IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Synchronization for Linear Networked Systems Subject to Input and Communication Delays

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***Abstract*— This paper investigates the synchronization problem for generic linear multi-agent systems with known or unknown heterogeneous input and communication delays. We propose two protocols that consist of consensus-based internal controller states and decentralized controllers. This kind of distributed dynamic control methodology is able to circumvent the interactive nature of two delays by translating the synchronization problem of agents into the stability of a set of delay differential equations. We examine the synchronization problem for two distinct cases, namely, known delays and unknown delays. When the delays are known, the stability criteria are satisfied by the feasibility of an input-delay-dependent linear matrix inequality and a communication-delay-dependent coupling strength bound. The margin on the communication delay is dependent not only on the network topology but also on the system matrix, which does not have any eigenvalues with positive real parts. We also develop a distributed dynamic control protocol that can handle unknown input and communication delays, and the stability criteria are realized by using the feasibility of a linear matrix inequality and a positive coupling strength. Synchronization is guaranteed even if the unknown communication delays are arbitrarily large but bounded and the upper bound on the heterogeneous input delays is known. The proposed control methodology guarantees that inaccurate measurements of the actual states of a particular agent will not lead to an irretrievable failure of the mission.**

***Index Terms*— Communication Delay, Input Delay, Linear Matrix Inequality, Linear Multi-agent Systems.**

I. INTRODUCTION

For many years, synchronization of multi-agent systems (MASs) has been a popular research topic due to the fact that it has numerous applications, including biological formations found in swarming insects [1], robot swarms [2], and the application of synchronization to power grids [3]. The term "synchronization problem" refers to the scenario in which the states of an ordinarily large number of agents converge on a common dynamics [4]. The research goals of the control community is to build distributed control protocols such that agents' states converge, resulting in synchronization. Synchronisation of multiple unmanned ground vehicles [5] is an example application of the problem posed in this work, where the goal is to ensure synchronization of the position and velocity states of vehicles as well as their angular position and angular velocity. Consensus is one of the well-known distributive protocols in which, based on a local interaction rule, all participating agents try to reach an agreement which depends on the collective state of the system [6]. In this work, we deploy consensus algorithms to accomplish synchronization for a network of linear systems.

One of the major challenges in the networked systems is determining the criteria for synchronization, which depends on the dynamics of individual agents and the constraints of the interaction network [7]. Authors in [8] construct a feedback coupling to guarantee that the agents will converge exponentially. Li *et al.* [9] propose a distributed observer-based consensus protocol to unify the synchronization of networked systems and establish the concept of synchronization region by utilizing the stability of matrix pencils. Using linear quadratic regulator based state variable feedback control protocol, it is demonstrated in [10] that unbounded synchronization regions can be ensured. It is possible to decouple the design of the synchronization gains from the communication graph structure using the control protocols described in [8]–[10]. This may not be the case if there are delays in the system, which motivates us to establish delay-dependent criteria for synchronization gains. By solving algebraic Riccati equations and utilizing weighting factors that are dependent on the Laplacian, [4] achieves robust synchronization of uncertain MASs through the use of observer-based dynamic protocols. Panteley and Loría [11] show that the synchronization error can be compensated by an increment of the coupling strength of the interconnections. Xu *et al.* [12] focus on synchronizing output rather than synchronizing state because the internal states of the agents may not be comparable to one another. In this work, we confine our investigation to the availability of internal controller (IC) states of neighbouring agents and use Luenberger observers for output synchronization.

Since the communication channel has a limited bandwidth, there is a time delay (communication delay) whenever actual states or controller states are being exchanged with neighbouring agents [13]. MAS may potentially suffer from input delay as the generation of control input for an agent requires some amount of processing time [14]. The interactive nature of communication and input delays is exemplified in [15], where authors investigate stability switches in double-integrator MAS via delay scheduling. De *et al.* [16], [17] focus on the synchronization problem of double-integrators in the presence of a leader and demonstrate how the interactive nature of communication and input delays can be mitigated. In this paper, we adopt a similar strategy and develop distributed dynamic control protocols for high-order linear agents.

