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Safety-Probability Analysis and Control for Stochastic Systems Based on Lyapunov Candidate Functions*

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Abstract— **In recent control theory, safety analysis and safety-critical control based on a (control) barrier function have been actively pursued. The barrier function is closely related to a Lyapunov function, which is an important property that guarantees asymptotic stability of the system, i.e., the settling to the target state, which is a fundamental control performance. Therefore, control strategies that simultaneously guarantee safety and stability are important in the recent control scene. In this paper, we propose a method for quantitative evaluation of safety probability for stochastic systems based on barrier functions generated from Lyapunov functions, and then develop control design methods to increase the safety probability. In particular, safety analysis and safety-critical control of linear stochastic systems having additive noises are performed based on linear algebra. We also discuss design methods for safety and safety-critical control for input-affine stochastic systems. The effectiveness of the proposed method is demonstrated based on a simple example.**

I. INTRODUCTION

In recent years, the advancement of human-machine interaction and automation technologies has been actively promoted, and the establishment of system safety has become an urgent issue in control engineering. Therefore, on the theoretical side, safety control based on control barrier functions has been vigorously studied in the past few years [1], [2], [3], [4]. In safe control theory based on control barrier functions, system safety is achieved by some kind of state restriction. That is, the control goal is to make a certain subset of the state space invariance (in forward time), and the (forward) invariance set is called the safety set. Various applications are expected in this policy, for example, the straightforward realization of seemingly complex commands in barrier functions in practice [1], [2] and human-assisted control [3], [4] are being promoted.

Then, the demand to maintain invariance of a safe set no matter how violently and irregularly disturbances are applied has led to various reports on barrier functions for stochastic systems [5], [6], [7], [8], [9], [10], [11], [12]; Clark [5] discusses conditions for safety with probability one, Tamba et al. [6] consider a strict type of stochatic barrier function for safety with probability one, and Wang et al. [7] provide a detailed discussion of Markov time until a trajectory leaves a safe set. Prajna et al. [8] discuss in detail the probability of staying in a certain set, and Santoyo et al. [9] develop this into a safe control problem based on control barrier functions. Wisniewski and Bujorianu [10] provide a detailed discussion of stochastic safety named *p*-safety. Stochastic safety-critical control of stochastic discrete-time systems has been studied in detail by Jagtap et al. [11] and further developed into a data-driven framework by Salamati and Zamani [12]. These methods are extremely powerful for complex systems such as hybrid systems and systems with linear temporal logic.

On the other hand, in many problem settings of stochastic systems, safety is not guaranteed with probability one, therefore, a quantitative evaluation of the degree of safety is essential. In the setting of quantitative analysis of a safe set, Prajna et al. [8] and Wisniewski and Bujorianu [10] provide excellent analysis methods, and recently Nishimura and Hoshino [13] have proposed a simple analysis method with sufficient conditions that directly includes diffusion coefficients.

Safety-critical control based on control barrier functions is generally in the nature of a compensator that is added to improve some control performance. Thus, the most basic application of safety-critical control is to add a compensator to a system in which asymptotic stabilization has already been achieved. Applying this policy to a stochastic system, we can conceive of the possibility of analyzing how much safety probability a level set of a Lyapunov function can retain under the influence of white noises. Based on this analysis, the safety probability can be improved by adding safety-critical control as a compensator. In particular, when white noise is introduced into a linear time-invariant system, a new analysis is to be explored using a quadratic Lyapunov candidate function. This analysis should aim to quantify safety probabilities and develop design methods to enhance safety probabilities using techniques from linear systems theory.

In this paper, we propose a simple and concrete analytical method based on the method in [13] to quantify the safety probability that the trajectory of an asymptotically stable system remains inside the safety set under stochastic disturbances, and show that the safety probability can be improved by designing compensators and with concrete numerical targets. First, since the goal of this paper is closely related to the invariance of a level set of a Lyapunov function in Lyapunov stability theory, we consider the situation where an asymptotically stable linear system is influenced by additive noises. We define a safe set, an initial value set, and a barrier function associated with the level set of the Lyapunov

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function, and propose a method to compute the safety probability of the safe set in linear algebra. We also give a control design guideline to improve the safety probability of the safety set. Next, an analysis method of safety probability for more general stochastic input-affine systems and a control design for improving the safety probability are derived. The effectiveness of the proposed method is verified by a simple numerical example.

