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A Unified Approach to Communication Delay and Communication Frequency in Distributed State Estimation of Linear Systems

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Abstract— **This paper introduces a novel method to address communication delay and communication frequency in distributed state estimation using a unified structure. Starting from a general type of delay, which can be time-varying, unknown, and different for each channel, two estimation strategies are proposed which use either a unified upper-bound of the delay in all channels, or a sender specific upper-bound in order to improve the performance. The strategies guarantee convergence of the estimation error, and they can be applied to study how communication frequency (or asynchronous communication) affects the convergence rate of the estimation. Thus, the task of determining a necessary communication frequency for obtaining specified performance of distributed estimation is considered, and this task is approached as a special case of the one tailored to delay. Effectiveness of the techniques is confirmed by numerical examples for both types of tasks.**

I. Introduction

Fast developments of cyber-physical systems (CPS) raise new challenges for controller synthesis, such as accounting for the complexity arising from the interaction of many subsystems, or the satisfaction of input and state constraints. For the latter challenge, model predictive control (MPC) has been identified as a suitable strategy, but it often requires that the overall state is known for all subsystems (or are least a subset of neighboring subsystems). If not all states can be measured due to system size, the missing states may be estimated by concepts such as Luenberger observers. However, the estimation error converges only if the networked system is detectable, which turns out to be very restrictive for CPS with many subsystems. To alleviate this requirement, the notion of distributed state estimation, in which each subsystem is only required to measure a part of the overall output, has attracted large interest in recent years.

The pioneering work in [1]–[3] has shown that, even if none of the local subsystems is detectable, the local estimation error can still be stabilized to zero by letting the observers communicate their local estimates. This scheme, however, requires real-time communication between observers. In practice, different types of communication problems may occur, such as intermittent communication restrictions [4], [5], limited communication bandwidth [6], or communication delay [7], [8]. Since communication delays inevitably occur in every channel, providing robustness against such delays has been addressed in several papers on the subject. Existing research on communication delay mainly distinguishes between different types of delay: In [7], the delay is assumed to be **time-varying** but **known** by the receiver. For this case, the authors propose a robust distributed estimation strategy by solving semi-definite programming (SDP) problems for discrete-time linear systems. The work in [9] switched the focus to the continuous-time case, and divided the time domain into different intervals according to **constant** and **known** delay values. Also for the continuous-time case, the work in [10] studied the problem of determining the largest **constant** delay in distributed state estimation for which the estimation is guaranteed to be stable. This result was extended in [8] in order to cope with **time-varying** and **unknown** communication and measurement delay. However, the communication delay there is assumed to be **identical** for all senders. The work in [11] considered a similar setting as in [8], but the main focus was shifted to graph properties for the case of extremely large delays. In the recent work [12], the authors also focused on **unknown** communication delay which is assumed to be **identical** for all channels.

In contrast to previous work, this paper starts from a more general delay setting, namely the case of **unknown**, **time-varying** delays which are **heterogeneous** over different channels with different bounds. To handle such delays, two novel methods are proposed in the first part, one based on a unifying upper delay bound over all channels, the other one working with sender-specific delay bounds. It is shown that both strategies can ensure the convergence of the estimation error in the presence of delay (by solving SDP problems), while the second strategy is less conservative (but more complex) and leads to faster convergence.

In the second part of the paper, a different and important aspect of distributed state estimation is addressed, namely the one of determining an appropriate frequency of communication [6], [13], [14]. Note that communication delay and communication frequency in distributed state estimation are usually treated separately in literature – in the present paper, the latter is regarded as a special case of the former, allowing to handle both problems with a common approach. It is shown that the question of how communication frequency affects the convergence rate can be cast into a problem of estimation under delay with using a common delay bound. In contrast, the question of how the frequency of asynchronous communication affects the convergence rate can be cast into a delay problem with sender-specific delay bounds. The joint solution path for these cases enables addressing the tasks of estimation with delay and of determining the communication frequency simultaneously and in a reliable manner.

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In the next section, the considered class of systems and delays are defined first, before the proposed solution strategy as well as variants to improve its performance are introduced in Sec. 3. The use for the solution of communication frequency problems is described in Sec. 4, and the effectiveness of the technique is evaluated in a series of tests in Sec. 5. The work is concluded in Sec. 6 with an outlook on future directions.

II. Problem Description

This paper considers continuous-time linear systems:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with time $t \in \mathbb{R}^{\geq 0}$, the system state $x(t) \in \mathbb{R}^n$, and the input $u(t) \in \mathbb{R}^m$. The system is equipped with a set $\mathcal{N} = \{1, \ldots, N\}$ of distributed observers, where each local observer $i \in \mathcal{N}$ can measure a part of the system output by:

$$y_i(t) = C_i x(t), \quad y_i(t) \in \mathbb{R}^{p_i}. \tag{2}$$

For this setting, the following are assumed:

Assumption 1: The matrices A and B, as well as the input $u(t)$ in any time t are known to all observers, and the matrix pair (C, A) for $C = [C_1^T, \ldots, C_N^T]^T$ is detectable.

