2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Optimal Phase Control of Limit-Cycle Oscillators with Strong Inputs through Phase-Amplitude Reduction

Norihisa Namura and Hiroya Nakao

Abstract— **We present a method for optimal phase control of limit-cycle oscillators using strong inputs. Based on the phase-amplitude reduction, which provides a concise representation of the oscillator dynamics, we design an optimal control input that quickly realizes the target phase while keeping the oscillator state close to the original limit cycle by penalizing the amplitude deviations. The derived scheme requires only a single one-dimensional phase equation even for the control of high-dimensional oscillators. We demonstrate the effectiveness of the proposed method by comparing the control performance with other methods, using the van der Pol oscillator and Willamowski-Rössler oscillator as examples.**

I. INTRODUCTION

Synchronization of rhythmic systems (self-sustained oscillators) is widely observed in the real world. It refers to dynamical phenomena in which rhythmic systems, either interacting or acting unilaterally, align their dynamics [1]. When a periodic input is given to the oscillator, *phase locking* to the input can take place. When two or more oscillators interact, mutual synchronization of the oscillators can occur.

Control of synchronization in rhythmic systems is a fundamental topic in various fields of engineering. Examples include power grids [2], frequency tuning or stabilization in electrical oscillators [3], and suppression of pulsus alternans in the heart [4]. Rhythmic systems can be mathematically modeled as limit-cycle oscillators, which are nonlinear dynamical systems with stable limit-cycle trajectories.

For the purpose of synchronization control, the phase reduction theory has been widely employed [5], [6], [7], [8], [9], which is useful for analyzing the dynamics of limit-cycle oscillators subjected to weak inputs. It approximately describes multidimensional dynamics of the oscillator by a one-dimensional equation for the phase defined along the limit cycle. Many studies have been conducted on controlling synchronization dynamics of limit-cycle oscillators based on phase reduction [10], [11], [12], [13], [14], [15]. However, their applicability is restricted to sufficiently weak inputs.

In real-world systems, it is often necessary to control the phase of oscillations without altering their waveforms, e.g., in adjusting the circadian rhythm [7] after a jet lag. Thus, it is important to devise methods for controlling the oscillator phase while keeping the oscillator state in the vicinity of the limit cycle. Recently, the phase reduction theory has been extended to the phase-amplitude reduction theory, which introduces amplitude variables to represent the deviations of the oscillator state from the limit cycle in addition to the phase variable [16], [17], [18], [19]. Using phase-amplitude reduction, the oscillator phase can be controlled more efficiently by applying stronger inputs while suppressing the amplitude variables to keep the oscillator state close to the original limit cycle [19], [20]. For example, in [19], an optimal periodic input for the phase-locking of the oscillator was designed by introducing an averaged penalty for the amplitude deviations in the optimization problem.

In many studies on synchronization, it is assumed that the oscillator receives periodic inputs. However, if the purpose is to steer the oscillator phase, the control input does not need to be periodic. By using a non-periodic input, optimal control of the oscillator phase was formulated by Moehlis *et al.* in [10] using phase reduction, where the input was assumed sufficiently small so that it does not kick the oscillator state away from the limit cycle. Later, Monga and Moehlis [20] generalized the theory to suppress the amplitude deviations from the limit cycle for faster synchronization.

In this study, we propose a new method of optimal phase control that allows stronger inputs by introducing a penalty on the amplitude deviations from the limit cycle. Our formulation differs from the previous methods [19], [20] in that the derived scheme requires only a single phase equation for the control, even for high-dimensional oscillators. This is useful, because the amplitude variables are difficult to evaluate numerically, and also the amplitude equations are more challenging to estimate than the phase equation from time series data when the mathematical models are unavailable [21]. To verify the effectiveness of the proposed control scheme, we compare its performance with the two methods, i.e., optimal control without amplitude penalty and optimal phase locking with averaged amplitude penalty, using the van der Pol and Willamowski-Rössler oscillators as examples.

This paper is organized as follows. We first describe phase-amplitude reduction and Floquet theory in Sec. II. In Sec. III, we propose an optimal control scheme for the oscillator phase with amplitude penalty. In Sec. IV, we show the effectiveness of the proposed method by numerical simulations. We conclude this study in Sec. V.

N. Namura is with the Department of Systems and Control Engineering, Institute of Science Tokyo, Tokyo 152-8552, Japan namura.n.60d9@m.isct.ac.jp (namura.n.aa@m.titech.ac.jp)

H. Nakao is with the Department of Systems and Control Engineering and Research Center for Autonomous Systems Materialogy, Institute of Science Tokyo, Tokyo 152-8552, Japan nakao.h.ee74@m.isct.ac.jp (nakao@sc.e.titech.ac.jp)

We acknowledge the support from JSPS KAKENHI (grant numbers: JP22K11919 and JP22H00516) and JST CREST (grant number: JPMJCR1913). N. Namura acknowledges the support from JST SPRING (grant number: JPMJSP2106) and AIP challenge program from JST CREST.

II. PHASE-AMPLITUDE REDUCTION AND FLOQUET THEORY

In this section, we briefly review phase-amplitude reduction [16], [17], [18], [19] and Floquet theory [22].

A. Phase-Amplitude Reduction

We consider a limit-cycle oscillator described by

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}), \tag{1}$$

where $\boldsymbol{X}(t) \in \mathbb{R}^N$ is the system state at time $t \geq 0$, $\boldsymbol{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth vector field, and $\dot{\boldsymbol{X}}$ is the time derivative of $\boldsymbol{X}$. We assume that the system has an exponentially stable limit cycle χ with a natural period T and frequency $\omega = 2\pi/T$ whose trajectory is represented as $\tilde{\boldsymbol{X}}_0(t)$, which is a T-periodic function of t satisfying $\tilde{\boldsymbol{X}}_0(t+T) = \tilde{\boldsymbol{X}}_0(t)$.

