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Relaxed Feasibility and Stability Criteria for Flexible-step MPC

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Abstract— **We provide extensions to the new flexible-step model predictive control (MPC) scheme, which is based on the idea of generalized discrete-time control Lyapunov functions. These facilitate the implementation of a flexible number of control inputs in each iteration of the MPC scheme. We present relaxed recursive feasibility and stability results and provide a converse Lyapunov result. These results combined simplify the design of the flexible-step MPC scheme. We demonstrate the capabilities of the flexible-step MPC algorithm for a nonholonomic system, where the standard one-step implementation may suffer from lack of asymptotic convergence.**

Index Terms— **Predictive control for nonlinear systems, Lyapunov methods, stability of nonlinear systems.**

I. Introduction

Recent advances in performance of optimization algorithms, along with the increased availability of computational power, have made model predictive control (MPC) one of the most utilized-in-practice branches of control theory. In order to deal with some key challenges, such as certificates for guaranteed stability, most MPC algorithms at their core heavily rely on imposing terminal costs and/or terminal constraints to achieve a strict decay of the Lyapunov function (optimal value function) along the trajectories of a system. In addition to potentially degenerating the performance of MPC, terminal conditions can lead to major technical issues, for instance when dealing with nonholonomic systems [11]. With the exception of linear-quadratic control, where the solution of the Riccati equation comes to rescue, it is also unclear how such terminal conditions need to be selected and imposed. There is a large literature devoted to the issues highlighted above, which we are unable to review here, but we point out that several modifications have been proposed to overcome some of these challenges.

The classical Lyapunov function with its classical decay condition was first replaced by higher-order Lyapunov functions (continuous time) [14] and later in discrete-time by non-monotonic Lyapunov functions [15]. In general, MPC approaches that incorporate a Lyapunov function decay constraint in the optimization problem are called Lyapunov-based [12], [13]. Most relevant to our work, so-called "finite-step" approaches [5], [6] instead consider the implementation of a finite number of steps, while inheriting stability from finite-step control Lyapunov functions. Notably, these functions may increase along the trajectories of a system, but are guaranteed to decrease compared to the function value at the beginning of the prediction. For instance, in [6], finite-step Lyapunov functions are used together with finite-step contractive sets, which are motivated by the stabilization of periodically time-varying systems [10] and allow the states of a system to leave the set for only a finite number of steps before returning to it.

Another approach worthy of mentioning here is called "flexible-step" [7], [8], where a flexible number of elements of the optimal input sequence is implemented. This flexible number is deduced from the prediction horizon, which is an integer decision variable in this scheme.

The computation of a controlled invariant set to be used as the terminal set is still a challenge in much of the literature on MPC. Putting linear systems aside, there are mainly two approaches to tackle this challenge. One where the machinery of terminal sets is avoided since stability can be obtained without a terminal constraint, if a sufficiently long prediction horizon is used [9] and another where the terminal set is relaxed by using the above-mentioned contractive sets.

The purpose of this work is to build on the new flexible-step MPC, an approach which revisits the core idea from Lyapunov theory that has led to the existing MPC schemes. Even though it has its roots in optimization theory, the requirement for ensuring stability has restricted the "exploration" capabilities of most MPC schemes, a fundamental feature of efficient optimization algorithms. The idea of flexible-step MPC, extensively outlined in [1], is to replace the strict descent requirement of the Lyapunov function values with an average descent through the novel notion of generalized Lyapunov functions, while still preserving stability. This way, our algorithm allows for the implementation of a flexible number of elements of the optimal input sequence. In [1], we already shed light on some of the benefits that come with this novel MPC scheme, in particular when working with nonholonomic systems where standard MPC may suffer from lack of stability guarantees. The focus of the current work is to present a collection of key relaxed stability and feasibility results in comparison to [1], which allow us to subsume finite-step MPC techniques, while demonstrating the added benefits of using our flexible-step MPC protocol. We also provide a converse Lyapunov result, and show the performance and versatility of our algorithm on a simulation example of navigating a nonholonomic system in an environment with obstacles.

The paper is organized as follows. We explain our notation in Section I-A. We give a summary of the flexible-step MPC scheme in Section II. The main results can be found in

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Section III and IV, covering feasibility, stability and converse results. In Section V, we apply the flexible-step MPC scheme to a nonholonomic system and present the numerical results. We discuss our future directions in Section VI.