Assumption 2: The local observers can communicate to each other via a strongly connected and directed graph $\mathcal{G}$.

Note that this paper deals with distributed estimation (rather than distributed control), and thus assumes that the input $u(t)$ is selected by a central controller and known by all observers. (Even for a system with single input, a set of distributed estimators may still be necessary to jointly estimate the state in case of distributed sensors.)

The adjacency matrix $A_g \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is determined by letting a_{ij} (the entry in the i-th row and j-th column of A_g) equal to one if the observer j can send information to i, and zero otherwise. Based on A_g, the Laplacian matrix of the graph is determined by $\mathcal{L}_g = D_g - A_g \in \mathbb{R}^{N\times N}$, where $D_g \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$ on the diagonal.

Given the two assumptions, it is known from [1], [2], that gain matrices $L_i \in \mathbb{R}^{n\times p_i}$ and $M_i \in \mathbb{R}^{n\times n}$ exist for which the estimation law:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - M_i \sum_{j\in\mathcal{N}} a_{ij}(\hat{x}_j(t) - \hat{x}_i(t))\end{aligned} \tag{3}$$

allows to update the local estimate $\hat{x}_i(t)$, $i \in \mathcal{N}$, and the estimation error $e_i(t) = x(t) - \hat{x}_i(t)$ converges to zero for $t \to \infty$ for all observers. By using (3), the local estimate $\hat{x}_j(t)$ sent from the observer j must be received by i at the same time t. In practice, however, a delay $\tau_{ji}(t) \geq 0$ of the communication is unavoidable, leading to:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - M_i \sum_{j\in\mathcal{N}} a_{ij}(\hat{x}_j(t - \tau_{ji}(t)) - \hat{x}_i(t)).\end{aligned} \tag{4}$$

The delay $\tau_{ji}(t)$ is assumed to be time-varying, unknown, heterogeneous for different channels, and bounded by:

$$0 \leq \tau_{ji}(t) \leq \tau_{ji,max}. \tag{5}$$

Note that the upper-bounds $\tau_{ji,max}$ can often be determined conservatively based on experiments. In the presence of communication delay, the gain matrices L_i and M_i synthesized for the delay-free case by (3) may fail to ensure the convergence of $e_i(t)$. Thus, a critical question is how to ensure convergence in the presence of delays. In addition, if the input $u(t)$ is determined according to the estimated state, the estimation error $e_i(t)$ should not only be stabilized to zero, but should also show a suitably high convergence rate to avoid negative impact on the control performance.

A. Common Upper Bound of Delays for all Channels

To handle the different delays $\tau_{ji}(t)$ occurring for the channels, a possible first step is to determine a common upper bound $\bar{\tau}$ over all channels:

$$\bar{\tau} := \max_{\forall i,j\in\mathcal{N},\ a_{ij}\neq 0} \tau_{ji,max}. \tag{6}$$

Starting from the initial time $t_0 = 0$, the continuous time domain is divided into intervals based on $\bar{\tau}$, leading to sampling times $(t_0, t_1, \ldots, t_k = k\cdot\bar{\tau}, \ldots)$, $k \in \mathbb{N}\cup\{0\}$. The local observers $i \in \mathcal{N}$ are required to send their local estimate $\hat{x}_i(t)$ to the others in each time t_k, see Fig. 1. Using this scheme, the local estimate $\hat{x}_j(t_{k-1})$ sent from j at t_{k-1} is ensured to be received by i before t_k. Accordingly, a new estimation law based on (4) is proposed for $t \in [t_k, t_{k+1})$:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - M_i \sum_{j\in\mathcal{N}} a_{ij}(\hat{x}_j(t_{k-1}) - \hat{x}_i(t_{k-1})).\end{aligned} \tag{7}$$

Compared to (4), the local estimate $\hat{x}_j(t - \tau_{ji}(t))$ of j is replaced by $\hat{x}_j(t_{k-1})$ in (7), as the latter must have been received by i before t_k. The local estimate $\hat{x}_i(t)$ being subtracted from $\hat{x}_j(t - \tau_{ji}(t))$ in (4) is also replaced by $\hat{x}_i(t_{k-1})$ in (7), in order to work with time-consistent data. By using (7) for (1) and (2), the dynamics of the estimation error $e_i(t)$ for $t \in [t_k, t_{k+1})$ is given by:

$$\dot{e}_i(t) = (A - L_iC_i)e_i(t) - M_i\sum_{j\in\mathcal{N}} a_{ij}(e_j(t_{k-1}) - e_i(t_{k-1})). \tag{8}$$