This paper is organized as follows. In Section II, we describe mathematical notations and target systems. In Section III, we preliminary discuss safety for stochastic systems. In Section IV, we present the main results on the analysis and design of safety probabilities for linera systems with additive noises. In Section V, we extend the results of the previous section to nonlinear systems. In Section VI, we demonstrate a numerical example. Section VII concludes this paper. The main results are shown in Sections IV and V.

II. PRELIMINARY

A. Notations

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and especially $\mathbb{R} := \mathbb{R}^1$. A Lie derivative of a smooth mapping $W : \mathbb{R}^n \to \mathbb{R}$ in a mapping $F = (F_1, \ldots, F_q) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ with $F_1, \ldots, F_q : \mathbb{R}^n \to \mathbb{R}^n$ is denoted by

$$\mathrm{L}_F W(x) = \left(\frac{\partial W}{\partial x} F_1(x), \ldots, \frac{\partial W}{\partial x} F_q(x) \right). \tag{1}$$

For constants $a, b > 0$, a continuous mapping $\alpha : [-b, a] \to \mathbb{R}$ is said to be an extended class $\mathscr{K}$ function if it is strictly increasing and satisfies $\alpha(0) = 0$. The boundary of a set $\mathscr{A}$ is denoted by $\partial\mathscr{A}$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where Ω is the sample space, $\mathscr{F}$ is the σ-algebra that is a collection of all the events, $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration of $\mathscr{F}$ and $\mathbb{P}$ is a probabilistic measure. In the filtered probability space, $\mathbb{P}[A|B]$ denotes the probability of some event A under some condition B and $w = (w_1, \ldots, w_d)^T$ is a d-dimensional standard Wiener process. For a Markov process $x(t) \in \mathbb{R}^n$ with an initial state $x(0) = x_0$, we often use the following notation $\mathbb{P}_{x_0}[A] = \mathbb{P}[A|x(0) = x_0]$. The differential form of an Itô integral of $\sigma : \mathbb{R}^n \to \mathbb{R}^d$ over w is represented by $\sigma(x)dw$. The trace of a square matrix X is denoted by $\mathrm{tr}[X]$. For a matrix X whose eigenvalues are all real, $\mathrm{eigmax}[X]$ and $\mathrm{eigmin}[X]$ are the maximum eigenvalue and the minimum eigenvalue, respectively.

B. Target system, the related functions, and a global solution

In this subsection, we describe a target system, the related functions frequently used throughout the paper, and the definition of a solution in global time.

In this paper, we consider a stochastic system

$$dx = f(x)dt + \sigma(x)dw \tag{2}$$

and a stochastic control system

$$dx = \{f(x) + g(x)u\}dt + \sigma(x)dw, \tag{3}$$

where $x \in \mathbb{R}^n$ is a state vector with a fixed initial value $x_0 = x(0) \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ is a control input vector, where U denotes an admissible control set, w is a d-dimensional standard Wiener process, and mappings f, g and $\sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$ are assumed to be locally Lipschitz and satisfy linear growth condition for ensuring that the system (3) has a strong solution for any $t \in [0, \infty)$; see, for example, [14], [15], [16].

For simplicity, we further define some functions. For a C^2 mapping $y : M \to \mathbb{R}$ with some $M \subset \mathbb{R}^n$, let

$$L^I_\sigma(y(x)) := \frac{1}{2}\mathrm{tr}\left[\sigma(x)\sigma(x)^T \left[\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\right]^T\right](x)\right], \tag{4}$$

$$H_\sigma(h(x)) := \frac{1}{2}\mathrm{L}_\sigma h(x)(\mathrm{L}_\sigma h(x))^T. \tag{5}$$

The infinitesimal operator for (2) is defined by $\mathscr{L}_{f,\sigma}$ as follows:

$$\mathscr{L}_{f,\sigma}(y(x)) := \mathrm{L}_f y(x) + L^I_\sigma(y(x)); \tag{6}$$

and the one for (3) is defined by $\mathscr{L}_{f,g,\sigma}$ as follows:

$$\mathscr{L}_{f,g,\sigma}(u, y(x)) := \mathrm{L}_f y(x) + \mathrm{L}_g y(x)u + L^I_\sigma(y(x)). \tag{7}$$

III. PROBABILITY OF SAFETY

A. Definitions of a safe set and safety for a stochastic system

Let us define a safe set $\chi \subset \mathbb{R}^n$ being open, and there exists a mapping $h : \mathbb{R}^n \to \mathbb{R}$ satisfying all the following conditions:

(Z1) $h(x)$ is C^2.