Linear stability of χ is characterized by the Floquet exponents $\{\lambda_n\}_{n=1}^N$ (see below) [22]. Since χ is exponentially stable, one Floquet exponent $\lambda_1 = 0$ and the real part of all other exponents are negative, i.e., $\mathrm{Re}\ \lambda_n < 0$ $(n = 2, 3, \ldots, n)$, where $\{\lambda_n\}_{n=2}^N$ are sorted so that they satisfy $\mathrm{Re}\ \lambda_2 \geq \cdots \geq \mathrm{Re}\ \lambda_N$. We assume that $\{\mathrm{Re}\ \lambda_n\}_{n=2}^N$ characterizing the decay rates of the system state towards χ are of $O(1)$.

We can introduce the asymptotic phase function $\Theta : \mathcal{B} \to [0, 2\pi)$ and amplitude function $R_n : \mathcal{B} \to \mathbb{C}$ $(n = 2, 3, \ldots, N)$ in the basin of attraction $\mathcal{B}$ of χ such that the following relationships hold [16], [17], [18], [19]:

$$\frac{d}{dt}\Theta(\boldsymbol{X}) = \langle \nabla\Theta(\boldsymbol{X}), \boldsymbol{F}(\boldsymbol{X}) \rangle = \omega, \tag{2}$$

$$\frac{d}{dt}R_n(\boldsymbol{X}) = \langle \nabla R_n(\boldsymbol{X}), \boldsymbol{F}(\boldsymbol{X}) \rangle = \lambda_n R_n(\boldsymbol{X}). \tag{3}$$

Here, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{n=1}^N a_n^* b_n : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ represents the scalar product of two complex vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$, where $*$ denotes the complex conjugate. Using the asymptotic phase and amplitude functions, we can define the phase by $\theta = \Theta(\boldsymbol{X})$ and amplitudes by $r_n = R_n(\boldsymbol{X})$ $(n = 2, 3, \ldots, N)$ for the state $\boldsymbol{X} \in \mathcal{B}$, respectively. Here, the phase values 0 and 2π are considered identical. The state of the oscillator on χ can be expressed by $\boldsymbol{X}_0(\theta) = \tilde{\boldsymbol{X}}_0(t = \theta/\omega)$ as a 2π-periodic function of the phase θ, i.e., $\boldsymbol{X}_0(\theta) = \boldsymbol{X}_0(\theta + 2\pi)$. Each amplitude measures the deviation of $\boldsymbol{X}$ from χ and vanishes on χ, i.e., $R_n(\boldsymbol{X}_0(\theta)) = 0$. We note that the amplitude values can be complex because the Floquet exponents can be complex in general.

We now consider that the limit-cycle oscillator receives an external input $\boldsymbol{u}(t) \in \mathbb{R}^N$, which is described by

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}(t)) + \boldsymbol{u}(t). \tag{4}$$

In contrast to the conventional phase reduction, we do not assume the input $\boldsymbol{u}$ to be sufficiently weak. Instead, we assume that each amplitude r_n is of $O(\delta)$ with $0 \leq \delta \ll 1$ even under the effect of $\boldsymbol{u}$, i.e., the deviation of the oscillator state from χ is kept sufficiently small. We can then obtain the following phase-amplitude equations [19]:

$$\dot{\theta}(t) = \omega + \langle \boldsymbol{Z}(\theta(t)), \boldsymbol{u}(t) \rangle + O(\delta/\omega), \tag{5}$$

$$\dot{r}_n(t) = \lambda_n r_n(t) + \langle \boldsymbol{I}_n(\theta(t)), \boldsymbol{u}(t) \rangle + O(\delta/\omega), \tag{6}$$

where $n = 2, 3, \ldots, N$. We can neglect the terms of $O(\delta/\omega)$ if δ/ω is sufficiently small. Here, $\boldsymbol{Z}$ is the phase sensitivity function (PSF) and $\{\boldsymbol{I}_n\}_{n=2}^N$ are the amplitude sensitivity functions (ASFs, also known as isostable sensitivity functions) of χ, which are defined by the gradients of the phase and amplitude functions at $\boldsymbol{X}_0(\theta)$ as $\boldsymbol{Z}(\theta) = \nabla\Theta(\boldsymbol{X})|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta)} \in \mathbb{R}^N$ and $\boldsymbol{I}_n(\theta) = \nabla R_n(\boldsymbol{X})|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta)} \in \mathbb{C}^N$ $(n = 2, 3, \ldots, N)$ for $\theta \in [0, 2\pi)$, respectively. The domains of $\boldsymbol{Z}$ and $\boldsymbol{I}_n$ are extended to $\mathbb{R}$ as 2π-periodic functions, $\boldsymbol{Z}(\theta + 2\pi) = \boldsymbol{Z}(\theta)$ and $\boldsymbol{I}_n(\theta + 2\pi) = \boldsymbol{I}_n(\theta)$, respectively. The PSF should satisfy a normalization condition $\langle \boldsymbol{Z}(\theta), d\boldsymbol{X}_0(\theta)/d\theta \rangle = 1$, while the scales of the ASFs can be chosen arbitrarily. We note that the ASFs can be complex.

B. Floquet Theory

Next, we explain the Floquet theory [22] for characterizing the linear stability of the limit cycle and the relationship of the PSF and ASFs with the left Floquet eigenvectors.