By collecting the local $e_i(t)$ of all observers into one global vector $e(t) := [e_1^T(t), e_2^T(t), \ldots, e_N^T(t)]^T \in \mathbb{R}^{N\cdot n\times 1}$, the dynamics of $e(t)$ for $t \in [t_k, t_{k+1})$ are governed by:

$$\dot{e}(t) = \tilde{A}e(t) + \tilde{M}\tilde{\mathcal{L}}e(t_{k-1}) \tag{9}$$

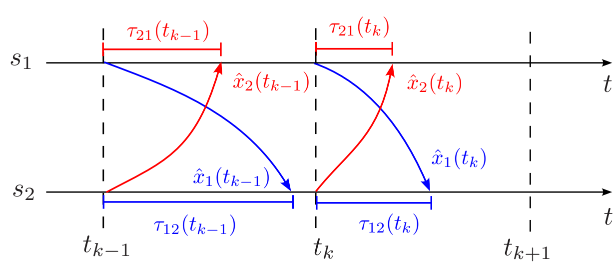

Fig. 1. For two observers s_1 and s_2, the local estimate $\hat{x}_i(t_k)$ is communicated to each other in every sampling time t_k, while the sampling interval $\bar{\tau}$ is selected according to (6).

according to (8), where:

$$\tilde{A} = diag(A - L_1C_1, \ldots, A - L_NC_N) \quad (10)$$

$$\tilde{M} = diag(M_1, \ldots, M_N), \quad \tilde{\mathcal{L}} = \mathcal{L}_g \otimes I_n. \quad (11)$$

Here, I_n represents an $n \times n$ identity matrix and $\otimes$ denotes the Kronecker product of two matrices. Based on (9), the following theorem is proposed, which provides a rule for synthesizing the gain matrices L_i and M_i in (7), such that the local estimation error $e_i(t)$, $i \in \mathcal{N}$ converges to zero with a certain rate despite of the communication delays.

Theorem 1: Given $\bar{\tau}$ from (6) and a constant $\eta \geq 0$, let (symmetric) positive-definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$, matrices $R_0, R_1 \in \mathbb{R}^{(nN)\times(nN)}$, as well as matrices $W_i \in \mathbb{R}^{n\times p_i}$, $Y_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$ exist, which satisfy the following linear matrix inequalities (LMIs):

$$\begin{bmatrix} P\tilde{A}+\tilde{A}^TP+Y\tilde{\mathcal{L}}+\tilde{\mathcal{L}}^TY^T+(4\bar{\tau}e^{2\eta\bar{\tau}}+\eta)P & Y\tilde{\mathcal{L}} \\ \tilde{\mathcal{L}}^TY^T & -\frac{1}{2\bar{\tau}}P \end{bmatrix} \preceq 0 \quad (12)$$

$$\begin{bmatrix} P & \tilde{A}^TP \\ P\tilde{A} & R_0 \end{bmatrix} \succeq 0 \quad (13)$$

$$\begin{bmatrix} P & \tilde{\mathcal{L}}^TY^T \\ Y\tilde{\mathcal{L}} & R_1 \end{bmatrix} \succeq 0 \quad (14)$$

$$P - (R_0 + R_1) \succeq 0 \quad (15)$$

with $P = diag(P_1, \ldots, P_N)$, $Y = P\tilde{M} = diag(Y_1, \ldots, Y_N)$, and $P\tilde{A} = diag(P_1A - W_1C_1, \ldots, P_NA - W_NC_N)$. When using the gain matrices L_i and M_i determined by:

$$L_i := P_i^{-1}W_i, \quad M_i := P_i^{-1}Y_i \quad (16)$$

in the local estimation laws (7) for all $i \in \mathcal{N}$, the global estimation error $e(t)$ in (9) satisfies:

$$||e(t)|| \leq c \cdot e^{-\frac{\eta}{2}(t-t_0)}||e(t_0)|| \quad (17)$$

for a constant $c > 0$ and $t \geq t_0$. □

Proof. First of all, by using the Newton-Leibniz formula, the error dynamics (9) is reformulated into:

$$\dot{e}(t) = (\tilde{A} + \tilde{M}\tilde{\mathcal{L}})e(t) - \tilde{M}\tilde{\mathcal{L}}(e(t) - e(t_{k-1})) \quad (18)$$

$$= (\tilde{A} + \tilde{M}\tilde{\mathcal{L}})e(t) - \tilde{M}\tilde{\mathcal{L}}\int_{t_{k-1}}^{t}(\tilde{A}e(s) + \tilde{M}\tilde{\mathcal{L}}e(t_{k-1}))ds.$$