(Z2) $h(x)$ is proper in $\mathbb{R}^n$; that is, for any $L \in [0, \infty)$, any superlevel set $\{x \in \mathbb{R}^n | h(x) \geq L\}$ is compact.

(Z3) The closure of χ is the 0-superlevel set of $h(x)$; that is, both of the following two conditions are satisfied:

$$\chi = \{x \in \mathbb{R}^n | h(x) > 0\}, \tag{8}$$

$$\partial\chi = \{x \in \mathbb{R}^n | h(x) = 0\}. \tag{9}$$

We also notice that the reciprocal function $B(x) := (h(x))^{-1}$ is often used after.

Let $p \in [0, 1]$ and $\chi' \subset \chi$. System (3) is said to be *safe in* (χ', χ, p) if, for any $x_0 \in \chi'$,

$$\mathbb{P}_{x_0}[x(t) \in \chi, \forall t \in [0, \infty)] \geq p \tag{10}$$

is satisfied. The definition of safety is influenced by the discussion of stochastic stability analysis by Kushner [17].

B. Sufficient Conditions for Stochastic Safety

In this subsection, based on [13], we show a barrier function for a stochastic system to yield a quantitative evaluation of how safe the system is from the viewpoint of probability.

We set some sets and stopping times used in this subsection. For $\mu > 0$, let

$$\chi_\mu := \{x \in \mathbb{R}^n | h(x) \in (0, \mu]\} \subset \chi, \tag{11}$$

$$\chi_{h>\mu} := \chi \setminus \chi_\mu = \{x \in \mathbb{R}^n | h(x) > \mu\}, \tag{12}$$

$$\mathbb{R}^n_{h\leq\mu} := \tilde{\chi} \cup \chi_\mu = \{x \in \mathbb{R}^n | h(x) \leq \mu\}, \tag{13}$$

be defined. For a solution to the system (3) with $x_0 \in \chi_\mu$, the first exit time from χ_μ is denoted by $\tau_{0\mu}$, and for the solution with $x_0 \in \chi$, the first exit time from χ is denoted by τ_0.

To analyze the safety of a stochastic system without inputs, we first define the following:

Definition 1 (Stochastic ZBF): Let (2) be considered with χ and $h(x)$ satisfying (Z1), (Z2) and (Z3). If, for all $x \in \mathbb{R}^n_{h \le \mu}$,

$$\mathscr{L}_{f,\sigma}(h(x)) \geq bH_\sigma(h(x)) \tag{14}$$

is satisfied with some $b > 0$, then $h(x)$ is said to be *a stochastic zeroing barrier function (ZBF).* □

Lemma 1: If there exists a stochastic ZBF $h(x)$ for the system (2), then it is safe in $(\chi_\mu, \chi, 1-e^{-bh(x_0)})$. ♦

The proof follows immediately from the results in [13].

If the initial value satisfies $h(x_0) > \mu$, then the above lemma is improved so that the probability is independent of the initial value.

Theorem 1: If there exists a stochastic ZBF for the system (2), then it is safe in $(\chi_{h>\mu}, \chi, 1-e^{-b\mu})$. ♦

The proof follows immediately from the results in [13].

Remark 1: The characteristic feature of the stochastic ZBF is the explicit inclusion of the diffusion coefficient in the given inequality (14). The feature has the advantage of simplifying the calculation of the safety probability and of allowing the safety probability to be recalculated in conjunction with changes in the diffusion coefficient. ♢

C. Safety-critical Control

In this subsection, we summarize the extension of the previous results to a stochastic system with control inputs (3).

