{yy} &= [(C\Sigma_{k,i}^{-}C^T + R)^{-1} + Y]^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} f(x_k, y_k|\mathcal{I}_k) &= \frac{1}{\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})} \\ &\quad \times \sum_{i=1}^{N_{k-1}} \frac{\alpha_{k,i}^{-} \exp(-\frac{c_{k,i}}{2})}{\sqrt{\det(I + Y\Sigma_{k,i}^{yy})}} \mathcal{N}_{x_k,y_k}(\bar{\eta}_{k,i}, \bar{\Phi}_{k,i}) \\ &= \frac{1}{\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})} \sum_{i=1}^{N_{k-1}} \beta_{k,i}^{0} \mathcal{N}_{x_k,y_k}(\bar{\eta}_{k,i}, \bar{\Phi}_{k,i}). \end{aligned}$$

Since $f(x_k, y_k|\mathcal{I}_k)$ is a PDF,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x_k, y_k|\mathcal{I}_k) dx_k dy_k = 1.$$

Hence,

$$\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1}) = \sum_{i=1}^{N_{k-1}} \beta_{k,i}^{0}. \tag{20}$$

On the other hand, since $\zeta_k = \gamma_k = 0$, $N_k = N_{k-1}$. As a consequence, $f(x_k, y_k|\mathcal{I}_k)$ is a GMM with N_k components:

$$f(x_k, y_k|\mathcal{I}_k) = \sum_{k=1}^{N_k} \frac{\beta_{k,i}^0}{\sum_{i=1}^{N_k} \beta_{k,i}^0} \mathcal{N}_{x_k,y_k}(\bar{\eta}_{k,i}, \bar{\Phi}_{k,i}).$$

Thus, (6)-(8) hold when $\zeta_k = 0$ and $\gamma_k = 0$.

2) When $\zeta_k = 1$ and $\gamma_k = 1$, the estimator successfully receives y_k with no error. Therefore,

$$\begin{aligned}
f(x_k|\mathcal{I}_k) &= f(x_k|y_k, \zeta_k = 1, \gamma_k = 1, \mathcal{I}_{k-1}) \\
&= \frac{\mathbb{P}(\zeta_k = 1, \gamma_k = 1|x_k, y_k, \mathcal{I}_{k-1}) f(x_k|y_k, \mathcal{I}_{k-1})}{\mathbb{P}(\zeta_k = 1, \gamma_k = 1|y_k, \mathcal{I}_{k-1})} \\
&= \frac{\mathbb{P}(\gamma_k = 1)\mathbb{P}(\zeta_k = 1|y_k) f(x_k|y_k, \mathcal{I}_{k-1})}{\mathbb{P}(\gamma_k = 1)\mathbb{P}(\zeta_k = 1|y_k)} \\
&= f(x_k|y_k, \mathcal{I}_{k-1}) \\
&= \frac{f(x_k, y_k|\mathcal{I}_{k-1})}{\mathbb{P}(y_k|\mathcal{I}_{k-1})} \\
&= \frac{\sum_{i=1}^{N_{k-1}} \alpha_{k,i}^- \mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i})}{\sum_{i=1}^{N_{k-1}} \alpha_{k,i}^- \mathcal{N}_{y_k}(\eta_{k,i}^y, \Phi_{k,i}^{yy})}.
\end{aligned}$$

Note that

$$\frac{\mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i})}{\mathcal{N}_{y_k}(\eta_{k,i}^y, \Phi_{k,i}^{yy})} = \mathcal{N}_{x_k}(\tilde{\mu}_{k,i}, \tilde{\Sigma}_{k,i}),$$

where $\tilde{\mu}_{k,i}^x$ and $\tilde{\Sigma}_{k,i}^{xx}$ is computed via conditional Gaussian distribution, i.e.,