A. Notation

We denote the set of non-negative (positive) integers by $\mathbb{N}$ ($\mathbb{N}_{>0}$), the set of (non-negative) reals by $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$), and the interior of a subset $S \subseteq \mathbb{R}^n$ by $\operatorname{int} S$. The set of real matrices of the size $p \times q$ is denoted by $\mathbb{R}^{p,q}$. We make use of boldface when considering a sequence of finite vectors, e.g. $\mathbf{u} = [u_0, \ldots, u_{N-1}] \in \mathbb{R}^{p,N}$, we refer to jth component by u_j and to the subsequence going from component i to j by $\mathbf{u}_{[i:j]}$. Similarly, $\mathbf{U}_{[0:N-1]} \subseteq \mathbb{R}^{p,N}$ denotes a set of (ordered) sequences of vectors with components zero to $N-1$. When such a set depends on the initial state x, it is expressed by $\mathbf{U}_{[0:N-1]}(x)$. As usual, $\mathbf{u}^*$ denotes the solution to the optimal control problem solved in the iteration of an MPC scheme, the symbol $\mathbf{u}^{*-}$ denotes the optimal control sequence of the *previous* iteration. For later use, we recall that a function $V : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ is called positive definite if $V(x, \mathbf{u}) = 0$ is equivalent to $(x, \mathbf{u}) = (0, \mathbf{0})$ and $V(x, \mathbf{u}) > 0$ for all $(x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^{p,q} \setminus \{(0, \mathbf{0})\}$. Furthermore, a function $\alpha : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ is called radially unbounded if $\|(x, \mathbf{u})\| \to \infty$ implies $\alpha(x, \mathbf{u}) \to \infty$, where $\|\cdot\|$ is the Euclidean norm. Note that x is a vector and $\mathbf{u}$ is a matrix here, so with $\|(x, \mathbf{u})\|$ we implicitly refer to the norm of $[x^T, \operatorname{vec}(\mathbf{u})^T]$, where $\operatorname{vec}(\mathbf{u})$ is the usual vectorization of a matrix into a column vector. We now provide the definitions of some comparison functions. A positive definite function $\overline{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ ($\overline{\alpha} \in \mathcal{K}$) if it is continuous and strictly increasing. It is of class-$\mathcal{K}_\infty$ ($\overline{\alpha} \in \mathcal{K}_\infty$) if $\overline{\alpha} \in \mathcal{K}$ and also $\overline{\alpha}(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $s \geq 0, \beta(\cdot, s) \in \mathcal{K}$ and for each $r \geq 0, \beta(r, \cdot)$ is decreasing with $\beta(r, s) \to 0$ as $s \to \infty$.

In the setting of MPC, it is important to distinguish between predictions and the actual implementations at a given time index $k \in \mathbb{N}$. In particular, we use x^k, u^k to refer to *predictions*, whereas we use $x(k), u(k)$ to refer to the *actual* states and *implemented* inputs, respectively.

II. Flexible-step MPC Scheme

The flexible-step MPC scheme introduced in [1] can handle nonlinear discrete-time control systems of the form

$$x^{k+1} = f(x^k, u^k), \tag{1}$$

where $k \in \mathbb{N}$ denotes the time index, $x^k \in X \subseteq \mathbb{R}^n$ is the state with the initial state $x^0 \in X$ and $u^k \in U \subseteq \mathbb{R}^p$ is an input. The state and input constraints satisfy $0 \in \operatorname{int} X$, $0 \in \operatorname{int} U$ and $f : X \times U \to \mathbb{R}^n$ is continuous with $f(0,0) = 0$. Before we can describe the flexible-step MPC scheme itself, we introduce the notion of g-dclfs.

Definition 1 (Set of Feasible Controls): For $x \in X$, we define a set of feasible controls as

$$\mathbf{U}_{[0:N-1]}(x) := \{\mathbf{u}_{[0:N-1]} \in \mathbb{R}^{p,N} : \text{ with } x^0 = x, \quad u_j \in U, x^{j+1} = f(x^j, u_j) \in X, j = 0, \ldots, N-1\}. \tag{2}$$

An infinite sequence of control inputs is called feasible when equation (2) is fulfilled for all times $j \in \mathbb{N}$.

Definition 2 (g-dclf): Consider the control system (1). Let $m \in \mathbb{N}_{>0}$ and $q \in \mathbb{N}$. We call $V : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ a generalized discrete-time control Lyapunov function of order m (g-dclf) for system (1) if V is continuous, positive definite and additionally:

When $q = 0$:

i) there exists a continuous, radially unbounded and positive definite function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that for any $x^0 \in \mathbb{R}^n$ we have $V(x^0) - \alpha(x^0) \geq 0$;

ii) for any $x^0 \in \mathbb{R}^n$ there exists $\boldsymbol{\nu}_{[0:m-1]} \in \mathbf{U}_{[0:m-1]}(x^0)$, which steers x^0 to some x^m, such that

$$\tfrac{1}{m}(\sigma_m V(x^m) + \cdots + \sigma_1 V(x^1)) - V(x^0) \leq -\alpha(x^0), \tag{3}$$

where $\sigma_m, \ldots, \sigma_1 \in \mathbb{R}_{\geq 0}$ and

$$\tfrac{1}{m}(\sigma_m + \sigma_{m-1} + \ldots + \sigma_1) - 1 \geq 0. \tag{4}$$

When $q \neq 0$:

i') there exists a continuous, radially unbounded and positive definite function $\alpha : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ such that for any $(x^0, \mathbf{u}_{[0:q-1]}) \in \mathbb{R}^n \times \mathbf{U}_{[0:q-1]}(x^0)$ we have

$$V(x^0, \mathbf{u}_{[0:q-1]}) - \alpha(x^0, \mathbf{u}_{[0:q-1]}) \geq 0; \tag{5}$$

ii') for any $(x^0, \mathbf{u}_{[0:q-1]}) \in \mathbb{R}^n \times \mathbf{U}_{[0:q-1]}(x^0)$ there exists $\boldsymbol{\nu}_{[0:q+m-1]} \in \mathbb{R}^{p,q+m}$ with $\boldsymbol{\nu}_{[l:q+l-1]} \in \mathbf{U}_{[0:q-1]}(x^l)$ for every $l \in \{0, 1, \ldots, m\}$, which steers x^0 to some x^m, such that

$$\tfrac{1}{m}(\sigma_m V(x^m, \boldsymbol{\nu}_{[m:q+m-1]}) + \cdots + \sigma_1 V(x^1, \boldsymbol{\nu}_{[1:q]})) - V(x^0, \mathbf{u}_{[0:q-1]}) \leq -\alpha(x^0, \mathbf{u}_{[0:q-1]}), \tag{6}$$

where $\sigma_m, \ldots, \sigma_1 \in \mathbb{R}_{\geq 0}$ satisfy (4).