For $P_i \succ 0$, $i \in \mathcal{N}$, a Lyapunov function candidate:

$$V(e(t)) := e(t)^TPe(t) \quad (19)$$

with $P = diag(P_1, \ldots, P_N)$ of $e(t)$ is determined. It is known that the convergence rate of $||e(t)||$ in (17) holds if:

$$\dot{V}(e(t)) \leq -\eta V(e(t)) \quad (20)$$

applies for any $t \geq t_0$. According to (18), the derivative $\dot{V}(e(t))$ in the time interval $t \in [t_k, t_{k+1})$ is given by:

$$\dot{V}(e(t)) = e^T(t)(P(\tilde{A} + \tilde{M}\tilde{\mathcal{L}}) + (\tilde{A} + \tilde{M}\tilde{\mathcal{L}})^TP)e(t) \quad (21)$$

$$- 2e^T(t)P\tilde{M}\tilde{\mathcal{L}}(\int_{t_{k-1}}^{t}\tilde{A}e(s)ds + (t - t_{k-1})\tilde{M}\tilde{\mathcal{L}}e(t_{k-1})).$$

Based on [15] and the Schur complement of (13), the term $-2e^T(t)P\tilde{M}\tilde{\mathcal{L}}\int_{t_{k-1}}^{t}\tilde{A}e(s)ds$ in (21) is known to satisfy:

$$- 2e^T(t)P\tilde{M}\tilde{\mathcal{L}}\int_{t_{k-1}}^{t}\tilde{A}e(s)ds \leq \int_{t_{k-1}}^{t}e^T(s)\tilde{A}^TPR_0^{-1}P\tilde{A}e(s)ds$$
$$+ (t - t_{k-1})e^T(t)P\tilde{M}\tilde{\mathcal{L}}P^{-1}R_0P^{-1}\tilde{\mathcal{L}}^T\tilde{M}^TPe(t)$$
$$\leq \int_{t_{k-1}}^{t}e^T(s)Pe(s)ds + 2\bar{\tau}e^T(t)P\tilde{M}\tilde{\mathcal{L}}P^{-1}R_0P^{-1}\tilde{\mathcal{L}}^T\tilde{M}^TPe(t) \quad (22)$$

for any positive-definite matrix $R_0 \in \mathbb{R}^{(nN)\times(nN)}$ and for $t \in [t_k, t_{k+1})$. For the term $\int_{t_{k-1}}^{t}e^T(s)Pe(s)ds$ in (22), which equals to $\int_{t_{k-1}}^{t}V(e(s))ds$, it must apply, if (20) holds for any $t \geq t_0$, that:

$$V(e(t)) \leq e^{-\eta(t-t_{k-1})}V(e(t_{k-1})) \quad (23)$$

for $t \in [t_k, t_{k+1})$. A constant $\beta > e^{2\eta\bar{\tau}}$ thus must exist, such that the relation:

$$\beta \cdot V(e(t)) \geq V(e(t_{k-1})) \quad (24)$$

holds for this time interval. Given β, an upper-bound of the term $\int_{t_{k-1}}^{t}e^T(s)Pe(s)ds$ in (22) is provided by:

$$\int_{t_{k-1}}^{t}e^T(s)Pe(s)ds \leq 2\bar{\tau}\beta e^T(t)Pe(t). \quad (25)$$

The term $-2e^T(t)P\tilde{M}\tilde{\mathcal{L}}(t - t_{k-1})\tilde{M}\tilde{\mathcal{L}}e(t_{k-1})$ in (21) can be similarly bounded from above according to the Schur complement of (14) and the constant β in (24) by:

$$- 2e^T(t)P\tilde{M}\tilde{\mathcal{L}}(t - t_{k-1})\tilde{M}\tilde{\mathcal{L}}e(t_{k-1}) \leq$$
$$e^T(t)(2\bar{\tau}(P\tilde{M}\tilde{\mathcal{L}}P^{-1}R_1P^{-1}\tilde{\mathcal{L}}^T\tilde{M}^TP + \beta P))e(t) \quad (26)$$

for any positive-definite matrix $R_1 \in \mathbb{R}^{(nN)\times(nN)}$. Based on (21), (22), (25) and (26), the inequality (20) now holds if:

$$P(\tilde{A} + \tilde{M}\tilde{\mathcal{L}}) + (\tilde{A} + \tilde{M}\tilde{\mathcal{L}})^TP + (4\bar{\tau}\beta + \eta)P$$
$$+2\bar{\tau}P\tilde{M}\tilde{\mathcal{L}}P^{-1}(R_0 + R_1)P^{-1}\tilde{\mathcal{L}}^T\tilde{M}^TP \preceq 0 \quad (27)$$

applies. In addition, due to the continuity of the constant β satisfying (24), it is known that if the inequality:

$$P(\tilde{A} + \tilde{M}\tilde{\mathcal{L}}) + (\tilde{A} + \tilde{M}\tilde{\mathcal{L}})^TP + (4\bar{\tau}e^{2\eta\bar{\tau}} + \eta)P$$
$$+2\bar{\tau}P\tilde{M}\tilde{\mathcal{L}}P^{-1}(R_0 + R_1)P^{-1}\tilde{\mathcal{L}}^T\tilde{M}^TP \preceq 0 \quad (28)$$

holds, there must exist a constant β sufficiently close to $e^{2\eta\bar{\tau}}$, such that (27) also holds (see [16] for more details). Finally, given the constraints (15) and (16), the inequality (28) can be cast into (12) by using the Schur complement, what finishes the proof. □

Theorem 1 can be applied to check if a given convergence rate η is realizable for the unified delay bound $\bar{\tau}$. If the outcome is affirmative, one can incrementally increase η and test the feasibility of the LMIs, until the largest η_{max} is found. The value of η_{max} thus represents the highest convergence rate that can be achieved by (7).

B. Sender-Specific Upper Delay Bounds

Although Theorem 1 provides a distributed estimation law that is robust to delay, the use of a unified delay upper-bound may lead to significant conservativeness, resulting in a small η_{max} value satisfying (12) and (13), thus leading to a small convergence rate. This can be unfavorable, in particular, if e.g. the delays in only a few channels are large, while the delays remain very small in most other channels. In regard to this problem, a new estimation law is now proposed in order to increase the convergence rate, while employing the solution of a similar SDP problem as the one in Theorem 1.

First of all, the largest upper-bound $\bar{\tau}_j$ for all channels from the sending observer j to any of its receivers i is determined:

$$\bar{\tau}_j := \max_{i \in \mathcal{N},\ a_{ij} \neq 0} \tau_{ji,max}. \tag{29}$$

Then, the smallest $\bar{\tau}_j$ over all $j \in \mathcal{N}$ is determined by:

$$\tau^* := \min_{j \in \mathcal{N}} \bar{\tau}_j \tag{30}$$

with the smallest multiple of τ^* as $\sigma_j \in \mathbb{Z}$ satisfying:

$$\bar{\tau}_j \leq \sigma_j \cdot \tau^* \tag{31}$$

for each observer $j \in \mathcal{N}$. Now, the local observers are required to send their local estimate $\hat{x}_j(t)$ after any step τ^*, i.e., at each sampling time $(t_0, t_1, \ldots, t_k, \ldots)$ with a sampling interval of τ^*, see Fig. 2. By using this scheme, the local estimates $\hat{x}_j(t_{k-\sigma_j})$ of the observer j at time $t_{k-\sigma_j}$, must have been received by i at t_k according to (31). A new estimation law for each time interval $t \in [t_k, t_{k+1})$ is thus obtained by:

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad - \sum\nolimits_{j \in \mathcal{N}} a_{ij} M_{ij}(\hat{x}_j(t_{k-\sigma_j}) - \hat{x}_i(t_{k-\sigma_j})). \end{aligned} \tag{32}$$

Compared to (7), two main differences can be noticed in (32), namely: 1.) For a different sender j, the local estimate $\hat{x}_i(t_{k-\sigma_j})$ subtracted from $\hat{x}_j(t_{k-\sigma_j})$ in (32) is also different, and 2.) a common gain matrix M_i in (7) is used by the observer i, while different matrices M_{ij} are adopted in (32), i.e., the local estimates from different senders (with different delays) are treated separately. The dynamics of the local estimation error $e_i(t)$ is thus given by:

$$\begin{aligned} \dot{e}_i(t) &= (A - L_iC_i)e_i(t) \\ &\quad - \sum\nolimits_{j \in \mathcal{N}} a_{ij} M_{ij}(e_j(t_{k-\sigma_j}) - e_i(t_{k-\sigma_j})). \end{aligned} \tag{33}$$

Now, by removing all edges from the graph $\mathcal{G}$ except those emerging from the observer j, a local Laplacian matrix $\mathcal{L}_{g,j}$ can be determined, and the set of such matrices satisfies:

$$\sum\nolimits_{j \in \mathcal{N}} \mathcal{L}_{g,j} = \mathcal{L}_g. \tag{34}$$

Given the matrix $\tilde{A}$ in (10) and by defining:

$$\tilde{M}_j = diag(M_{1j}, \ldots, M_{Nj}), \quad \tilde{\mathcal{L}}_j = \mathcal{L}_{g,j} \otimes I_n \tag{35}$$

for all $j \in \mathcal{N}$, the global error $e(t)$ follows to:

$$\dot{e}(t) = \tilde{A}e(t) + \sum\nolimits_{j \in \mathcal{N}} \tilde{M}_j \tilde{\mathcal{L}}_j e(t_{k-\sigma_j}) \tag{36}$$

for the time interval $t \in [t_k, t_{k+1})$. Based on (36), the following result can be obtained similarly to Theorem 1.