In comparison to single- or double-integrator agents,

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synchronization of delayed MAS with high-order dynamic agents is more complex [18]. Wang *et al.* [19] discuss communication delay-dependent synchronization among a group of agents that are, at most, marginally unstable and the dependency of the delay margin on the agent dynamics and communication topology is established. The study investigated in [20] describes the progress made in synchronization for high-order delayed MAS. A dynamic controller based on synchronization protocol is constructed in [21] to incorporate communication delays, and it is shown that this approach has less computational complexity than [19]. The synchronization protocol described in [22] can accommodate arbitrary large communication delays for marginally unstable MAS, with information exchanged only at sampling instants and not continuously. Zhang *et al.* [23] investigate the synchronization problem for continuous- or discrete-time MASs and establish a sufficient condition on the tolerated input delay that depends on agent dynamics but not the network graph. Jiang *et al.* [24] use the generalized Nyquist criterion to develop a distributed observer that allows agents with general linear dynamics to achieve consensus with arbitrary large but bounded input or communication delays. Nevertheless, these studies consider the relative difference between the actual states or outputs of an agent and those of its neighbours. This results in the possibility that the cooperative mission will be unsuccessful due to an imprecise measurement of the states or outputs. This motivates us to construct a distributed dynamic control protocol capable of preventing mission failure due to inaccurate measurements of actual states or outputs. The contributions of our work are:

- Sufficient conditions are obtained to guarantee synchronization of marginally unstable agents with known or unknown delays. We find that the choice of the dynamic control protocol leads to a feasibility check of a linear matrix inequality (LMI) with the same order as the order of the system matrix and the determination of the bound on the coupling strength of the IC state dynamics.
- The proposed synchronization protocols simplify design by avoiding the interactive nature of input and communication delays. This is possible because of the correlation between the bound on the input delay and the feedback gain matrix in the decentralized controller, and the dependence/independence of communication delays is related to the coupling strength of the IC state dynamics.
- As IC state dynamics are decoupled from the decentralized controller, the mission will not fail definitively if the actual states or outputs of a given agent are not accurately measured.

II. Preliminaries and Problem Formulation

Consider a MAS with N agents whose dynamics is

$$\dot{x}_i(t) = Ax_i(t) + Bu_i\left(t - \tau_i^{in}\right), \quad y_i(t) = Cx_i(t), \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are state and control input of i^{th} agent, respectively. The input delay τ_i^{in} is constant. The input delays may be unknown, but the upper bound of input delays $\bar{\tau}^{in} = \max\limits_i \tau_i^{in}$ is known to all agents. We assume that the pairs (A, B) and (A, C) are stabilizable and detectable, respectively. Eigenvalues of A are denoted by $\lambda_l(A)$. The eigenvalues $\lambda_l(A)$ for $l = 1, \ldots, p$ are purely imaginary while $(n - p)$ eigenvalues have negative real parts.

The information flow is represented by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. $\mathcal{A} = [a_{ik}]$ denotes the adjacency matrix, where $a_{ik} > 0$, if edge $e_{k,i} \in \mathcal{E}$ and $a_{ik} = 0$, otherwise. An edge $e_{k,i} \in \mathcal{E}$ means i^{th} agent receives information from k^{th} agent. The set of neighbours of node v_i is $\mathcal{N}_i = \{v_k \in \mathcal{V} : e_{k,i} \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L}_N$ is defined by (self-cycles not allowed)

$$\mathcal{L}_N = (l_{i,k})_{N \times N}, \quad l_{i,k} = \begin{cases} \sum\limits_{k=1, k \neq i}^{N} a_{ik} & k = i \\ -a_{ik} & k \neq i. \end{cases}$$

Agent i receives information about its neighbour k with a delay of $\tau_{ik}^{com} (\geq 0)$, which may be known or unknown.

Assumption 1: [25] The graph $\mathcal{G}$ contains a directed spanning tree, leading to a simple zero eigenvalue for $\mathcal{L}_N$.

Synchronization problem involves developing a distributed control protocol so that eventually the agents will share a common dynamics, i.e., the solutions of individual agents asymptotically synchronize with the solution of $\dot{x}_0(t) = Ax_0(t)$ for some $x_0(0) \in \mathbb{R}^n$. In this work, we construct protocols for achieving both state and output synchronization. Our first goal is to achieve state synchronization for any initial conditions of the agents, i.e., $\forall i, k \in \{1, 2, \ldots, N\}$, $\lim_{t \to \infty} \parallel x_i(t) - x_k(t) \parallel = 0$. Our second objective is to achieve output synchronization among all the agents, i.e., $\forall i, k \in \{1, 2, \ldots, N\}$, $\lim_{t \to \infty} \parallel y_i(t) - y_k(t) \parallel = 0$. Furthermore, We intend to accomplish state and output synchronization in the event of unknown delays.