The interpretation of Definition 2 is that the sequence of Lyapunov function values decreases on average every m steps (condition (6)) and not at every step like a classical Lyapunov function. As we will see later, this allows us to relax the assumptions on the terminal conditions used in standard MPC schemes. Definition 2 also allows for a g-dclf V to depend on the state *and* on a q-long control sequence.

We now utilize the concept of g-dclfs to present a novel MPC scheme. Whereas the inequality (5) is satisfied for all $x^0 \in \mathbb{R}^n$ and all $\mathbf{u}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}$, Definition 2 only guarantees the *existence* of $\boldsymbol{\nu}_{[0:q+m-1]}$ satisfying (6). These controls need to be found through an optimization problem. More precisely, the flexible-step MPC scheme is based on the optimal control Problem 3, where the condition (6) has been added as a constraint. The fraction in (6) (or (3) in the case $q = 0$) represents a weighted average, which is why we will refer to this constraint as the average decrease constraint (adc). Recall that such finite-horizon problems in MPC schemes are solved in a receding horizon fashion. The q-long control strategy of the *previous* iteration is denoted by $\mathbf{u}^{*-}_{[0:q-1]}$. To be precise, $\mathbf{u}^{*-}_{[0:q-1]} =$

$\mathbf{u}^*_{[\ell_{\text{decr}}:q+\ell_{\text{decr}}-1]}$, where $\mathbf{u}^*_{[\ell_{\text{decr}}:q+\ell_{\text{decr}}-1]}$ was the optimal control strategy with the index of descent ℓ_{decr} of the g-dclf in that previous iteration. The adc constraint guarantees, that there exists an index $1 \leq \ell_{\text{decr}} \leq m$ at which a descent of the g-dclf is achieved, that is $V(x^{*\ell_{\text{decr}}}, \mathbf{u}^*_{[\ell_{\text{decr}}:q+\ell_{\text{decr}}-1]}) - V(x, \mathbf{u}^{*-}_{[0:q-1]}) \leq -\alpha(x, \mathbf{u}^{*-}_{[0:q-1]})$. Otherwise, a contradiction arises [1, Proposition 3.1]. The current state is $x = x(k)$ and the decision variables are $[u^0, u^1, \ldots, u^{N-1}, x^1, \ldots, x^{N_p}]$, where $N = \max\{q+m, N_p\}$. This means if $q+m > N_p$ we will consider control inputs which go beyond the prediction horizon and help us decrease V on average.

Problem 3: Choose the following parameters a-priori: $N_p \in \mathbb{N}, q \leq N_p$ with $q \in \mathbb{N}, m \leq N_p$ with $m \in \mathbb{N}_{>0}, X^{N_p} \subseteq X \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} X^{N_p}, \sigma_m, \ldots, \sigma_1 \in \mathbb{R}_{\geq 0}$ satisfying (4), a positive semi-definite function $\phi : \mathbb{R}^n \to \mathbb{R}$ and a g-dclf $V : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ of order m. Solve the finite-horizon optimal control problem (7), where the function $f_0 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is positive definite and $0 \in \operatorname{int} U$, $U \subseteq \mathbb{R}^p$.

$$
\begin{aligned}
\min \sum_{j=0}^{N_p-1} & f_0(x^j, u^j) + \phi(x^{N_p}) \\
\text{s.t. } & x^{j+1} = f(x^j, u^j),\ j = 0, \ldots, N_p - 1, \\
& x^0 = x, \\
& u^j \in U,\, x^j \in X,\, x^{N_p} \in X^{N_p},\ j = 0, \ldots, N_p - 1 \\
& [u^l, \ldots, u^{l+q-1}] \in \mathbf{U}_{[0:q-1]}(x^l) \text{ for } l = 0, 1, \ldots, m \\
& \frac{1}{m} \sum_{j=1}^{m} \sigma_j V(x^j, [u^j, \ldots, u^{j-1+q}]) - V(x^0, \mathbf{u}^{*-}_{[0:q-1]}) \\
& \qquad \leq -\alpha(x^0, \mathbf{u}^{*-}_{[0:q-1]})
\end{aligned}
\tag{7}
$$

The flexible-step MPC scheme is given in Algorithm 1. Note that within Algorithm 1, Problem 3 needs to be solved. Like in any MPC scheme, any suitable solver for this nonlinear constrained optimization problem can be used. Similar to standard [2] or finite-step MPC [5] certain design parameters need to be chosen a-priori in order to use Algorithm 1. We anticipate that the flexibility, coming from our novel scheme, simplifies the design of said parameters.