We represent the oscillator state $\boldsymbol{X}(t)$ near the limit cycle χ as $\boldsymbol{X}(t) = \tilde{\boldsymbol{X}}_0(t) + \boldsymbol{y}(t)$. The small variation $\boldsymbol{y}(t)$ approximately obeys a linearized periodic system,

$$\dot{\boldsymbol{y}}(t) = \tilde{\boldsymbol{J}}(t)\boldsymbol{y}(t), \tag{7}$$

where $\tilde{\boldsymbol{J}}(t) \in \mathbb{R}^{N\times N}$ is a T-periodic Jacobian matrix of $\boldsymbol{F}$ at $\boldsymbol{X}_0(\theta(t))$ on χ, i.e., $\tilde{\boldsymbol{J}}(t) = \nabla\boldsymbol{F}(\boldsymbol{X})|_{\boldsymbol{X}=\boldsymbol{X}_0(\theta(t))}$. For the linear periodic system (7), we can define a monodromy matrix $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ satisfying $\boldsymbol{V}(T) = \boldsymbol{V}(0)\boldsymbol{M}$, where $\boldsymbol{V} \in \mathbb{R}^{N\times N}$ is the fundamental matrix of Eq. (7) [8], [22]. Since $\boldsymbol{M}$ is regular, there exists a matrix $\boldsymbol{\Lambda} \in \mathbb{C}^{N\times N}$ such that $\boldsymbol{M} = \exp(\boldsymbol{\Lambda}T)$. From Floquet theory, the fundamental matrix can be expressed as $\boldsymbol{V}(t) = \boldsymbol{R}(t)\exp(\boldsymbol{\Lambda}t)$, where $\boldsymbol{R}(t) \in \mathbb{C}^{N\times N}$ is a T-periodic matrix satisfying $\boldsymbol{R}(t+T) = \boldsymbol{R}(t)$ and $\boldsymbol{R}(0) = \boldsymbol{E}$, and $\boldsymbol{\Lambda} \in \mathbb{C}^{N\times N}$ is a constant matrix. Here, $\boldsymbol{E}$ is an N-dimensional identity matrix.

We next consider the eigensystem of $\boldsymbol{\Lambda}$: $\left\{\lambda_n \in \mathbb{C},\ \boldsymbol{p}_n \in \mathbb{C}^N, \boldsymbol{q}_n \in \mathbb{C}^N\right\}_{n=1}^N$, where $\boldsymbol{\Lambda}_n\boldsymbol{p}_n = \lambda_n\boldsymbol{p}_n$ and $\boldsymbol{\Lambda}_n^\dagger\boldsymbol{q}_n = \lambda_n^\dagger\boldsymbol{q}_n$ for $n = 1, 2, \ldots, N$. Here, $\dagger$ denotes the Hermitian conjugate. Each eigenvalue λ_n is the Floquet exponent, i.e., $\exp(\lambda_n T)$ is the Floquet multiplier, where the principal value is used when λ_n is complex. The eigenvectors $\boldsymbol{p}_k$ and $\boldsymbol{q}_j$ can be bi-orthonormalized to satisfy $\langle \boldsymbol{q}_j, \boldsymbol{p}_k \rangle = \delta_{jk}$ for $j, k = 1, 2, \ldots, N$, where δ_{jk} is the Kronecker delta [19].

We further define the time evolution of $\boldsymbol{p}_n$ and $\boldsymbol{q}_n$ as $\boldsymbol{p}_n(t) = \boldsymbol{R}(t)\boldsymbol{p}_n$ and $\boldsymbol{q}_n(t) = \left(\boldsymbol{R}(t)^\dagger\right)^{-1}\boldsymbol{q}_n$ for $0 \leq t < T$, where we call $\boldsymbol{p}_n(t)$ and $\boldsymbol{q}_n(t)$ the right and left Floquet vectors, respectively. These vectors are T-periodic, i.e., $\boldsymbol{p}_n(t+T) = \boldsymbol{p}_n(t)$ and $\boldsymbol{q}_n(t+T) = \boldsymbol{q}_n(t)$, and satisfy the bi-orthonormality condition $\langle \boldsymbol{q}_j(t), \boldsymbol{p}_k(t) \rangle = \delta_{jk}$ for all t. They can be calculated as the T-periodic solution to the following linear system and its adjoint, respectively [19]:

$$\dot{\boldsymbol{p}}_n(t) = \left(\tilde{\boldsymbol{J}}(t) - \lambda_n \boldsymbol{E}\right) \boldsymbol{p}_n(t), \tag{8}$$

$$\dot{\boldsymbol{q}}_n(t) = -\left(\tilde{\boldsymbol{J}}(t)^\dagger - \lambda_n^\dagger \boldsymbol{E}\right) \boldsymbol{q}_n(t). \tag{9}$$

If we take $\boldsymbol{p}_1(0) = \frac{1}{\omega}\boldsymbol{F}(\boldsymbol{X}_0(0))$, $\boldsymbol{p}_1$ can be represented as

$$\boldsymbol{p}_1(\theta/\omega) = \frac{1}{\omega}\boldsymbol{F}(\boldsymbol{X}_0(\theta)) = \frac{d}{d\theta}\boldsymbol{X}_0(\theta). \tag{10}$$

We also normalize the right Floquet vectors $\{\boldsymbol{p}_m(t)\}_{m=2}^N$ to be of $O(1)$. The PSF and ASFs are represented by the left Floquet eigenvectors as [19]

$$\boldsymbol{Z}(\theta) = \boldsymbol{q}_1(\theta/\omega), \tag{11}$$

$$\boldsymbol{I}_n(\theta) = \boldsymbol{q}_n(\theta/\omega), \quad n = 2, 3, \ldots, N. \tag{12}$$

We note that each ASF $\boldsymbol{I}_n$ is orthogonal to $d\boldsymbol{X}_0(\theta)/d\theta$.

III. OPTIMAL CONTROL WITH AMPLITUDE PENALTY

In this section, we explain the control objective and propose an optimal control method with amplitude penalty.

A. Control Objective

Our control objective is to drive the oscillator phase at a given frequency $\Omega = 2\pi/T_\Omega$ with a given phase shift α $(0 \le \alpha < 2\pi)$ from the reference phase Ωt until the final time $t = t_\mathrm{f}$, while keeping the oscillator state always in the close vicinity of χ. That is, we aim to steer the phase $\theta(t)$ of the oscillator state $\boldsymbol{X}(t)$ towards $\Omega t + \alpha \mod 2\pi$ while suppressing the amplitude variables sufficiently small.