Definition 2 (Stochastic ZCBF): Let (3) be considered with χ and $h(x)$ satisfying (Z1), (Z2) and (Z3). If there exists a continuous mapping $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that $h(x)$ becomes a stochastic ZBF; that is, for all $x \in \mathbb{R}^n_{h \le \mu}$,

$$\mathscr{L}_{f,g,\sigma}(\phi(x), h(x)) \geq bH_\sigma(h(x)) \tag{15}$$

is satisfied with some $b > 0$, then $h(x)$ is said to be *a stochastic zeroing control barrier function (ZCBF).* □

Using the notion of a stochastic ZCBF, the results of the previous subsection are described as follows:

Theorem 2: Let the system (3) be considered. If there exists a stochastic ZCBF $h(x)$, then the system becomes safe in both $(\chi_\mu, \chi, 1-e^{-bh(x_0)})$ and $(\chi_{h>\mu}, \chi, 1-e^{-b\mu})$ by designing $u = \phi(x)$ satisfying all the conditions in Definition 2. ♦

The proof of the theorem follows immediately from the results in [13].

IV. SAFETY PROBABILITY ANALYSIS FOR STOCHASTIC LINEAR SYSTEMS

A. Safety for Closed-loop Systems

In this section, we propose an analysis using stochastic ZCBFs to quantitatively evaluate the behavior of trajectories in stochastic linear systems.

Here we consider a deterministic system whose origin is asymptotically stable and its Lyapunov function $V(x)$, and suppose that the system loses asymptotic stability due to the application of white noises. In this case, $V(x)$ is no longer a stochastic Lyapunov function, but can be used as a tool to evaluate the behavior of the trajectory as a "former Lyapunov function." Specifically, we can define a sublevel set of $V(x)$ as a safe set and calculate the probability that the trajectory stays within the safe set. In the following, we describe the analytical method for linear systems.

Let us consider a stochastic linear system

$$dx = Axdt + Gdw, \tag{16}$$

where $x \in \mathbb{R}^n$ is a state vector with a fixed initial value $x_0 = x(0) \in \mathbb{R}^n$, w is a d-dimensional standard Wiener process, and $A \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times d}$ are constant matrices.

Theorem 3: Assume that there exist positive definite and symmetric matrices $P, Q \in \mathbb{R}^{n\times n}$ satisfying a Lyapunov equation

$$PA + A^T P = -Q. \tag{17}$$

Let us also consider a candidate for a stochastic ZBF

$$h(x) = -x^T Px + M, \; M > 0, \tag{18}$$

a safe set χ and the related sets χ_μ, $\chi_{h>\mu}$ and $\mathbb{R}^n_{h\le\mu}$ with $\mu \in (0, M)$. If

$$L := \mathrm{eigmin}[Q] - \mathrm{eigmax}[P]\frac{\mathrm{tr}[G^T PG]}{M-\mu} > 0 \tag{19}$$

is satisfied, then the system (16) is safe in $(\chi_{h>\mu}, \chi, 1-e^{-b\mu})$, where

$$b \leq \frac{L}{2\mathrm{eigmax}[PGG^T P]}. \tag{20}$$

♦

The proof is shown in Appendix.

The above theorem can be explained based on the Lyapunov function $V(x) = x^T Px$ as follows. By (18), the safe set is the open set of the M-level set of $V(x)$ minus the boundary; that is, $\chi = \{x \in \mathbb{R}^n | V(x) < M\}$. Also, the set of initial states $\chi_{h>\mu}$ is the open set of the μ-level set of $V(x)$ minus the boundary; that is, $\chi_{h>\mu} = \{x \in \mathbb{R}^n | V(x) < \mu\}$. For a deterministic system, any level set of a Lyapunov function is an invariance set, therefore it is surely safe. However, for a stochastic system, a level set of a Lyapunov function is generally not an invariance set, therefore safety is not guaranteed with probability one. Using the above theorem, we can quantify the degree to which a particular level set of a Lyapunov function guarantees safety.

B. Safety-Probability Compensators

In this subsection, we propose two control designs for improving the quantity of safety of a safe set.