Algorithm 1 Flexible-step MPC scheme

1: set $k = k_0$, measure the initial state $x(k_0)$ and choose an arbitrary $\mathbf{u}^{*-}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}(x(k_0))$
2: measure the current state $x(k)$ of (1)
3: solve Problem 3 with $x = x(k)$ and obtain the optimal input $\mathbf{u}^*_{[0:N-1]}$, where $N = \max\{q+m, N_p\}$
4: choose an index $1 \leq \ell_{\text{decr}} \leq m$ for which $V(x^{*\ell_{\text{decr}}}, \mathbf{u}^*_{[\ell_{\text{decr}}:q+\ell_{\text{decr}}-1]}) - V(x, \mathbf{u}^{*-}_{[0:q-1]}) \leq -\alpha(x, \mathbf{u}^{*-}_{[0:q-1]})$
5: implement $\mathbf{u}^*_{[0:\ell_{\text{decr}}-1]} =: [c^k_{mpc}, \ldots, c^{k+\ell_{decr}-1}_{mpc}]$ and redefine $\mathbf{u}^{*-}_{[0:q-1]} := \mathbf{u}^*_{[\ell_{\text{decr}}:q+\ell_{\text{decr}}-1]}$
6: increase $k := k + \ell_{\text{decr}}$ and go to 2

To explain the idea behind our implementation, we borrow the function values generated in Section V by Algorithm 1 for the g-dclf $V(x) = \|x\|$ with $m = 6$. Fig. 1 focuses on the evolution of the Lyapunov function values between time step 24 and 44 for the sake of clarity. At $k = 24$, Problem 11 is solved with the initial state $x(24)$. We obtain a trajectory of predicted states $(x^0, x^1, x^2, x^3, x^4, x^5, x^6)$ and the corresponding Lyapunov function values seen in Fig. 1. By definition of adc, at least one of the six future Lyapunov function values will be strictly less than $V(x(24))$, Algorithm 1 finds the index that achieves that descent in step 4, here, it chooses the index with the greatest descent. In step 5, we implement the optimal control sequence until said descent of the Lyapunov function values occurs. In Fig. 1, the minimal Lyapunov function value occurs at $V(x^6)$, which is displayed in red, thus, we implement the optimal control sequence for six time steps and the actual states are $x(25) = x^1, \ldots, x(30) = x^6$. The new initial state for the finite-horizon optimal control problem becomes $x(30)$, with the corresponding Lyapunov function value circled in black. The arrows in Fig. 1 indicate the process of making the current actual state the new initial state for the optimization, during which time is frozen. We repeat this process and decide to implement the control sequence for five steps, $x(31) = x^1, \ldots, x(35) = x^5$. The predicted state x^6, whose Lyapunov function value is colored in gray, is discarded in this optimization instance. We define the new initial state and proceed as before. The greatest descent is achieved after five time steps, see Fig. 1. We observe that the Lyapunov function value along the actual states is not monotonically decreasing.

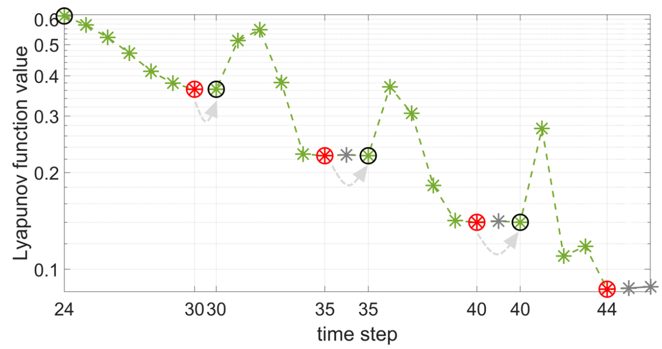

Fig. 1. The Lyapunov function value between the time steps 24 and 44 according to the solution of Problem 11

III. Feasibility and Stability Results

In this section, we will discuss stability results with g-dclfs. To this end, we recall the asymptotic stability of a time-varying system [4]. Consider

$$
x^{k+1} = F(k, x^k), \quad k \geq k_0, \tag{8}
$$

where $F : \mathbb{N} \times X \to \mathbb{R}^n$ and $F(k, 0) = 0$ for all $k \geq k_0$. The origin is an asymptotically stable equilibrium of (8) (with region of attraction X) if

1) for all $\varepsilon > 0$ and any $k_0 \in \mathbb{N}$, there exists $\delta = \delta(\varepsilon, k_0) > 0$ such that $x^{k_0} \in X$ with $0 < \|x^{k_0}\| < \delta$ implies $\|x^k\| < \varepsilon$ for all $k \geq k_0$;

2) for any $k_0 \in \mathbb{N}$ and $x^{k_0} \in X$, it holds $\lim_{k\to\infty} x^k = 0$.

As a first result in this section, we give a recursive feasibility result, which generalizes [1, Theorem 2.11]. For this, we need the following generalized invariance assumption.

Assumption 4: Consider the sets $U \subseteq \mathbb{R}^p, X^{N_p} \subseteq X \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} U$ and $0 \in \operatorname{int} X^{N_p}$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a g-dclf of order $m = N_p$ for system (1) and $1 < s \in \mathbb{N}$. Assume that for any $(x^0, \mathbf{u}_{[0:N_p-1]}) \in X \times \mathbf{U}_{[0:N_p-1]}(x^0)$ and any $l \in \{s, \ldots, N_p\}$ there exists a feedback $\mathbf{c} : X^{N_p} \to \mathbb{R}^{p,l}, \tilde{x} \mapsto \mathbf{c}(\tilde{x})$ such that for all $\tilde{x} \in X^{N_p}$

1) $c_0(\tilde{x}), c_1(\tilde{x}), \ldots, c_{l-1}(\tilde{x}) \in U$;
2) with $x^0 = \tilde{x}$, we have $x^{j+1} = f(x^j, c_j(\tilde{x})) \in X$ for all $j = 0, \ldots, s-2$ and $x^{j+1} = f(x^j, c_j(\tilde{x})) \in X^{N_p}$ for all $j = s-1, \ldots, l-1$;
3) $[\mathbf{u}_{[l:N_p-1]}, \mathbf{c}(\tilde{x})]$ satisfies adc (3) with $\sigma_1 = \cdots = \sigma_{s-1} = 0$, i.e. the control sequence $[\mathbf{u}_{[l:N_p-1]}, \mathbf{c}(\tilde{x})]$ steers x^l to some x^{N_p+l} such that $\frac{1}{N_p}\sum_{j=s}^{N_p} \sigma_j V(x^{j+l}) - V(x^l) \le -\alpha(x^l)$.