$$\begin{aligned}
\tilde{\mu}_{k,i} &= \eta_{k,i}^x + \Phi_{k,i}^{xy}(\Phi_{k,i}^{yy})^{-1}(y_k - \eta_{k,i}^y) \\
&= \mu_{k,i}^- + \Sigma_{k,i}^- C^T (C\Sigma_{k,i}^- C^T + R)^{-1}(y_k - C\mu_{k,i}^-), \\
\tilde{\Sigma}_{k,i} &= \Phi_{k,i}^{xx} - \Phi_{k,i}^{xy}(\Phi_{k,i}^{yy})^{-1}(\Phi_{k,i}^{xy})^T \\
&= \Sigma_{k,i}^- - \Sigma_{k,i}^- C^T (C\Sigma_{k,i}^- C^T + R)^{-1} C\Sigma_{k,i}^-.
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x_k|\mathcal{I}_k) &= \frac{\sum_{i=1}^{N_{k-1}} \alpha_{k,i}^- \mathcal{N}_{y_k}(\eta_{k,i}^y, \Phi_{k,i}^{yy}) \mathcal{N}_{x_k}(\hat{\mu}_{k,i}, \hat{\Sigma}_{k,i})}{\sum_{i=1}^{N_{k-1}} \alpha_{k,i}^- \mathcal{N}_{y_k}(\eta_{k,i}^y, \Phi_{k,i}^{yy})} \\
&= \sum_{i=1}^{N_{k-1}} \frac{\beta_{k,i}^1}{\sum_{i=1}^{N_{k-1}} \beta_{k,i}^1} \mathcal{N}_{x_k}(\tilde{\mu}_{k,i}, \tilde{\Sigma}_{k,i}).
\end{aligned}$$

Since $\zeta_k = \gamma_k = 1$, $N_k = N_{k-1}$. As a result, $f(x_k|\mathcal{I}_k)$ is a GMM with N_k components, each component of which has mean $\tilde{\mu}_{k,i}$ and covariance $\tilde{\Sigma}_{k,i}$. Thus, (6)-(8) hold.

3) When $\zeta_k = 1$ and $\gamma_k = 0$, the measurement y_k is transmitted by the sensor yet not received successfully by the remote estimator. Thus, by substituting (20) into the following equation, the joint conditional PDF of x_k and y_k is given by

$$\begin{aligned}
f(x_k, y_k|\mathcal{I}_k) =& \frac{1 - \exp(-\frac{1}{2} y_k^T Y y_k)}{1 - \sum_{i=1}^{N_{k-1}} \beta_{k,i}^0} \\
& \times \sum_{i=1}^{N_{k-1}} \alpha_{k,i}^- \mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i}),
\end{aligned} \tag{21}$$

where the derivation is obtained by replacing $\mathbb{P}(\zeta_k = 0|y_k)$ and $\mathbb{P}(\zeta_k = 0|\mathcal{I}_{k-1})$ in (17) with $\mathbb{P}(\zeta_k = 1|y_k)$ and $\mathbb{P}(\zeta_k = 1|\mathcal{I}_{k-1})$, respectively. Analogously, by reorganizing (21) and applying [6, Lemma 1], the PDF can be rewritten as

$$\begin{aligned}
f(x_k, y_k|\mathcal{I}_k) =& \frac{1}{1 - \sum_{i=1}^{N_{k-1}} \beta_{k,i}^0} \\
& \times \sum_{i=1}^{N_{k-1}} \left(\alpha_{k,i}^- \mathcal{N}_{x_k,y_k}(\eta_{k,i}, \Phi_{k,i}) - \beta_{k,i}^0 \mathcal{N}_{x_k,y_k}(\bar{\eta}_{k,i}, \bar{\Phi}_{k,i}) \right).
\end{aligned}$$

Since $\zeta_k = 1$ and $\gamma_k = 0$, $N_k = 2N_{k-1}$ and (9)-(14) hold.

Finally, for time update, the distribution of x_{k+1} conditioned on $\mathcal{I}_k$ is

$$\begin{aligned}
f(x_{k+1}|\mathcal{I}_k) &= f(Ax_k + w_k|\mathcal{I}_k) \\
&= \sum_{i=1}^{N_k} \alpha_{k,i} \mathcal{N}_{x_{k+1}}(A\mu_{k,i}, A\Sigma_{k,i}A^T + Q).
\end{aligned}$$

Therefore, (3)-(5) hold and the proof is now complete. ■

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