Theorem 2: Given τ^* according to (30) and a constant $\eta \geq 0$, assume that (symmetric) positive-definite matrices $P_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{N}$ and $R_{i,j} \in \mathbb{R}^{(nN) \times (nN)}$, $i \in \mathcal{N}$, $j \in \{0, 1, \ldots, N\}$, as well as matrices $W_i \in \mathbb{R}^{n \times p_i}$, $i \in \mathcal{N}$ and $Y_{ij} \in \mathbb{R}^{n \times n}$, $i, j \in \mathcal{N}$ exist, which satisfy the LMIs:

$$\begin{bmatrix} \Gamma & Y_{[1]}\tilde{\mathcal{L}}_1 & \ldots & Y_{[N]}\tilde{\mathcal{L}}_N \\ \tilde{\mathcal{L}}_1^T Y_{[1]}^T & -\frac{P}{(\sigma_1+1)\tau^*} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \tilde{\mathcal{L}}_N^T Y_{[N]}^T & 0 & 0 & -\frac{P}{(\sigma_N+1)\tau^*} \end{bmatrix} \preceq 0 \tag{37}$$

$$\begin{bmatrix} P & \tilde{A}^T P \\ P\tilde{A} & R_{i,0} \end{bmatrix} \succeq 0, \ \forall i \in \mathcal{N} \tag{38}$$

$$\begin{bmatrix} P & \tilde{\mathcal{L}}_j^T Y_{[j]}^T \\ Y_{[j]}\tilde{\mathcal{L}}_j & R_{i,j} \end{bmatrix} \succeq 0, \ \forall i \in \mathcal{N}, \forall j \in \mathcal{N} \tag{39}$$

$$P - \sum_{j=0}^{N} R_{i,j} \succeq 0, \ \forall i \in \mathcal{N} \tag{40}$$

where:

$$\begin{aligned} \Gamma &:= P\tilde{A} + \sum_{j \in \mathcal{N}} Y_{[j]}\tilde{\mathcal{L}}_j + \tilde{A}^T P + \sum_{j \in \mathcal{N}} Y_{[j]}^T \tilde{\mathcal{L}}_j^T \\ &+ ((\sum_{i \in \mathcal{N}} (\sigma_i + 1)\tau^*(e^{\eta(\sigma_i+1)\tau^*} + \sum_{j \in \mathcal{N}} e^{\eta(\sigma_j+1)\tau^*})) + \eta)P \end{aligned} \tag{41}$$

with $P = diag(P_1, \ldots, P_N)$, $P\tilde{A} = diag(P_1A - W_1C_1, \ldots, P_NA - W_NC_N)$ and $Y_{[j]} = P\tilde{M}_j = diag(Y_{1j}, \ldots, Y_{Nj})$ for all $j \in \mathcal{N}$. With gain matrices L_i and M_{ij} according to:

$$L_i := P_i^{-1} W_i, \quad M_{ij} := P_i^{-1} Y_{ij} \tag{42}$$

in the local estimation laws (32), the global estimation error $e(t)$ in (36) satisfies:

$$||e(t)|| \leq c \cdot e^{-\frac{\eta}{2}(t-t_0)} ||e(t_0)|| \tag{43}$$

for a constant $c > 0$ and $t \geq t_0$. □

The proof of Theorem 2 follows the same path as that of Theorem 1 and is thus omitted. Note that the main reason why the largest η from Theorem 2 can be higher than that from Theorem 1 is due to the circumstance, that for Theorem 2, more gain matrices M_{ij} are synthesized and thus more degrees of freedom are present. This improvement, however, also leads to a higher computational complexity, since a

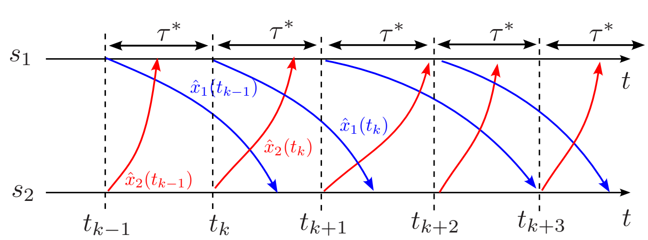

Fig. 2. For two observers s_1 and s_2 with $\bar{\tau}_1 \leq 2\tau^*$ and $\bar{\tau}_2 \leq \tau^*$., the local estimates are exchanged every τ^* time units.

number of N^2+N-2 matrices $R_{i,j}$ and a number of N^2-N matrices Y_{ij} must be synthesized in addition.