Like the standard MPC invariance assumption [2], this assumption consists of three conditions: 1) the feedback $\mathbf{c}(\tilde{x})$ satisfies the control constraints, 2) the feedback $\mathbf{c}(\tilde{x})$ renders X^{N_p} invariant in a generalized sense and 3) the control sequence $[\mathbf{u}_{[l:N_p-1]}, \mathbf{c}(\tilde{x})]$ satisfies adc where σ_s is the first non-zero weight. To see that this assumption is a generalization of [1, Theorem 2.11], let us elaborate more on condition 2). It allows the states to leave the terminal region X^{N_p}, as long as they return after at most s steps, i.e. invariance is given after s steps and not after one step. Meanwhile, condition 3) guarantees that the g-dclf decreases after at least s steps, meaning that, if this assumption is satisfied, then Algorithm 1 will implement at least s steps in each iteration. The following theorem guarantees recursive feasibility as defined in e.g. [2, p. 797].

Theorem 5 (Relaxed Recursive Feasibility): Let $V : \mathbb{R}^n \to \mathbb{R}$ be a g-dclf of order $m = N_p$ for system (1) with weights $\sigma_1 = \cdots = \sigma_{s-1} = 0$ in (3). Suppose that Assumption 4 is satisfied and that Problem 3 with $x = x(k_0)$ is feasible. If the index of descent in Algorithm 1 is chosen between $s \le \ell_{\text{decr}} \le N_p$, then the resulting MPC scheme is recursively feasible.

Proof: To prove recursive feasibility, we make use of the terminal set X^{N_p}. As a first step, we add the terminal constraint $x^{N_p} \in X^{N_p}$ of Problem 3 to the set of feasible controls $\mathbf{U}_{[0:N_p-1]}(x^0)$ of length $N = N_p$, compare (2). Consider now Algorithm 1 starting at some non-negative integer k, and suppose that Problem 3 was feasible in the previous iteration. We show that Problem 3 stays feasible after executing Algorithm 1. To that end, suppose that we arrived at the state $x(k) \in X$, the previous control strategy is given by $\mathbf{u}^{*-}_{[0:N_p-1]}$ and the previous index of descent is given by ℓ^-_{decr}. By assumption, the states steered by $\mathbf{u}^{*-}_{[0:N_p-1]}$ satisfy $x^1, \ldots, x^{N_p-1} \in X, x^{N_p} \in X^{N_p}$. Additionally, all components of $\mathbf{u}^{*-}_{[0:N_p-1]}$ are elements of U. For the optimization instance starting at $x(k)$, let us focus on the last $N_p - \ell^-_{\text{decr}}$ components of $\mathbf{u}^{*-}_{[0:N_p-1]}$. By considering $\tilde{x} = x^{N_p} \in X^{N_p}$ for Assumption 4, the control sequence $[\mathbf{u}^{*-}_{[\ell^-_{\text{decr}}:N_p-1]}, \mathbf{c}(x^{N_p})]$ and the corresponding states starting at $x(k)$ satisfy adc. Moreover, the controller guarantees that the states return to X^{N_p} after at most $s \le \ell^-_{\text{decr}}$ steps. Hence, the choice of $[\mathbf{u}^{*-}_{[\ell^-_{\text{decr}}:N_p-1]}, \mathbf{c}(x^{N_p})]$ yields a feasible controller. □

In [1, Theorem 2.5], we have shown convergence of the state to zero and asymptotic stability for strictly positive weights σ_i by using the framework of g-dclfs. The strictly positive weights are a mean for governing the state and control pairs coming out of adc. In light of our relaxed feasibility result, we next state an asymptotic stability result without an additional restriction on σ_i. If we want to remove the positivity restriction on the weights, we are in need of a new assumption on the g-dclf for which we orient ourselves to finite-step control Lyapunov functions [5]. The following assumption ensures that the state and control pairs are $\mathcal{K}$-bounded.

Assumption 6: There exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that for any $\boldsymbol{\omega} = (x^0, \mathbf{u}_{[0:q-1]}) \in \mathbb{R}^n \times \mathbf{U}_{[0:q-1]}(x^0)$ we have

$$\underline{\alpha}(\|\boldsymbol{\omega}\|) \le V(\boldsymbol{\omega}) \le \overline{\alpha}(\|\boldsymbol{\omega}\|). \tag{9}$$

Furthermore, assume that there exist class-$\mathcal{K}$ functions $\{\kappa_j\}_{j=0}^{m-1}$ such that for any $(x^0, \mathbf{u}_{[0:q-1]}) \in \mathbb{R}^n \times \mathbf{U}_{[0:q-1]}(x^0)$ and $j = 0, \ldots, m-1$ there exists $\boldsymbol{\nu}_{[0:q+m-1]} \in \mathbb{R}^{p,q+m}$ with $\boldsymbol{\nu}_{[j:q+j-1]} \in \mathbf{U}_{[0:q-1]}(x^j)$ and $\|(x^j, \boldsymbol{\nu}_{[j:q+j-1]})\| \le \kappa_j(x^0, \mathbf{u}_{[0:q-1]})$.