III. Synchronization in Delayed MAS

In this section, the distributed synchronization protocol is developed to address state synchronization in MAS when input and communication delays are present simultaneously. The following dynamic control protocol has traditionally been used [8], [9], [21] with or without delay:

$$\left.\begin{aligned} \dot{\zeta}_i(t) &= (A + BK)\zeta_i(t) + c \sum_{k \in \mathcal{N}_i} a_{ik} \left(x_i(t - \bar{\tau}^c)\right. \\ &\quad \left. - x_k(t - \bar{\tau}^c) - \zeta_i(t - \bar{\tau}^c) + \zeta_k(t - \bar{\tau}^c)\right) \\ u_i(t) &= K\zeta_i(t). \end{aligned}\right\} \quad (2)$$

Here, $K \in \mathbb{R}^{m \times n}$ is the feedback gain matrix, and $c \in \mathbb{R}$ is the coupling strength. For appropriate design parameters, the IC state $\zeta_i(t)$ converge to zero. In addition to relative delayed IC states, the IC state dynamics also involves relative delayed agent states. The interactive nature of the delays can be avoided by modifying the dynamic control protocol described in [8], [9], [21] appropriately. However, synchronization among agents will not occur if the actual states of an agent contain any measurement errors. Our goal in this work is not only to avoid the interactive nature of input and communication delays but also to provide a control architecture in which, in the event of an inaccurate measurement of an agent's state, only that agent is affected while the other agents synchronize.

To overcome this, the current work employs a distributed dynamic control protocol that entails the exchange of information on the IC states. As a result, inaccurate measurement of the actual state of an agent has no effect on the convergence of the IC states. Consequently, only the agent with inaccurate measurement will be impacted, while the remaining agents will continue to synchronize with one another. Following the works of [16], [17], we have proposed a novel synchronization protocol that can avoid the interaction of input and communication delays and guarantees synchronization between agents with the exception of the agent susceptible to measurement error. The decentralized controller for i^{th} agent takes the form:

$$u_i(t) = K\left(x_i(t) - \zeta_i(t)\right). \tag{3}$$

In this paper, we develop two protocols for dynamic control: the first protocol (Design I) assumes that the delay values are known, while the second protocol (Design II) does not require this information. For both designs, the decentralised controller (3) remains unchanged.

A. Design I: Dynamic Control with Known Delays

When communication delays are known, the states of an agent and its neighbours can be delayed, and the delay differential equations with heterogeneous delays can eventually be transformed into delay differential equations with homogeneous delay $\bar{\tau}^c = \max_{i,k\in\mathcal{I}} \tau_{ik}^{com}$, where $\mathcal{I} = \{1, 2, \dots, N\}$. The dynamics of $\zeta_i(t)$ is

$$\dot{\zeta}_i(t) = A\zeta_i(t) + c\sum_{k\in\mathcal{N}_i} a_{ik}\left(\zeta_k(t-\bar{\tau}^c) - \zeta_i(t-\bar{\tau}^c)\right). \tag{4}$$

We now present Lemma 1 which will be useful to find the stability conditions under the synchronization protocol (3)-(4).

Lemma 1: Consider a delayed linear system

$$\dot{z}(t) = Az(t) + BKz\left(t - \tau^{in}\right) \tag{5}$$

with a feedback gain matrix $K = -B^\top Q^{-1}$, where $Q \succ 0$ is a solution of

$$AQ + QA^\top - 2BB^\top + 2\alpha Q \prec 0 \tag{6}$$

with $\alpha < \alpha_0$, where $-\alpha_0$ is the maximum real part of the uncontrollable modes of A. The system (5) with this feedback gain matrix will be stable for $0 \le \tau^{in} \le \bar{\tau}^{in}$ if there exist $P \succ 0$, $Q_1 \succ 0$, $Q_2 \succ 0$ such that

$$\begin{bmatrix} \Psi & \bar{\tau}^{in}PBK & \bar{\tau}^{in}PBK & \bar{\tau}^{in}A^\top Q_1 & \bar{\tau}^{in}A_d^\top Q_2 \\ \star & -\bar{\tau}^{in}Q_1 & 0 & 0 & 0 \\ \star & \star & -\bar{\tau}^{in}Q_2 & 0 & 0 \\ \star & \star & \star & -\bar{\tau}^{in}Q_1 & 0 \\ \star & \star & \star & \star & -\bar{\tau}^{in}Q_2 \end{bmatrix} \prec 0, \tag{7}$$

where $\Psi = (A+BK)^\top P + P(A+BK)$. Furthermore, there always exist a $\bar{\tau}^{in} > 0$ such that (7) has a feasible solution.