Introducing a relative phase $\phi(t) = \theta(t) - \Omega t \in \mathbb{R}$, we obtain the dynamics of ϕ by neglecting the terms of $O(\delta/\omega)$ from Eq. (5) as

$$\begin{aligned}\dot{\phi}(t) &= \Delta_\Omega + \langle \boldsymbol{Z}(\phi(t) + \Omega t), \boldsymbol{u}(t)\rangle \\ &:= f(\phi(t), \boldsymbol{u}(t), t),\end{aligned} \tag{13}$$

where we defined $\Delta_\Omega = \omega - \Omega$ (frequency mismatch between the controlled and uncontrolled oscillators in the steady state) and denoted the right-hand side as f. In terms of the relative phase $\phi(t)$, our control objective is to realize and maintain $\phi(t) = \alpha \mod 2\pi$ until $t = t_\mathrm{f}$.

B. Cost Function for Proposed Optimal Control

In our method, to penalize the amplitude deviations of the oscillator state from the limit cycle at any moment of time, we do not perform averaging approximation to derive a time-invariant phase equation [5], [6], [19] but use an optimal control scheme that can handle a time-variant phase equation.

We obtain an optimal control input that achieves the control objective while keeping the oscillator state close to χ by introducing the following cost function $\mathcal{J}$:

$$\mathcal{J}[\boldsymbol{u}] = \mathcal{L}_\mathrm{f}(\phi(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} \mathcal{L}(\phi(t), \boldsymbol{u}(t), t)dt, \tag{14}$$

where $\mathcal{L}_\mathrm{f} : \mathbb{R} \to \mathbb{R}$ is the final cost and $\mathcal{L} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the stage cost given by

$$\begin{aligned}&\mathcal{L}(\phi(t), \boldsymbol{u}(t), t) \\ &= \mathcal{L}_\Phi(\phi(t)) + \mathcal{L}_U(\boldsymbol{u}(t)) + w_P P(\phi(t), \boldsymbol{u}(t), t).\end{aligned} \tag{15}$$

Here, $\mathcal{L}_\Phi : \mathbb{R} \to \mathbb{R}$ is a phase cost characterizing the difference of the relative phase ϕ from the control objective, $\mathcal{L}_U : \mathbb{R}^N \to \mathbb{R}$ is an input cost, and $P : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ denotes a penalty for the amplitude deviations from χ with $w_P > 0$ representing its weight.

Specifically, we choose the cost functions as

$$\mathcal{L}_\mathrm{f}(\phi(t_\mathrm{f})) = w_\mathrm{f} \sin^2\left(\frac{\phi(t_\mathrm{f}) - \alpha}{2}\right), \tag{16}$$

$$\mathcal{L}_\Phi(\phi(t)) = w_\Phi \sin^2\left(\frac{\phi(t) - \alpha}{2}\right), \tag{17}$$

$$\mathcal{L}_U(\boldsymbol{u}(t)) = \boldsymbol{u}(t)^\top \boldsymbol{W} \boldsymbol{u}(t), \tag{18}$$

respectively, where w_f, $w_\Phi > 0$ are weight coefficients and $\boldsymbol{W} \in \mathbb{R}^{N\times N}$ is a positive definite matrix. Here, we introduced squared sinusoidal functions, which are minimized when $\phi(t_\mathrm{f})$ or $\phi(t)$ is equal to α in modulo 2π. Our cost functions are different from those used in the previous studies [10], [20] in that we explicitly introduce the stage cost with respect to ϕ to take account of the transient dynamics. We aim to maintain the target oscillation frequency during the control by introducing the stage cost.

The function P is the penalty for the control input $\boldsymbol{u}(t)$ causing amplitude deviations from the state $\boldsymbol{X}_0(\theta(t))$ on χ,

$$\begin{aligned}&P(\phi(t), \boldsymbol{u}(t), t) \\ &= \|\boldsymbol{u}(t)\|^2 - \left\langle \frac{\frac{d}{d\phi}\boldsymbol{X}_0(\phi(t) + \Omega t)}{\left\|\frac{d}{d\phi}\boldsymbol{X}_0(\phi(t) + \Omega t)\right\|}, \boldsymbol{u}(t)\right\rangle^2,\end{aligned} \tag{19}$$

where $\|\cdot\|$ denotes the Euclidean norm. Note that the first argument in the scalar product represents a unit vector tangent to χ, so the right-hand side characterizes the magnitude of the non-tangential components in $\boldsymbol{u}(t)$; it vanishes when $\boldsymbol{u}(t)$ is strictly tangent to χ.

Theorem 1: The perturbations given to the amplitudes r_n $(n = 2, 3, \ldots, N)$ caused by the control input $\boldsymbol{u}(t)$ in Eq. (6) are of $O(\delta/\omega)$ if the penalty P is of $O(\delta^2/\omega^2)$.

Proof: If the penalty function P is of $O(\delta^2/\omega^2)$ for $0 \le \delta \ll 1$, the input $\boldsymbol{u}(t)$ can be represented as

$$\boldsymbol{u}(t) = a\frac{d}{d\theta}\boldsymbol{X}_0(\theta(t)) + \frac{\delta}{\omega}\sum_{m=2}^N b_m \boldsymbol{p}_m(\theta(t)/\omega), \tag{20}$$

where $a \in \mathbb{R}$ and $b_m \in \mathbb{C}$ $(m = 2, \ldots, N)$ are of $O(1)$ and the right Floquet vectors $\{\boldsymbol{p}_m(\theta(t)/\omega)\}_{m=2}^N$ are also of $O(1)$. Here, we used $\boldsymbol{p}_1(\theta(t)/\omega) = d\boldsymbol{X}_0(\theta(t))/d\theta = d\boldsymbol{X}_0(\phi(t) + \Omega t)/d\phi$. For such $\boldsymbol{u}(t)$, using the bi-orthonormality condition, the scalar product of $\boldsymbol{u}(t)$ and $\boldsymbol{I}_n(\theta(t))$ is calculated as

$$\langle \boldsymbol{I}_n(\theta(t)), \boldsymbol{u}(t)\rangle = \frac{\delta}{\omega}b_n \quad (n = 2, \ldots, N), \tag{21}$$

which is $O(\delta/\omega)$. Therefore, the amplitude perturbations caused by $\boldsymbol{u}(t)$ in Eq. (6) are of $O(\delta/\omega)$. Hence, the deviations of the oscillator state from χ remain sufficiently small, provided that the decay rates $\{\mathrm{Re}\ \lambda_n\}_{n=2}^{N}$ are of $O(1)$. ■

Thus, by introducing the penalty function P and keeping it small, we can expect that the control input is almost in the same direction as the original velocity vector at each point on the limit cycle χ even for high-dimensional oscillator, hence we can suppress the amplitude deviation of the oscillator state from the limit cycle and keep the validity of the approximate phase equation while using strong inputs.