Theorem 7 (Relaxed Asymptotic Stability): Let $V : \mathbb{R}^n \times \mathbb{R}^{p,q} \to \mathbb{R}$ be a g-dclf of order m for (1), suppose that Assumption 6 and for $q \neq 0$ the Small Control Property [1, Assumption 2.4] holds. Then there exists a feasible control strategy rendering the origin asymptotically stable.

Proof: To ease notation, we will present the proof for the case $m = 2$, while keeping some of the notations for a generic m for illustrative purposes. Let us consider the case $\sigma_1 = 0$, which implies $\sigma_2 \ge 2$. By [1, Theorem 2.3] it is guaranteed that any state x^{k_0} can be steered to zero. It remains to show asymptotic stability. To this end, fix $\varepsilon > 0$ and $k_0 \in \mathbb{N}$. Due to the radial unboundedness of V by (5), we can choose $\mathbf{0} \neq \mathbf{p} \in \mathbb{R}^n \times \mathbb{R}^{p,q}$ such that there exists $r \in (0, \varepsilon)$, where

$$V(x, \mathbf{u}) \le V(\mathbf{p}) \Rightarrow \|(x, \mathbf{u})\| \le r. \tag{10}$$

Now define $B_r := \{(x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^{p,q} : \|(x, \mathbf{u})\| \le r\}$, $\beta := V(\mathbf{p}) > 0$ and $\Omega_\beta := \{(x, \mathbf{u}) \in B_r : V(x, \mathbf{u}) \le \beta\}$. It follows $\Omega_\beta \subseteq B_r$. Once we choose initial data $(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]})$, a whole sequence $\{(x^k, \mathbf{u}^k_{[0:q-1]})\}_{k \ge k_0}$ is generated through the control strategy characterized by [1, Proposition 3.2], whose subsequence achieves a strict descent of the Lyapunov function values. Since we are investigating the case $\sigma_1 = 0$ and $\sigma_2 \ge 2$, we may assume, without loss of generalization, that the descent always occurs after two time steps, i.e. the subsequence $\{(x^{k_0+2l}, \mathbf{u}^{k_0+2l}_{[0:q-1]})\}_{l \in \mathbb{N}}$ achieves

$$V(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) > V(x^{k_0+2}, \mathbf{u}^{k_0+2}_{[0:q-1]}) > \cdots \ge 0. \tag{11}$$

Let $0 < \hat{\beta} < \beta$. Due to the continuity of V, there exists $k_N < \infty$ such that $V(x^k, \mathbf{u}^k_{[0:q-1]}) < \hat{\beta}$ for all $k \ge k_N$.

Recall that, by Assumption 6, we have the two bounds

$$V(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]}) \leq \overline{\alpha}(\|(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]})\|)$$
$$\|(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]})\| \leq \kappa_j(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]})$$

for all $j = 0, \ldots, m-1$. This implies $V(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]}) \leq \overline{\alpha}(\kappa_j(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}))$ for all $j = 0, \ldots, m-1$. Motivated by [3], we define

$$V^{\max}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) := \max_{j=0,\ldots,m-1} V(x^{k_0+j}, \mathbf{u}^{k_0+j}_{[0:q-1]})$$
$$\kappa^{\max}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) := \max_{j=0,\ldots,m-1} \kappa_j(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}).$$

Let $\tilde{\beta}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) := \min\{\hat{\beta}, \min_{j=0,\cdots,N-1} b^j\}$, where

$$b^j = \{\beta - (\overline{\alpha}(\kappa^{\max}(x^{k_j}, \mathbf{u}^{k_j}_{[0:q-1]})) - V(x^{k_j}, \mathbf{u}^{k_j}_{[0:q-1]}))\}.$$

Recall that $\overline{\alpha} \in \mathcal{K}_\infty$, $\kappa_j \in \mathcal{K}$ and V is positive definite and continuous. Hence, we have $\tilde{\beta}(0, \mathbf{0}) > 0$. Together with the continuity of the minimum function, this implies that $\tilde{\beta}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]})$ is continuous and for small enough $(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]})$, it is greater than zero. Suppose for now we have

$$V(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) < \tilde{\beta}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) \tag{12}$$

for some $x^{k_0} \in X \setminus \{0\}$ and $\mathbf{u}^{k_0}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}(x^{k_0})$. Then we claim that the control strategy characterized by [1, Proposition 3.2] achieves $V(x^k, \mathbf{u}^k_{[0:q-1]}) \leq \beta$ for all $k \geq k_0$. Together with (10), this would imply that $(x^k, \mathbf{u}^k_{[0:q-1]})$ stays in Ω_β for $k \geq k_0$. We first observe, that by equation (11)

$$V(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]}) < V(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) < \tilde{\beta}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) < \beta$$

for all $k_n = k_0 + 2l, l \in \mathbb{N}$. It remains to guarantee that $V(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})$, $k_n = k_0 + 2l + 1$, stays within the bound for all $l \in \mathbb{N}$. Observe that for any $k_n = k_0 + 2l < k_N$ and any $j = 0, \ldots, m-1$, $V(x^{k_n+j}, \mathbf{u}^{k_n+j}_{[0:q-1]}) \leq V^{\max}(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})$, which is strictly less than

$$\begin{aligned}
&\tilde{\beta}(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]}) + \overline{\alpha}(\kappa^{\max}(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})) - V(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]}) \\
&\leq \beta - (\overline{\alpha}(\kappa^{\max}(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})) - V(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})) \\
&\quad + \overline{\alpha}(\kappa^{\max}(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]})) - V(x^{k_n}, \mathbf{u}^{k_n}_{[0:q-1]}).
\end{aligned}$$