Proof: The dynamics of $z(t)$ can be represented as

$$\dot{z}(t) = (A+BK)\,z(t) - BK\int_{t-\tau^{in}}^{t} Az(\sigma)d\sigma - BK\int_{t-\tau^{in}}^{t} BKz(\sigma-\tau^{in})d\sigma \tag{8}$$

We will proceed based on Lyapunov-Krasovski approach [26]. Let the Lyapunov candidate is

$$V = z^\top Pz + \int_{-\tau^{in}}^{0}\int_{t+\theta}^{t} z^\top(\sigma)A^\top Q_1 Az(\sigma)d\sigma d\theta + \int_{-\tau^{in}}^{0}\int_{t-\tau^{in}+\theta}^{t} z^\top(\sigma)K^\top B^\top Q_2 BKz(\sigma)d\sigma d\theta,$$

where $P \succ 0$, $Q_1 \succ 0$, $Q_2 \succ 0$. Using (8) and according to L-K method, we can determine

$$\begin{aligned}\dot{V} =& z^\top(t)\left[(A+BK)^\top P + P(A+BK)\right]z(t) + \tau^{in}z^\top(t) \\ & \left[A^\top Q_1 A + K^\top B^\top Q_2 BK\right]z(t) - 2\int_{t-\tau^{in}}^{t} z^\top(t)PBK \\ & Az(\sigma)d\sigma - 2\int_{t-\tau^{in}}^{t} z^\top(t)PBKBKz\left(\sigma-\tau^{in}\right)d\sigma \\ & - \int_{t-\tau^{in}}^{t} z^\top(\sigma)A^\top Q_1 Az(\sigma)d\sigma - \int_{t-\tau^{in}}^{t} z^\top\left(\sigma-\tau^{in}\right) \\ & K^\top B^\top Q_2 BKz\left(\sigma-\tau^{in}\right)d\sigma.\end{aligned} \tag{9}$$

Following Lemma 1 in [27], we can express (9) as

$$\begin{aligned}\dot{V} \le& z^\top(t)\left[(A+BK)^\top P + P(A+BK)\right]z(t) + \bar{\tau}^{in}z^\top(t) \\ & \left[A^\top Q_1 A + K^\top B^\top Q_2 BK\right]z(t) + \bar{\tau}^{in}z^\top(t)PBKQ_1^{-1} \\ & K^\top B^\top Pz(t) + \bar{\tau}^{in}z^\top(t)PBKQ_2^{-1}K^\top B^\top Pz(t).\end{aligned} \tag{10}$$

Hence, $\dot{V} < 0$ if the following inequality is satisfied

$$(A+BK)^\top P + P(A+BK) + \bar{\tau}^{in}X \prec 0, \tag{11}$$

where $X = A^\top Q_1 A + K^\top B^\top Q_2 BK + PBKQ_1^{-1}K^\top B^\top P + PBKQ_2^{-1}K^\top B^\top P$). When $\bar{\tau}^{in} = 0$, (11) becomes $(A+BK)^\top P + P(A+BK) \prec 0$. This inequality can also be written as

$$Q(A+BK)^\top + (A+BK)Q \prec 0, \quad Q \succ 0. \tag{12}$$

Following Lemma 3 in [28], as (A, B) pair is stabilizable, there exist a matrix $Q \succ 0$ such that

$$AQ + QA^\top - 2BB^\top + 2\alpha Q \prec 0, \quad \alpha > 0. \tag{13}$$

with $\alpha < \alpha_0$, where $-\alpha_0 < 0$ is the maximum real part of the uncontrollable modes of A, if any. Note that $\alpha > 0$ can be arbitrarily large if pair (A, B) is controllable. Let us choose $K = -B^\top Q^{-1}$. With this feedback gain matrix, (11) simplifies to $AQ + QA^\top - 2BB^\top \prec 0, \quad Q \succ 0$, which always has a solution as per (13). Therefore, there always exist a $\bar{\tau}^{in} > 0$ such that (11) has a feasible solution. The matrix $K = -B^\top Q^{-1}$ can be computed from (13). After obtaining K, we use the Schur-complement to express (11) as LMI (7). ■

Remark 1: The LMI (7) is checked only after solving the LMI (6) a priori. The design of the matrix K is realized through solving (6). Several techniques, such as pole placement, can be used to design K.

Remark 2: The L-K functional employed in Lemma 1 may be more conservative than recent work [29]. However, the objective of this research is to achieve a guaranteed feasible solution for the LMI, which is doable with the selected L-K functional. We prove that there is always a $\bar{\tau}^{in} > 0$ such that (7) can be solved.

We will now investigate the delay-dependent criteria required to accomplish state synchronization. Although the problem in consideration is a two-delay problem, the architecture of the proposed synchronization protocol simplifies its design.