C. Optimization

Our optimal control problem is formulated as follows:

$$\begin{aligned}\min_{\boldsymbol{u}} \quad & \mathcal{J}[\boldsymbol{u}] + \int_{t_0}^{t_\mathrm{f}} \mu(t)\left(f(\phi(t), \boldsymbol{u}(t), t) - \dot{\phi}(t)\right) dt \\ \text{s.t.} \quad & \phi(t_0) = \phi_0,\end{aligned} \tag{22}$$

where $\mu(t) \in \mathbb{R}$ is a Lagrange multiplier. Here, we assumed that the initial oscillator state $\boldsymbol{X}(t_0)$ is sufficiently close to the limit cycle χ and we denote the initial relative phase as ϕ_0. Defining the Hamiltonian H by

$$\begin{aligned}& H(\phi(t), \boldsymbol{u}(t), \mu(t), t) \\ &= \mathcal{L}(\phi(t), \boldsymbol{u}(t), t) + \mu(t) f(\phi(t), \boldsymbol{u}(t), t),\end{aligned} \tag{23}$$

we can obtain the following Euler-Lagrange equations for the optimal $\boldsymbol{u}$:

$$\dot{\phi}(t) = f(\phi(t), \boldsymbol{u}(t), t), \quad \phi(t_0) = \phi_0, \tag{24}$$

$$\dot{\mu}(t) = -\frac{\partial H}{\partial \phi}(\phi(t), \boldsymbol{u}(t), \mu(t), t), \quad \mu(t_\mathrm{f}) = \frac{d}{d\phi}\mathcal{L}_\mathrm{f}(\phi(t_\mathrm{f})), \tag{25}$$

$$\frac{\partial H}{\partial \boldsymbol{u}}(\phi(t), \boldsymbol{u}(t), \mu(t), t) = 0, \tag{26}$$

where we only need the phase information when solving the optimization problem although we have derived the optimal control based on phase-amplitude reduction. Since we cannot solve these equations analytically, we discretize $\phi(t)$ and $\boldsymbol{u}(t)$ using $L+1$ grid points with a time interval of $\Delta_t = (t_\mathrm{f} - t_0)/L$ and seek the optimal solution $\{\boldsymbol{u}(t_0 + \ell\Delta_t)\}_{\ell=0}^{L}$ by using a nonlinear optimization tool `fminunc` in MATLAB.

IV. RESULTS

A. van der Pol Oscillator

We first demonstrate the effectiveness of the proposed method by using the van der Pol (vdP) oscillator [23], [24]. The vector field of the vdP oscillator is represented as

$$\boldsymbol{F}(\boldsymbol{X}) = \begin{bmatrix} x \\ \nu(1 - x^2)y - x \end{bmatrix}, \tag{27}$$

where $\boldsymbol{X} = [x\ y]^\top$. Here, we assume that $\nu = 1$. We obtain a limit cycle with a period $T = 6.665$ and natural frequency $\omega = 0.9427$. We discretize the time with an interval of $\Delta_t = 5 \times 10^{-3}$ for numerical integration with the fourth-order Runge-Kutta method, where we used linear interpolation for calculating the inputs at $\{t_0 + (\ell + 1/2)\Delta_t\}_{\ell=0}^{N-1}$ for the numerical integration. We show the trajectory of the limit cycle χ in Fig. 1 (a), velocity on the limit-cycle $d\boldsymbol{X}_0(\theta)/d\theta$ in Fig. 1 (b), and PSF $\boldsymbol{Z}(\theta)$ calculated as the left Floquet eigenvector in Fig. 1 (c).

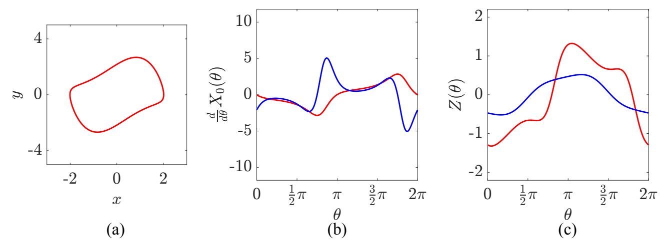

Fig. 1. Limit cycle and PSF of the vdP oscillator. (a) Limit-cycle trajectory χ. (b) Velocity on the limit cycle $d\boldsymbol{X}_0(\theta)/d\theta$ as a function of θ (x and y components). (c) PSF $\boldsymbol{Z}(\theta)$ (x and y components). In (b) and (c), the red and blue curves represent the x and y components, respectively.

Our control objective is to drive the oscillator phase at the frequency $\Omega = \omega + 0.01 = 0.9527$ and period $T_\Omega = 6.595$ (i.e., to drive the relative oscillator phase ϕ at a frequency $\Delta_\Omega = -0.01$) with a phase shift $\alpha = 0$ from the reference phase Ωt until the final time $t_\mathrm{f} = 6T_\Omega$ while keeping the oscillator state close to χ, starting from an initial state $\phi_0 = -\pi/2$ at $t_0 = 0$. We set the weight parameters as $w_\mathrm{f} = 10$, $w_\Phi = 10$, $w_P = 10^2$, and $\boldsymbol{W} = \mathrm{diag}(1, 1)$. For the optimal control without amplitude penalty, we use the same weight parameters but for $w_P = 0$. For the averaged amplitude penalty method, we obtain the optimal T_Ω-periodic input with the same total power as that of the control input within $[t_0, t_\mathrm{f}]$ obtained by the present method.