Let us consider a stochastic linear system

$$dx = (Ax + Bu)dt + Gdw, \tag{21}$$

where $x \in \mathbb{R}^n$ is a state vector with a fixed initial value $x_0 = x(0) \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ is a control input vector, where U denotes an acceptable control set, w is a d-dimensional standard Wiener process, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $G \in \mathbb{R}^{n\times d}$ are constant matrices.

Corollary 1: Assume that the conditions in Theorem 3 are all satisfied. If there exists $K^T \in \mathbb{R}^m$ such that the minimum eigenvalue for

$$\bar{Q} := Q + PBK + K^T B^T P \tag{22}$$

is larger than the minimum eigenvalue for Q, there exists $\bar{b} > b$ such that the system (21) with $u = -Kx$ is safe in $(\chi_{h>\mu}, \chi, 1 - e^{-\bar{b}\mu})$. ♦

The proof is shown in Appendix.

Corollary 2: Assume that the conditions in Theorem 3 are all satisfied. If there exist a positive definite and symmetric matrix $R \in \mathbb{R}^{m\times m}$ and $b^+ > 0$ such that

$$BR^{-1}B^T = b^+ GG^T, \tag{23}$$

then, the system (21) with $u = \phi_{po}(x)$, where

$$\phi_{po}(x) := -R^{-1}B^T Px, \tag{24}$$

is safe in $(\chi_{h>\mu}, \chi, 1 - e^{-(b+b^+)\mu})$. ♦

The proof is shown in Appendix.

We can use either of the two theorems above to improve the safety probability of a safe set χ. In the procedure, we can design the feedback gain (K in Corollary 1 or $R^{-1}B^T P$ in Corollary 2) by specifying the value of the safety probability as, for example, 90%. This will be illustrated in a numerical example in Section VI.

V. Safety-critical Control for Stochastic Nonlinear Systems

In this section, we consider a quantitative analysis of the safety of a nonlinear stochastic system (2) and a way to improve the safety probability of a nonlinear controlled system (3). Note that in the previous section we used the "former Lyapunov function," while in this section we analyze the safety probability based on the positive definite and proper function, which is not necessarily a Lyapunov function.

Lemma 2: Let (2) be considered. Let also $V : \mathbb{R}^n \to \mathbb{R}$ be a proper, C^2 and positive definite mapping. For $M > 0$, if there exist $\mu \in (0, M]$ and $b > 0$ such that

$$\mathscr{L}_{f,\sigma}(V(x)) \le -bH_\sigma(V(x)) \tag{25}$$

for all x satisfying $V(x) \ge M - \mu$, then

$$h(x) = -V(x) + M \tag{26}$$

is a stochastic ZBF. ♦

The proof is shown in Appendix.

Theorem 4: Let us assume that $h(x)$ in (26) is a stochastic ZBF for (3) with $u = 0$. Let us also design $u = \phi_N(x)$ with

$$\phi_N(x) := \begin{cases} -\frac{(\mathrm{L}_\sigma V)^T R^{-1} (\mathrm{L}_\sigma V)}{2\mathrm{L}_g V \cdot (\mathrm{L}_g V)^T} (\mathrm{L}_g V)^T, & \mathrm{L}_g V \ne 0, \\ 0, & \mathrm{L}_g V = 0, \end{cases} \tag{27}$$

where $R \in \mathbb{R}^{d\times d}$ is a positive definite and symmetric matrix. If there exists $b' > 0$ satisfies

$$\mathscr{L}_{f,g,\sigma}(0, V(x)) \le -b' H_\sigma(V(x)) \tag{28}$$

for all x satisfying both $V(x) \ge M - \mu$ and $\mathrm{L}_g V = 0$, (3) with $u = \phi_N(x)$ is safe in $(\chi_{h>\mu}, \chi, 1 - e^{-(b+b^+)\mu})$, where

$$b^+ = \min(b', \mathrm{eigmin}[R^{-1}]). \tag{29}$$

♦

The proof is shown in Appendix.