Theorem 1: Consider a MAS with linear dynamics (1). State synchronization is accomplished with the distributed dynamic control protocol (3)-(4), if $\bar{\tau}^c \in \left[0, \frac{\frac{\pi}{2}+arg(\lambda_i(\mathcal{L}_N))}{\max\limits_{l=1,\dots,p} \beta_l}\right)$, and coupling strength is bounded as

$$0 < c < \min_{i=2,\dots,N} \frac{1}{|\lambda_i(\mathcal{L}_N)|} \left\{ \min_{l=p+1,\dots,n} |Re(\lambda_k(A))|, \left(\frac{\pi}{2} + arg(\lambda_i(\mathcal{L}_N)) - \max_{l=1,\dots,p} \beta_l \bar{\tau}^c\right) / \bar{\tau}^c \right\}, \quad (14)$$

and (6)-(7) hold. Here, β_l, $l = 1,\dots,p$ are the absolute values of eigenvalues of A located on the imaginary axis. The feedback gain matrix K is constructed as $K = -B^\top Q^{-1}$, where $Q \succ 0$ is a solution of (6).

Proof: Let us define $\zeta = \left[\zeta_1^\top\ \zeta_2^\top \dots \zeta_N^\top\right]^\top$. With the synchronization protocol (3)-(4), the dynamics of IC states can be expressed as

$$\dot{\zeta}(t) = [I_N \otimes A]\,\zeta(t) - c\,[\mathcal{L}_N \otimes I_n]\,\zeta(t - \bar{\tau}^c). \quad (15)$$

We denote $r^\top$ as the left eigenvector of $\mathcal{L}_N$ associated with zero eigenvalue such that $r^\top \mathbf{1}_N = 1$, where $\mathbf{1}_N = \underbrace{[1, 1, \dots, 1]}_{N-\text{times}}^\top$. There always exists an invertible matrix $M \in \mathbb{C}^{N\times N}$ such that $J = M^{-1}\mathcal{L}_N M = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \Delta \end{bmatrix}$, where diagonal elements of Δ are given by $\lambda_i(\mathcal{L}_N)$, $i = 2,\dots,N$. We define $\epsilon(t) = [\epsilon_1^\top, \epsilon_2^\top, \dots, \epsilon_N^\top]^\top = \left[M^{-1} \otimes I_n\right]\left[\left(I_N - \mathbf{1}_N r^\top\right) \otimes I_n\right]\zeta(t)$. Using (15), time-derivative of $\epsilon(t)$ can be derived as

$$\dot{\epsilon}(t) = [I_N \otimes A]\left[M^{-1} \otimes I_n\right]\left[\left(I_N - \mathbf{1}_N r^\top\right) \otimes I_n\right]\zeta(t) - c\left[\left(M^{-1}\mathcal{L}_N - M^{-1}\mathbf{1}_N r^\top \mathcal{L}_N\right) \otimes I_n\right]\zeta(t - \bar{\tau}^c). \quad (16)$$

As $r^\top \mathcal{L}_N = 0$, (16) can be simplified as

$$\dot{\epsilon}(t) = [I_N \otimes A]\,\epsilon(t) - c\left[\left(M^{-1}\mathcal{L}_N\right) \otimes I_n\right]\zeta(t - \bar{\tau}^c). \quad (17)$$

Using the fact $\mathcal{L}_N \mathbf{1}_N = 0$, (17) can be expressed as

$$\dot{\epsilon}(t) = [I_N \otimes A]\,\epsilon(t) - c\,[J \otimes I_n]\,\epsilon(t - \bar{\tau}^c). \quad (18)$$

From the construction of the variable $\epsilon(t)$, it is evident that $\epsilon_1(t) = r^\top\left(I_N - \mathbf{1}_N r^\top\right)\zeta(t)$. Since $r^\top \mathbf{1}_N = 1$,

$$\epsilon_1(t) = \left(r^\top - r^\top\right)\zeta(t) = 0_{N\times 1}. \quad (19)$$

We define a new variable $\bar{\epsilon}(t) = [\epsilon_2^\top, \epsilon_3^\top, \dots, \epsilon_N^\top]^\top$. From (18) and (19), dynamics of $\bar{\epsilon}(t)$ can be found as

$$\dot{\bar{\epsilon}}(t) = [I_{N-1} \otimes A]\,\epsilon(t) - c\,[\Delta \otimes I_n]\,\bar{\epsilon}(t - \bar{\tau}^c). \quad (20)$$