We conclude that $(x^k, \mathbf{u}^k_{[0:q-1]}) \in \Omega_\beta$ and therefore, $(x^k, \mathbf{u}^k_{[0:q-1]}) \in B_r$ for all $k \geq k_0$. Thus, we obtain $\|x^k\| \leq \|(x^k, \mathbf{u}^k_{[0:q-1]})\| \leq r < \varepsilon$ for all $k \geq k_0$. It remains to guarantee that we can find $x^{k_0} \in X \setminus \{0\}$ and $\mathbf{u}^{k_0}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}(x^{k_0})$ with (12). Similarly to before, we make use of a continuity argument: The functions V and $\tilde{\beta}$ are continuous and $V(0, \mathbf{0}) = 0$, $\tilde{\beta}(0, \mathbf{0}) = \hat{\beta} > 0$. Hence, there exists $\delta > 0$ such that $\|(x, \mathbf{u}_{[0:q-1]})\| < \delta$ implies $\tilde{\beta}(x, \mathbf{u}_{[0:q-1]}) - V(x, \mathbf{u}_{[0:q-1]}) > 0$. If $q = 0$, then the continuity of V is enough, for $q \neq 0$ we need to evoke Small Control Property [1, Assumption 2.4]. It guarantees that for all $\bar{\varepsilon} > 0$, there exists $\bar{\delta} > 0$ such that for all $0 < \|x^{k_0}\| < \bar{\delta}$ there exists $\mathbf{u}^{k_0}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}(x^{k_0})$ with $\|\mathbf{u}^{k_0}_{[0:q-1]}\| < \bar{\varepsilon}$ satisfying the inequalities (5) and (6). Thus, for small enough $\bar{\varepsilon}$ we have $\bar{\varepsilon} + \bar{\delta} < \delta$ and obtain that for any $x^{k_0} \in X$ with $0 < \|x^{k_0}\| < \bar{\delta}$ there exists $\mathbf{u}^{k_0}_{[0:q-1]} \in \mathbf{U}_{[0:q-1]}(x^{k_0})$ with

$$\|(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]})\| \leq \|x^{k_0}\| + \|\mathbf{u}^{k_0}_{[0:q-1]}\| < \bar{\delta} + \bar{\varepsilon} < \delta.$$

We can deduce that the Lyapunov function value of $(x^{k_0}, \mathbf{u}^{k_0}_{[0:q-1]})$ satisfies the desired bound (12). In summary, if $x^{k_0} \in X$ satisfies $0 < \|x^{k_0}\| < \bar{\delta}$, then $\|x^k\| < \varepsilon$ for all $k \geq k_0$. This completes the proof. □

Remark 8: Unlike standard MPC [2], where the optimal value function is used as the Lyapunov function to show stability, the proof of Theorem 7 is utilizing the novel concept of a g-dclf and does not rely on the positive-definiteness of the cost function f_0. This assumption allows us, however, to recover standard MPC from flexible-step MPC [1, Remark 2.14].

IV. Converse Results

We now give a converse result. Here, we rely on the notion of a feasible flexible-step control law; to define this precisely, one requires some rather cumbersome notations, which we omit here for reasons of space, and instead refer the reader to [1, Remark 2.8] where a similar treatment is presented.

Theorem 9 (Converse Result): Consider system (1) for $X = \mathbb{R}^n$ with a feasible flexible-step control law. Assume that the origin of the resulting closed-loop system is asymptotically stable, i.e. there exists a $\mathcal{KL}$ function β such that for all $k_0 \in \mathbb{N}$ and $x^{k_0} \in \mathbb{R}^n$

$$\|x^k\| \leq \beta(\|x^{k_0}\|, k) \quad \forall k \geq k_0. \tag{13}$$

Furthermore, assume that there exists a continuous, radially unbounded and positive definite function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\tfrac{1}{m}(\sigma_1\beta(r,1) + \sigma_2\beta(r,2) + \cdots + \sigma_m\beta(r,m)) - r \leq -\gamma(r) \tag{14}$$

for all $r > 0$, with weights $\sigma_1, \ldots, \sigma_m \in \mathbb{R}_{\geq 0}$ satisfying (4). Then $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x \mapsto \eta\|x\|, \eta > 0$ is a g-dclf for the closed-loop system.

Proof: We will show that adc with $V(x) = \eta\|x\|, \eta > 0$ holds for all x^0, for a continuous, radially unbounded and positive definite function $\alpha : \mathbb{R}^n \to \mathbb{R}$. For all x^0, we have

$$\begin{aligned}
&\frac{\sigma_1\eta\|x^1\| + \sigma_2\eta\|x^2\| + \cdots + \sigma_m\eta\|x^m\|}{m} - \eta\|x^0\| \\
&\leq \eta\left(\frac{\sigma_1\beta(\|x^0\|,1) + \cdots + \sigma_m\beta(\|x^0\|,m)}{m} - \|x^0\|\right) \\
&\leq -\eta\gamma(\|x^0\|) =: -\alpha(x^0),
\end{aligned}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}, x \mapsto \eta\gamma(\|x\|)$ has the desired properties. This shows that $V(x) = \eta\|x\|$ is a g-dclf. □

Proposition 10: Consider system (1) for $X = \mathbb{R}^n$ with a feasible flexible-step control law. Assume that the origin of the resulting closed-loop system is exponentially stable, i.e. the $\mathcal{KL}$ function in (13) can be chosen as $\beta(r,t) = C\mu^t r$ with $C \geq 1$ and $\mu \in [0,1)$. Then if $m \in \mathbb{N}$ satisfies $\frac{C\bar{\sigma}\mu}{1-\mu} < m$, for $\bar{\sigma} = \max(\sigma_1, \ldots, \sigma_m)$, the condition (14) is satisfied.