III. Communication Frequency Analysis

Even if assuming that delays were not present, one should recognize that the continuous-time estimation rule (3) is an idealized setting, which cannot be applied in this form exactly, as it would require that information is exchanged infinitely often on a bounded time interval. Thus, the use of discretized time for communication, as underlying in the previous section, is mandatory – this raises the question of an appropriate communication frequency and its impact on the convergence rate.

Assume for this section that no delay is present in all channels and that the communication among observers is carried out after any time step $\delta_t (> 0)$, i.e. at sampling times $(t_0, t_1, \ldots, t_k, \ldots)$, $k \geq 0$, see Fig. 3. In this case, the local estimate $\hat{x}_j(t_k)$ sent from the observer j is received by i at the same time, and is held constant by i until t_{k+1}. This leads to the following estimation law for each observer for $t \in [t_k, t_{k+1})$:

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &- M_i \sum\nolimits_{j\in\mathcal{N}} a_{ij}(\hat{x}_j(t_k) - \hat{x}_i(t_k)). \qquad (44)\end{aligned}$$

It should be noticed, that this law is a special case of (7), i.e., the local estimate $\hat{x}_j(t_{k-1})$ of observer j is used on $[t_k, t_{k+1})$ in (7), while the more recent estimate $\hat{x}_j(t_k)$ of j is used in (44) for the same time interval. To this end, Theorem 1 is extended to study the influence of communication frequency on the convergence rate:

Corollary 1: Given the communication interval δ_t and a constant $\eta \geq 0$, let the value $\overline{\tau}$ in the LMIs (12) to (15) be replaced by $\frac{\delta_t}{2}$. If the resulting LMIs have a feasible solution, then the global estimation $e(t)$ in (9) satisfies the convergence rate (17) when using the estimation rule (44). □

This fact results from the similarity of the structures of (7) and (44): Note that $\overline{\tau}$ represents the sampling time in (7), while δ_t is the sampling time in (44). The corollary is obtained if $2\overline{\tau}$ (which refers to the largest value of $t - t_{k-1}$ for $t \in [t_k, t_{k+1})$ and sampling time $\overline{\tau}$) is replaced in (22) and (25) by δ_t (referring to the largest value of $t - t_{k-1}$ for $t \in [t_{k-1}, t_k)$ with sampling time δ_t). Thus, the highest convergence rate η_{max} for the communication frequency (referring to δ_t) can also be determined.

Asynchronous communication results if the observers do not use the same communication frequency, see Fig. 4. This may occur if one or more observers need to communicate with lower rate due to a malfunction, or if event-triggered schemes are applied [17]. To study the influence of asynchronous communication on the convergence rate (with the objective to conclude on measures to enhance the reliability of estimation), the method established by Theorem 2 can be further exploited: Let δ_i, $i \in \mathcal{N}$, denote the different communication intervals (thus different frequencies) of the observers. Then, the smallest interval of all observers:

$$\delta^* := \min_{i\in\mathcal{N}} \delta_i \qquad (45)$$

is determined, as well as the smallest integers $\sigma_i \in \mathbb{Z}$ satisfying: $\delta_i \leq \sigma_i \cdot \delta^*$ for each observer $i \in \mathcal{N}$.

Discretizing the continuous-time domain by use of δ^* leads to sampling times $(t_0, t_1, \ldots, t_k, \ldots)$, where all observers execute their first communication at $t_0 = 0$. An estimation law similar to (32) for the interval $t \in [t_k, t_{k+1})$ is formulated:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &- \sum\nolimits_{j\in\mathcal{N}} a_{ij}M_{ij}(\hat{x}_j(t_{\lfloor \frac{k}{\sigma_j}\rfloor\cdot\sigma_j}) - \hat{x}_i(t_{\lfloor \frac{k}{\sigma_j}\rfloor\cdot\sigma_j})). \quad (46)\end{aligned}$$

In here, $\lfloor \frac{k}{\sigma_j} \rfloor$ denotes rounding $\frac{k}{\sigma_j}$ to the next smaller integer. Given (46), the dynamics of $e_i(t)$ and $e(t)$ can be similarly determined as in (33) and (36), thus leading to LMIs which are very similar to those in Theorem 2. By this procedure, the highest convergence rate η_{max} can be determined by solving the corresponding SDP problem.