According to Theorem 1 in [21], the delayed system (20) is stable if the coupling strength is bounded as (14). The variable $\lim\limits_{t\to\infty} \epsilon(t) = 0$ if and only if $\lim\limits_{t\to\infty} \zeta_i(t) = \lim\limits_{t\to\infty} \zeta_k(t)$, $\forall i, k \in \{1, 2, \dots, N\}$. Therefore, if c satisfies (14), $\zeta_i(t)$ synchronizes to a common trajectory $\zeta_0(t)$, where $\dot{\zeta}_0(t) = A\zeta_0(t)$ for some $\zeta_0(0) \in \mathbb{R}^n$. Let $\delta_i(t) = x_i(t) - \zeta_i(t)$. The derivative of $\delta_i(t)$ yields

$$\dot{\delta}_i(t) = A\delta_i(t) + BK\delta_i\left(t - \tau_i^{in}\right) - c\sum_{k\in\mathcal{N}_i} a_{ik}\left(\zeta_k(t - \bar{\tau}^c) - \zeta_i(t - \bar{\tau}^c)\right). \quad (21)$$

As $\zeta(t)$ asymptotically converges to $\mathbf{1}_N \otimes \zeta_0(t)$, $\delta_i(t)$ will decay to zero if the system $\dot{\delta}_i(t) = A\delta_i(t) + BK\delta_i\left(t - \tau_i^{in}\right)$ is stable. Following Lemma 1, $\delta_i(t)$ will converge to zero if (6)-(7) hold. Thus, $x_i(t)$, $\forall i$, synchronize to a solution of $\dot{x}_0(t) = Ax_0(t)$ for some $x_0(0) \in \mathbb{R}^n$. ■

Remark 3: The stability conditions for the individual decentralized controller given by (6)-(7) are independent of the network topology whereas the coupling strength depends on $\mathcal{L}_N$. The stability conditions remain effective when the number of agents increases since we do not need to compute the zeros of quasipolynomial functions. The sole issue in this situation is calculating the eigenvalues of the $\mathcal{L}_N$, which is not as difficult as finding the zeros of quasipolynomial functions.

Remark 4: With the aid of a Luenberger observer, we can expand the concept of state synchronization to output synchronization. The IC state dynamics (4) remains unchanged and the control input for i^{th} agent is modified as $u_i(t) = K\left(x_i^{ob}(t) - \zeta_i(t)\right)$. The observer state for i^{th} agent has the following dynamics:

$$\dot{x}_i^{ob}(t) = Ax_i^{ob}(t) + BK\left(x_i^{ob}\left(t - \tau_i^{in}\right) - \zeta_i\left(t - \tau_i^{in}\right)\right) + L(C\zeta_i(t) - y_i(t))$$

The dynamics of $\tilde{x}_i(t) = x_i(t) - x_i^{ob}(t)$ can be evaluated as $\dot{\tilde{x}}_i(t) = (A + LC)\tilde{x}_i(t)$. Since pair (A, C) is detectable, there always exists a L such that $(A + LC)$ is Hurwitz. Therefore, $\lim\limits_{t\to\infty} \tilde{x}_i(t) = 0$, $\forall i$. The observer states finally converge to the actual states of agents. Consequently, the problem of output synchronization is reduced to the state synchronization. Therefore, the design of coupling strength c and feedback gain matrix is identical to Theorem 1.

B. Design II: Dynamic Control with Unknown Delays

For real world applications, the information on the heterogeneous communication delays may not be known. Hence, we cannot cast the synchronization problem with heterogeneous delays in a homogeneous delay platform. For unknown delays, we propose a control protocol in which the decentralized controller for i^{th} agent is given by (3) and the IC state dynamics for i^{th} agent is modified as:

$$\left.\begin{aligned} \dot{\zeta}_i(t) &= A\zeta_i(t) + ce^{At}\sum_{k\in\mathcal{N}_i} a_{ik}\left(\omega_k(t - \tau_{ik}^{com}) - \omega_i(t)\right) \\ \dot{\omega}_i(t) &= c\sum_{k\in\mathcal{N}_i} a_{ik}\left(\omega_k(t - \tau_{ik}^{com}) - \omega_i(t)\right), \end{aligned}\right\} \quad (22)$$

where $\omega_i \in \mathbb{R}^n$. Now, we will discuss the conditions for achieving state synchronization with the dynamic control protocol (3)-(22).