First, we show the actual optimal control inputs and typical dynamics of the driven oscillator states. Figure 2 shows the optimal inputs obtained by the proposed optimal control with amplitude penalty (a1), by the optimal control without amplitude penalty (a2), and by the averaged amplitude penalty method (a3) and the dynamics of the oscillator states driven by these inputs (b1)-(b3). The oscillator state driven by the optimal input with amplitude penalty has almost no deviation from χ in the whole time domain as shown in Fig. 2 (b1). In contrast, the oscillator state driven by the optimal input without amplitude penalty exhibits large deviations from χ, leading to breakdown of phase-only approximation, as shown in Fig. 2 (b2). The averaged amplitude penalty method also exhibits small amplitude deviations with the penalty parameter $k = 10^4$ (see [19]) as shown in Fig. 2 (b3), because this method evaluates only the averaged amplitude deviations in the phase-locked state, not the deviations at other instants in time, especially during the transient.

Next, we compare the convergence times of the relative phases between proposed method of optimal control with amplitude penalty and the averaged amplitude penalty method. We do not consider the input without amplitude penalty because the oscillator state deviates too largely from the limit cycle and does not synchronize as desired for the parameter values used here. Our proposed method can achieve considerably faster convergence as shown in Fig. 3 (a), where

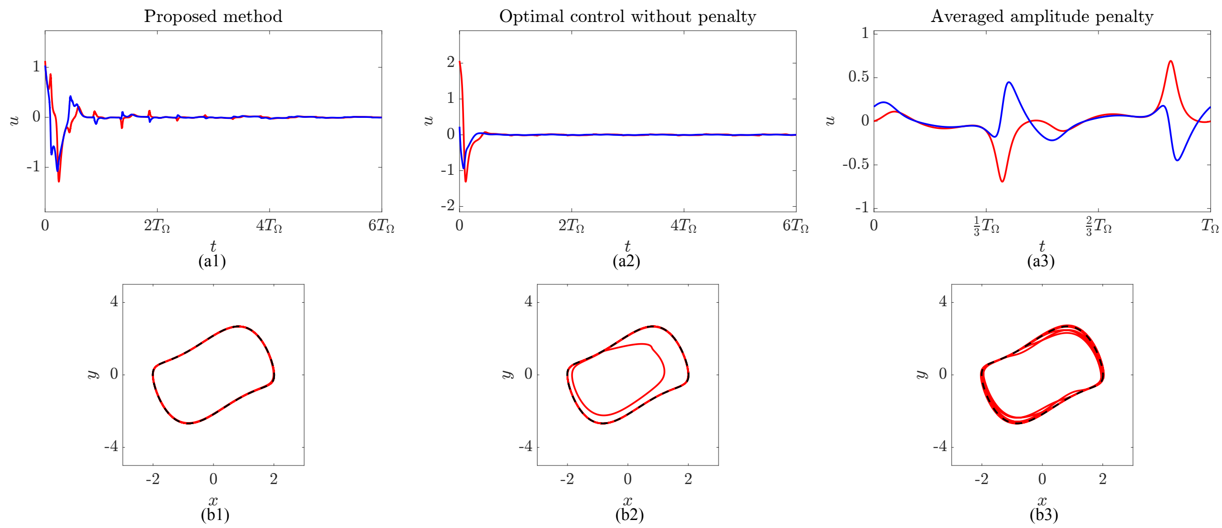

Fig. 2. (a1)-(a3) Optimal input $\boldsymbol{u}(t) = [u_x(t)\ u_y(t)]^\top$ to the vdP oscillator derived by each control method. (a1) Proposed method (optimal control with amplitude penalty). (a2) Optimal control without amplitude penalty. (a3) Averaged amplitude penalty method. In (a3), the optimal T_Ω-periodic input within $[0, T_\Omega]$ is shown. In each figure, the red and blue curves represent u_x and u_y, respectively. (b1)-(b3) Limit cycle and trajectory of the oscillator state of the vdP oscillator driven by each optimal input. (b1) Proposed method. (b2) Optimal control without amplitude penalty. (b3) Averaged amplitude penalty method. In each figure, the red curve shows the trajectory of the oscillator state and the black curve shows the original limit cycle.

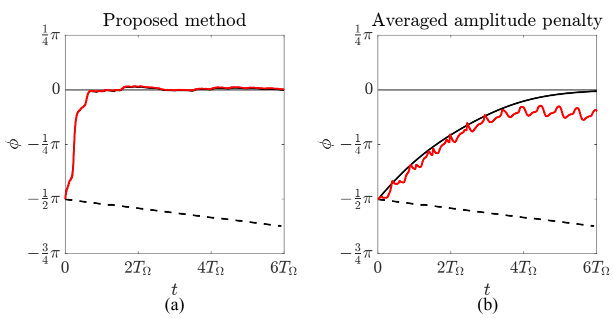

Fig. 3. Dynamics of the relative phases ϕ of the vdP oscillator driven by the inputs obtained by the proposed method (optimal control with amplitude penalty) and the averaged amplitude penalty method. (a) Proposed method. (b) Averaged amplitude penalty method. In (a), the black curve behind the red curve shows the relative phase obtained from the reduced phase equation (13), and the red curve shows the actual relative phase obtained by measuring the phase θ of the oscillator state $\boldsymbol{X}$ driven by Eq. (4); slight discrepancy is due to small errors in the phase-amplitude reduction. In (b), the black curve shows the relative phase obtained from the averaged phase equation (see [19]). In both figures, the dashed lines show the results without control.