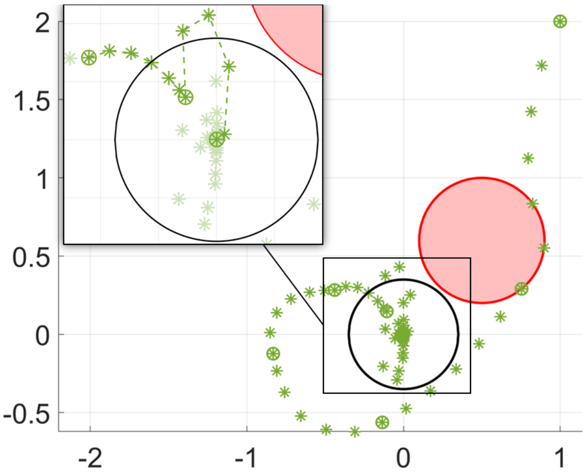

Fig. 2. Evolution of the first and second component of the state in the x_1x_2-plane according to the solution of Problem 11. The terminal region is a four dimensional sphere, but for any fixed $x_3 \in \mathbb{R}$ and $x_4 \in \mathbb{R}$ this sphere reduces to a circle, where the circle with $x_3 = x_4 = 0$ is the biggest of such circles, displayed here in black. The obstacle is shown in red. The initial states, where the next optimization instance starts, are circled. On the left, we focus on the two consecutive optimization instances, where the actual states first enter the terminal region.

With Theorem 5 and 7 at hand, recursive feasibility and asymptotic stability with a relaxed requirement on the terminal region (Assumption 4) is guaranteed. Additionally, Theorem 9 yields that we can focus on g-dclfs of the form $V(x) = \eta\|x\|$. This simplifies the design of our MPC scheme, as we will see in the next section.

V. Simulation Results

In this section, we demonstrate the performance of the flexible-step MPC scheme on a nonholonomic system with state constraints and a terminal region. The dynamics of the state $x^{j+1} = [x_1^{j+1} x_2^{j+1} x_3^{j+1} x_4^{j+1}]^T \in \mathbb{R}^4$ with the sampling time $h = 0.01$ and the control input $u^j = [u_1^j \; u_2^j]^T \in \mathbb{R}^2$ are given by

$$x^{j+1} = x^j + h\begin{bmatrix} 1 \\ 0 \\ -x_2^j \\ x_3^j \end{bmatrix} u_1^j + h\begin{bmatrix} 0 \\ 1 \\ x_1^j \\ x_2^j \end{bmatrix} u_2^j + h\begin{bmatrix} 0 \\ 0 \\ 0 \\ |x_4^j| \end{bmatrix}. \quad (15)$$

The state constraints are imposed through an obstacle, described by $(x_1^j - 0.5)^2 + (x_2^j - 0.6)^2 - 0.16 < 0$. Thanks to our relaxed invariance Assumption 4, the terminal region is simply taken to be $X^{N_p} := \{x \in \mathbb{R}^4 : \|x\| \leq 0.35\}$. We now solve the following optimal control problem with the initial state $[1\ 2\ 3\ 5]^T$ and the prediction horizon $N_p = 19$ within Algorithm 1 by using `fmincon` from MATLAB.

Problem 11:

$$\min \sum_{j=0}^{N_p-1} \|x^j\|^2 + 5\,\|u^j\|^2$$

$$\text{s.t. (15) holds with } x^0 = x$$
$$0.1 \cdot (6\|x^6\| + 5\|x^5\| + 5\|x^4\|) - \|x^0\| \leq -\varepsilon\|x^0\|^4$$
$$(x_1^j - 0.5)^2 + (x_2^j - 0.6)^2 - 0.16 \geq 0, \quad x^{N_p} \in X^{N_p}.$$

The evolution of the first and second component of the state are depicted in Fig. 2. We see that the states successfully avoid the obstacle and remain in the terminal region after some time. A closer look of the terminal region is given in the left-hand corner of Fig. 2. Once the states reach the terminal region, they leave this region for two time steps in the next optimization instance and then return. This is precisely where we see our relaxed invariance Assumption 4 at play, with invariance achieved after multiple steps. All in all, we successfully utilized the simple g-dclf $V(x) = \|x\|$ and the simple terminal region, here described by the Euclidean distance. The flexibility in the number of implemented steps can be seen in Fig. 1, where during the time steps 24 and 44, six, five, five and four steps were implemented.

VI. Conclusion

The main objective of this manuscript has been to demonstrate some advantageous features of the flexible-MPC scheme, particularly regarding stability and feasibility, in addition to the ease in the selection of terminal constraints. There are many avenues of future research, including the connection of classical Lyapunov functions to g-dclfs, the study of the trade-offs between exploration and robustness, and the applications of the proposed methods in data-driven settings.

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