Theorem 2: Consider a MAS with linear dynamics (1). State synchronization is accomplished with the distributed dynamic control protocol (3)-(22), if the initial condition of the IC states are designed as

$$\phi_i(\theta) = e^{A\theta}\omega_i(\theta), \quad \forall i, \quad \theta \in [-\bar{\tau}^c, 0], \tag{23}$$

the coupling strength $c > 0$, and (6)-(7) hold. The feedback gain matrix K is constructed as $K = -B^\top Q^{-1}$, where $Q \succ 0$ is a solution of (6).

Proof: The proof for the convergence of this protocol is along the lines of [30]. According to Theorem 1 in [31], $\lim_{t\to\infty} \omega_i(t) = \bar{\omega}$ for $c > 0$. As none of the eigenvalues has positive real part and $\lim_{t\to\infty} \omega_i(t) = \bar{\omega}$, the virtual state $\phi_i(t) = e^{At}\omega_i(t)$ will synchronize to a common solution $\phi_0(t)$, where $\dot{\phi}_0(t) = A\phi_0(t)$ for some $\phi_0(0) \in \mathbb{R}^n$. We can derive the dynamics of $\phi_i(t)$ as

$$\dot{\phi}_i(t) = A\phi_i(t) + ce^{At}\textstyle\sum_{k\in\mathcal{N}_i} a_{ik}\left(\omega_k(t-\tau_{ik}^{com}) - \omega_i(t)\right). \tag{24}$$

Note that the dynamics of $\zeta_i(t)$ and $\phi_i(t)$ given by (22) and (24), respectively are similar. Therefore, if the initial conditions of $\zeta_i(t)$ and $\phi_i(t)$, $\forall i$ are kept same, then $\zeta_i(t)$ and $\phi_i(t)$ will be identical. Therefore, $\zeta_i(t)$ synchronize to a common trajectory $\zeta_0(t)$, where $\dot{\zeta}_0(t) = A\zeta_0(t)$ for some $\zeta_0(0) \in \mathbb{R}^n$ if we choose $\phi_i(\theta) = e^{A\theta}\omega_i(\theta), \quad \theta \in [-\bar{\tau}^c, 0]$. Let us define $\delta_i(t) = x_i(t) - \zeta_i(t)$. The derivative of $\delta_i(t)$ yields $\dot{\delta}_i(t) = A\delta_i(t) + BK\delta_i\left(t - \tau_i^{in}\right) - ce^{At}\sum_{k\in\mathcal{N}_i} a_{ik}\left(\omega_k(t-\tau_{ik}^{com}) - \omega_i(t)\right)$. As $\lim_{t\to\infty}\omega_i(t) = \bar{\omega}$ for $c > 0$, $\delta_i(t)$ will decay to zero if the system $\dot{\delta}_i(t) = A\delta_i(t) + BK\delta_i\left(t-\tau_i^{in}\right)$ is stable. Following Lemma 1, $\delta_i(t)$ will converge to zero if (6)-(7) hold. Therefore, $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty}\zeta_i(t)$, $\forall i$. We already establish that $\zeta_i(t)$ will synchronize to a common trajectory ζ_0, where $\dot{\zeta}_0(t) = A\zeta_0(t)$. Thus, $x_i(t)$, $\forall i$, synchronize to a solution of $\dot{x}_0(t) = Ax_0(t)$ for some $x_0(0) \in \mathbb{R}^n$. ■

Remark 5: For output synchronization, the IC state dynamics (22) is kept unaltered, and the control input and observer state for i^{th} agent can be designed as done in Remark 4. From Remark 4, the rest of the proof follows. Note that the construction of the observer states necessitates knowledge of the heterogeneous input delays. In the event that these delays are not known, we need to make an estimate of the input delays [32]. Hence, the problem of output synchronization will become challenging.

IV. Simulations

Adapted from [33], we consider a group of 3 harmonic oscillators whose dynamics is given by

$$\dot{x}_{i1} = x_{i2}, \quad \dot{x}_{i2} = -x_{i1} + u_i, \quad i = 1, 2, 3.$$