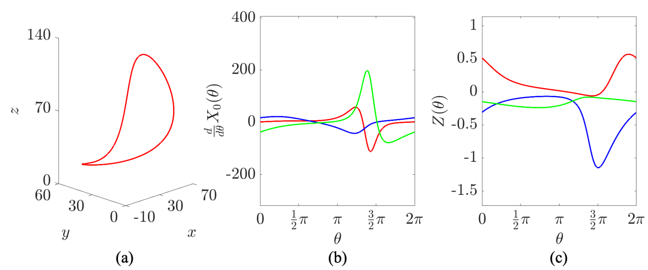

Fig. 4. Limit cycle and PSF of the WR oscillator. (a) Limit-cycle trajectory χ. (b) Velocity on the limit-cycle $d\boldsymbol{X}_0(\theta)/d\theta$ as a function of θ (x, y, and z components). (c) PSF $\boldsymbol{Z}(\theta)$ (x, y, and z components). In (b) and (c), the red, blue, and green curves represent the x, y, and z components, respectively.

the convergence time is about $t = T_\Omega$. By contrast, as shown in Fig. 3 (b), the convergence time predicted by the averaged phase equation (see [19]) in the averaged amplitude penalty method is about $t = 6T_\Omega$, which is much longer than the proposed method. Moreover, the relative oscillator phase in the case of the averaged penalty method shows a considerable discrepancy from the prediction by the averaged phase equation due to relatively large amplitude deviations. Thus, our proposed optimal control with amplitude penalty can control the oscillator phase more efficiently and quickly.

B. Willamowski-Rössler Oscillator

Next, we demonstrate that higher-dimensional oscillators with complex Floquet exponents can also be controlled by the proposed method. As an example, we use the Willamowski-Rössler (WR) oscillator [25], [26]. The vector field of the WR oscillator is represented in the expression of [26] as

$$\boldsymbol{F}(\boldsymbol{X}) = \begin{bmatrix} x(b_1 - d_1 x - y - z) \\ y(b_2 - d_2 y - x) \\ z(x - d_3) \end{bmatrix}, \tag{28}$$

where $\boldsymbol{X} = [x\ y\ z]^\top$. Here, we assume that $b_1 = 80$, $b_2 = 20$, $d_1 = 0.16$, $d_2 = 0.13$, and $d_3 = 16$. We obtain a limit cycle with a period $T = 0.3643$ and natural frequency $\omega = 17.25$. We use the same numerical procedure as before, where the interval of time discretization is $\Delta_t = 10^{-4}$. The Floquet exponents are calculated as $\lambda_1 \simeq 0$, $\lambda_2 = -3.2792 + 4.3283i$, and $\lambda_3 = -3.2792 - 4.3283i$, where i denotes the imaginary unit. Here, λ_2 and λ_3 are complex conjugate, so the associated amplitude variables and ASFs are also complex. We show the trajectory of the limit cycle χ in Fig. 4 (a), velocity on the limit cycle $d\boldsymbol{X}_0(\theta)/d\theta$ in Fig. 4 (b), and PSF $\boldsymbol{Z}(\theta)$ calculated as the left Floquet

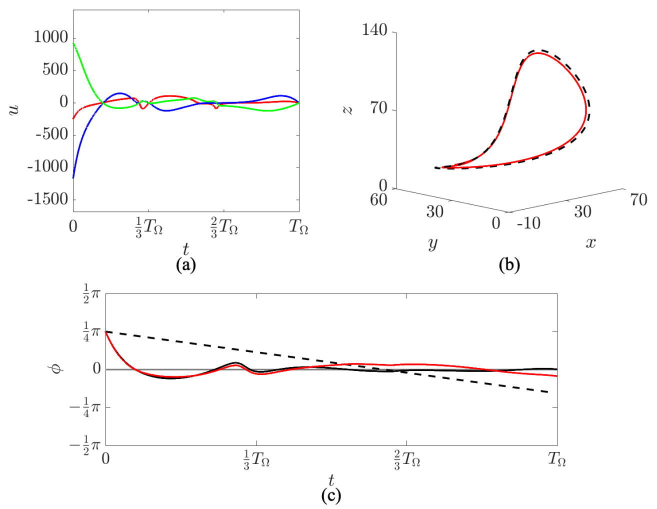

Fig. 5. (a) Optimal input $\boldsymbol{u}(t) = [u_x(t)\ u_y(t)\ u_z(t)]^\top$ to the WR oscillator derived by the proposed method. The red, blue, and green curves represent the x, y, and z component, respectively. (b) Limit cycle and trajectory of the oscillator state of the WR oscillator driven by the optimal input in (a). The red curve shows the oscillator state trajectory and the black curve shows the limit cycle. (c) Dynamics of the relative phase ϕ of the WR oscillator driven by the optimal input in (a). The black curve shows the relative phase ϕ obtained from the phase equation (13), and the red curve shows the relative phase ϕ obtained by measuring the phase θ of the oscillator state $\boldsymbol{X}$ described by Eq. (4); slight discrepancy is due to phase-amplitude approximation. The dashed line shows the result without control.

eigenvector in Fig. 4 (c).

Our control objective here is to steer the relative phase ϕ to $\alpha = 0$ from the initial state $\phi_0 = \pi/4$ and drive the oscillator state at the relative frequency $\Delta_\Omega = -3.5$ with $\phi = \alpha$ (i.e., at the frequency $\Omega = \omega + 3.5 = 20.75$ and $T_\Omega = 0.3028$) by the final time $t_{\mathrm{f}} = T_\Omega$ starting from $t_0 = 0$, while keeping the oscillator state close to the limit cycle. We set the weight parameters as $w_{\mathrm{f}} = 10^3$, $w_\Phi = 10^5$, $w_P = 10$, and $\boldsymbol{W} = \mathrm{diag}(10^{-5}, 10^{-5}, 10^{-5})$. The optimal input obtained by the proposed method is shown in Fig. 5 (a) and the dynamics of the oscillator state is shown in Fig. 5 (b). The relative phase dynamics is plotted in Fig. 5 (c). Although there are slight differences between the phase dynamics obtained from the reduced phase equation (red curve) and from the oscillator states (black curve) due to small deviations from χ, our proposed method can quickly achieve the control objective around $t = T_\Omega/3$.

V. CONCLUSIONS

We proposed an optimal control scheme for steering the phase of limit-cycle oscillators using strong inputs based on phase-amplitude reduction. Introducing a penalty function on the magnitude of the non-tangential components of the control input causing the amplitude deviation of the oscillator state from the limit cycle, we could obtain a control input that quickly realizes the control objective by using only the approximate phase equation. We presented an example in which our proposed method can synchronize the vdP oscillator within about one period of oscillation, which outperforms the previous control methods. We also showed that the proposed method can also control the relative phase of the WR oscillator with complex Floquet multipliers efficiently.