In this section, the proposed method is evaluated for a communication delay and frequency problem referring to $n = 4$ observers. The system dynamics in (1) and (2) are parameterized by:

$$A = \begin{bmatrix} 0.1 & -0.2 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ -0.1 & 0 & 0.5 & -1 \\ 0.1 & 0.1 & 0.05 & -0.1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 5 & 2 & 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \ C_4 = \begin{bmatrix} 1 & 0 & -5 & 3 \end{bmatrix}$$

Note that the matrix pair (C_i, A) is not detectable for two out of four observers. The initial estimate of each observer is randomly selected. For an instance of the delay problem,

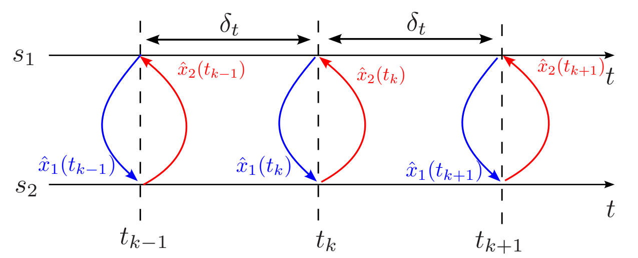

Fig. 3. To analyse the influence of the communication frequency, the local estimation is assumed to be received without delay in each sampling time.

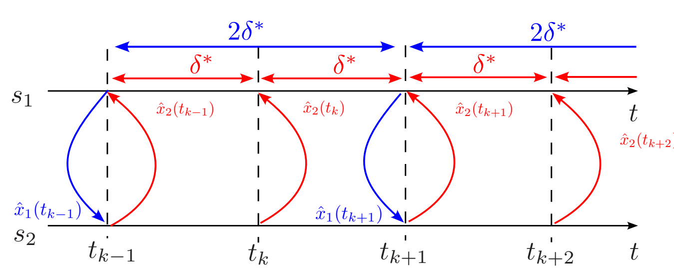

Fig. 4. The local estimate of the observer s_2 is sent to s_1 after time δ^*, and after time $2\delta^*$ from s_1 to s_2.

the upper delay bounds for the senders (29) are selected to: $\bar{\tau}_1 = 0.1,\ \bar{\tau}_2 = \bar{\tau}_3 = 0.01,\ \bar{\tau}_4 = 0.02$. The common upper bound for all channels in (6) is thus $\bar{\tau} = \bar{\tau}_1 = 0.1$. By using the law (7) with $\bar{\tau}$, the highest convergence rate of $e(t)$ obtained from Theorem 1 is $\eta_{max} = 0.414$, see the evolution of $e_i(t)$ in Fig. 5. The local estimation error converges to zero after approximately 15 seconds. By using the law (32) with $\tau^* = 0.01$ from Theorem 2, a significant improvement of the highest convergence rate is observed with $\eta_{max} = 0.927$, see Fig. 6. In this case, the local estimates converge to zero within approximately 8 seconds. For the same system, the influence of the communication frequency on the highest convergence rate is evaluated in Fig. 7.

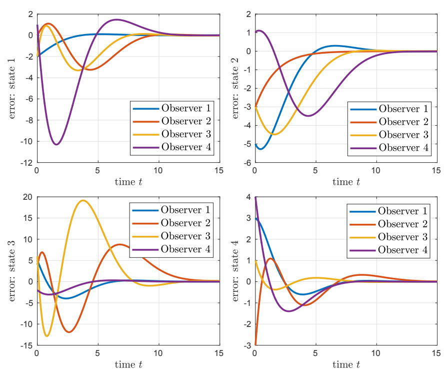

Fig. 5. Evolution of each dimension of the local estimation error $e_{i,[l]}(t)$, $i \in \mathcal{N}$ by using the estimation law (7).

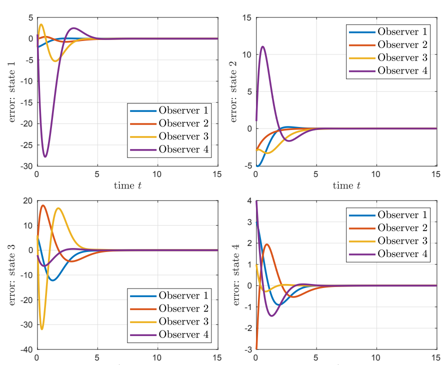

Fig. 6. Evolution of $e_{i,[l]}(t)$, $i \in \mathcal{N}$ by using the estimation law (32).

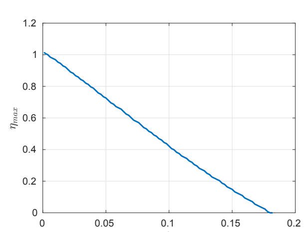

Fig. 7. Relation between the communication frequency determined by δ_t and the highest convergence rate η_{max}.

IV. Conclusions

This paper has addressed two important classes of communication problems for distributed state estimation, namely the robust estimation under communication delay and the choice of the communication frequency. Given a general type of communication delay, the first part of this paper has introduced two efficient methods to synthesize distributed estimation laws with different performances. In the second part, it is shown that the communication frequency problem is a special case of the delay problem, and can thus be studied using very similar principles. Current work aims at extending the proposed structure to nonlinear systems and to fault detection in distributed state estimation [18].

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