In [33], the control input for i^{th} agent is formulated as $u_i(t) = K\sum_{k\in\mathcal{N}_i} a_{ik}\left(x_k\left(t-\tau_k\right) - x_i\left(t-\tau_i\right)\right)$. The feedback gain matrix K is designed as $K = [0 \quad -1]$ and the delays are taken as $\tau_1 = 0$, $\tau_2 = \frac{\pi}{15}$, and $\tau_3 = \frac{2\pi}{15}$. Initial conditions of the harmonic oscillators are: $x_1(0) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $x_2(0) = (0, 1)$, $x_3(0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The trajectories of agent states x_{i1} for $i = 1, 2, 3$ are shown in Fig. 1(a).Figure 1(a) demonstrates that a synchronization error has occurred due to the heterogeneity of the delays. In our protocols (Design I and Design II), we have kept all the parameters same as [33]. Hence, the input delays $\tau_i^{in} = 0$, $\forall i$, and the communication delays are $\tau_{12}^{com} = \frac{\pi}{15}$, $\tau_{21}^{com} = 0$, $\tau_{23}^{com} = \frac{2\pi}{15}$, $\tau_{32}^{com} = \frac{\pi}{15}$. As there is no input delays, we can choose $K = [0 \quad -1]$ which ensures that $(A + BK)$ becomes Hurwitz. According to Theorem 1, communication delay margin is determined as 1.1072. Hence, $\bar{\tau}^c$ can be chosen as $\frac{2\pi}{15}$ for Design I. Following Theorem 1, the coupling strength c is selected as 0.2. The states of the agents x_{i1} for $i = 1, 2, 3$ synchronize approximately in 10 seconds. as can be seen from Fig. 1(b). For Design II, we keep all the design parameters same as Design I, which satisfies the conditions in Theorem 2. The states of agents x_{i1} for $i = 1, 2, 3$ are shown in Fig. 1(c). Figure 1(c) shows that agents synchronize approximately in 15 seconds.

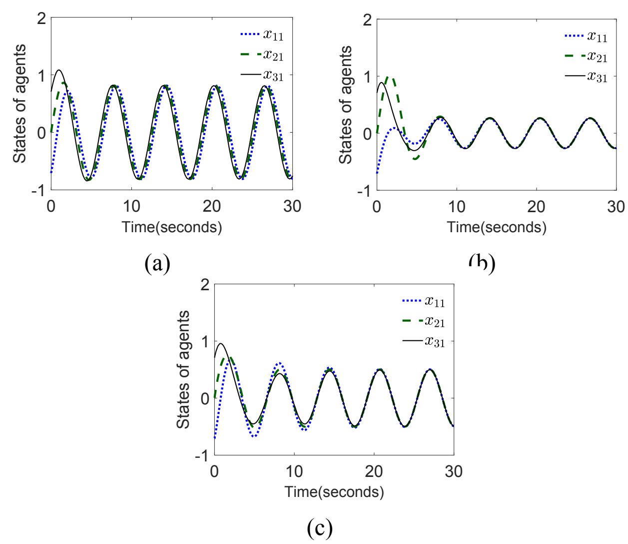
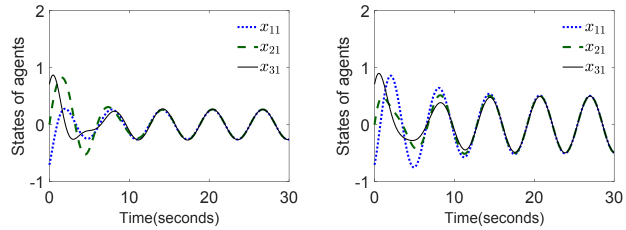


Fig. 1: *Synnchronization of agent states $x_{i1}(t)$ with heterogeneous communication delays: (a) following protocol in [33] (b) present protocol for known delays (c) present protocol for unknown delays.*

Next, we focus on synchronization with heterogeneous input delays together with the communication delays. The initial condition of agents, communication topology, and the communication delays are kept same as the previous example. The input delays are taken as $\tau_1^{in} = 0.1$, $\tau_1^{in} = 0.5$, and $\tau_1^{in} = 0.3$. For $\sigma = 0.1$, (6) gives a feasible solution of $Q \succ 0$, from which K can be computed as $K = [-0.2586 \;\; -0.8222]$. The states of the agents $x_{i1}(t)$ for Design I and Design II are shown in Figs. 2(a) and 2(b), respectively. It can be observed that synchronization occurs for both the designs. The same conclusion can be drawn from the agent state trajectories x_{i2} for $i = 1, 2, 3$, which is omitted due to space constraints.



(a) (b)

Fig. 2: *Synnchronization of agent states $x_{i1}(t)$ with heterogeneous delays: (a) Design I (b) Design II.*

V. Conclusions

We examine the synchronization problem in generic linear MASs with unknown heterogeneous input and communication delays. The proposed protocol employs a distributed dynamic control protocol which aids to avoid the interactive nature of two delays. This kind of approach ensures that the mission will not fail entirely, even if measurements of a given agent's actual states are imprecise. Our result can be extended to determine the conditions for network-based control when sampling and delays are present. In the future, it will be interesting to explore how the proposed framework can handle arbitrarily large yet bounded heterogeneous delays. Although it is achievable via predictor control [34], the inclusion of integral term increases computing complexity.

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