In this study, we assumed that the control input affects all components of the oscillator's state variable. We can reformulate our method for the case where the control input affects only a single component (assuming controllability). Compared to the previous study [20], our formulation provides a simpler algorithm as we only need to solve a 2D two-point boundary value problem, i.e., for the phase equation and its Lagrange multiplier (adjoint), whereas a 4D two-point boundary value problem for the phase and amplitude equations and their adjoints needs to be solved in [20].

One future task is to assess the control performance of the proposed method with additional constraints on the input (e.g., when only one vector component can be controlled) and to compare the performance quantitatively with other methods, particularly with [20]. Another future task is to keep the oscillator state synchronized after the final time by using model predictive control. We can also consider applying our method to the control of ensembles of oscillators, as discussed in [27].

REFERENCES

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A universal concept in nonlinear science. Cambridge University Press, 2001.

[2] F. Dörfler and F. Bullo, "Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators," SIAM Journal on Control and Optimization, vol. 50, no. 3, pp. 1616–1642, 2012.

[3] A. Daryoush, "Optical synchronization of millimeter-wave oscillators for distributed architecture," IEEE Transactions on Microwave Theory and Techniques, vol. 38, no. 5, pp. 467–476, 1990.

[4] D. Wilson and J. Moehlis, "Spatiotemporal control to eliminate cardiac alternans using isostable reduction," Physica D: Nonlinear Phenomena, vol. 342, pp. 32–44, 2017.

[5] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence. Berlin: Springer, 1984.

[6] F. C. Hoppensteadt and E. M. Izhikevich, Weakly Connected Neural Networks. New York: Springer, 1997.

[7] A. T. Winfree, The Geometry of Biological Time. New York: Springer, 2001.

[8] G. B. Ermentrout and D. H. Terman, Mathematical Foundations of Neuroscience. New York: Springer, 2010.

[9] H. Nakao, "Phase reduction approach to synchronisation of nonlinear oscillators," Contemporary Physics, vol. 57, no. 2, pp. 188–214, 2016.

[10] J. Moehlis, E. Shea-Brown, and H. Rabitz, "Optimal inputs for phase models of spiking neurons," Journal of Computational and Nonlinear Dynamics, vol. 1, no. 4, pp. 358–367, 2006.

[11] A. Zlotnik, Y. Chen, I. Z. Kiss, H.-A. Tanaka, and J.-S. Li, "Optimal waveform for fast entrainment of weakly forced nonlinear oscillators," Phys. Rev. Lett., vol. 111, p. 024102, 2013.

[12] V. Novičenko, "Delayed feedback control of synchronization in weakly coupled oscillator networks," Phys. Rev. E, vol. 92, p. 022919, 2015.

[13] B. Monga, D. Wilson, T. Matchen, and J. Moehlis, "Phase reduction and phase-based optimal control for biological systems: a tutorial," Biological cybernetics, vol. 113, no. 1, pp. 11–46, 2019.

[14] Y. Kato, A. Zlotnik, J.-S. Li, and H. Nakao, "Optimization of periodic input waveforms for global entrainment of weakly forced limit-cycle oscillators," Nonlinear Dynamics, vol. 105, no. 3, pp. 2247–2263, 2021.

[15] N. Namura and H. Nakao, "Optimal coupling functions for fast and global synchronization of weakly coupled limit-cycle oscillators," Chaos, Solitons & Fractals, vol. 185, p. 115080, 2024.

[16] A. Mauroy, I. Mezić, and J. Moehlis, "Isostables, isochrons, and Koopman spectrum for the action-angle representation of stable fixed point dynamics," Physica D: Nonlinear Phenomena, vol. 261, pp. 19–30, 2013.

[17] D. Wilson and J. Moehlis, "Isostable reduction of periodic orbits," Phys. Rev. E, vol. 94, p. 052213, 2016.

[18] S. Shirasaka, W. Kurebayashi, and H. Nakao, "Phase-amplitude reduction of transient dynamics far from attractors for limit-cycling systems," Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 27, no. 2, p. 023119, 2017.

[19] S. Takata, Y. Kato, and H. Nakao, "Fast optimal entrainment of limit-cycle oscillators by strong periodic inputs via phase-amplitude reduction and Floquet theory," Chaos: An Interdisciplinary Journal of Nonlinear Science, vol. 31, no. 9, p. 093124, 2021.

[20] B. Monga and J. Moehlis, "Optimal phase control of biological oscillators using augmented phase reduction," Biological Cybernetics, vol. 113, no. 1, pp. 161–178, 2019.

[21] N. Namura, S. Takata, K. Yamaguchi, R. Kobayashi, and H. Nakao, "Estimating asymptotic phase and amplitude functions of limit-cycle oscillators from time series data," Phys. Rev. E, vol. 106, p. 014204, 2022.

[22] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. New York: Springer, 1983.

[23] B. van der Pol, "Forced oscillations in a circuit with non-linear resistance. (Reception with reactive triode)," The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, vol. 3, no. 13, pp. 65–80, 1927.

[24] B. van der Pol and J. van der Mark, "Frequency demultiplication," Nature, vol. 120, no. 3019, pp. 363–364, 1927.

[25] K.-D. Willamowski and O. E. Rössler, "Irregular oscillations in a realistic abstract quadratic mass action system," Zeitschrift für Naturforschung A, vol. 35, no. 3, pp. 317–318, 1980.

[26] R. P. Boland, T. Galla, and A. J. McKane, "Limit cycles, complex Floquet multipliers, and intrinsic noise," Phys. Rev. E, vol. 79, p. 051131, 2009.

[27] A. Zlotnik, R. Nagao, I. Z. Kiss, and J.-S. Li, "Phase-selective entrainment of nonlinear oscillator ensembles," Nature Communications, vol. 7, p. 10788, 2016.