VI. Numerical Example

Let us consider (21) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}. \tag{30}$$

Because the following pair of Q and P satisfies the Lyapunov equation (17), it ensures the origin of the system is asymptotically stable if there is no diffusion term:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \tag{31}$$

Therefore, L in (19) and b in (20) are

$$L = 1 - \frac{2+\sqrt{2}}{4}\frac{1}{M-\mu}, \tag{32}$$

$$b \le 4L =: \bar{b}, \tag{33}$$

respectively. Therefore, if $M = 1$ and $\mu = 1/10$, the system with $u = 0$ is safe in $(\chi_{h>\mu}, \chi, p_1)$ with

$$p_1 = 1 - \exp(-\bar{b}\mu) = 1 - \exp(-2L/5) \approx 0.263. \tag{34}$$

In other words, using the related Lyapunov function $V_1(x) = x^T Px$, if an initial state satisfies $x_0 \in \chi_{h>\mu} = \{x \in \mathbb{R}^2 | V_1(x) < 1/10$, a sample path $x(t)$ stays $x \in \chi = \{x \in \mathbb{R}^2 | V_1(x) < 1\}$ for all $t \ge 0$ with probability at least 26.3%.

Next, let us consider improving the safety probability by adding a state feedback law $u = -Kx$. Because the target system has a single input, we can easily apply Corollary 2 to design the input. Specifically, we obtain $u = \phi_{po}(x)$ with (24) and $R = 4/b^+$; that is,

$$u = -\frac{b^+}{4}B^T Px. \tag{35}$$

The parameter b^+ is determined according to the control objective. Here, aiming to achieve a safety probability of 90% or greater for the safety set, the parameter should satisfy

$$1 - \exp(-(\bar{b} + b^+)\mu) > \frac{9}{10}; \tag{36}$$

therefore,

$$b^+ > -\bar{b} - \log(1/10)/\mu. \tag{37}$$

Substituting $M = 1$ and $\mu = 1/10$ to the above inequality, we obtain $b^+ > 19.97$. Therefore, setting $b^+ = 20$, we obtain

$$u = -40x_1 - 40x_2. \tag{38}$$

Using the above parameters, we confirm the validity of the proposed analysis and control by numerical simulation. We use Euler-Maruyama scheme [18], which is a popular method for computer simulation of stochastic differential equations. The initial value is set to $x_0 = (0,0)^T$ and the number of trials

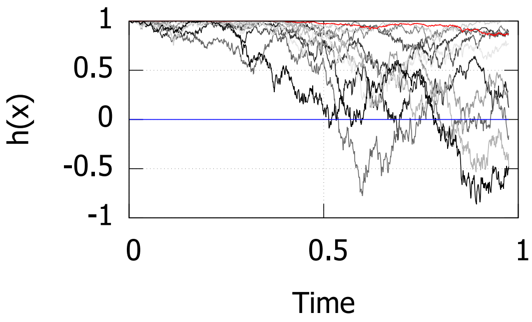

Fig. 1. Time responses of a ZBF $h(x) = -x^T Px + M$ with $u = 0$.

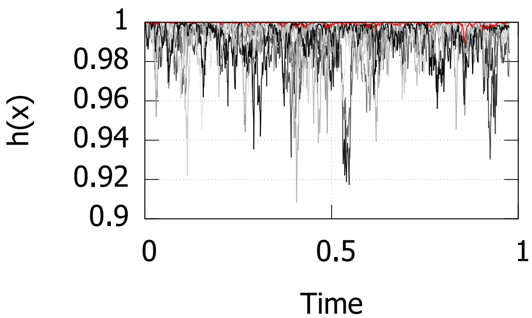

Fig. 2. Time responses of a ZCBF $h(x) = -x^T Px + M$ with $u = -40x_1 - 40x_2$.

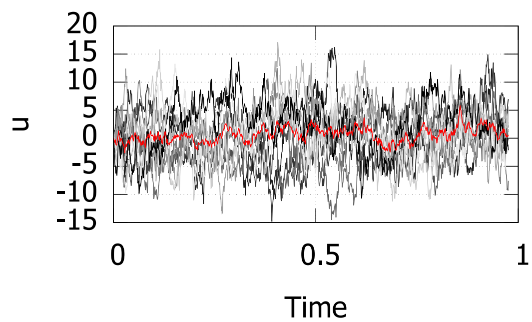

Fig. 3. Time responses of a controller $u = -40x_1 - 40x_2$.

is 10. Fig. 1 shows sample paths of $h(x) = -V_1(x) + M$ when $u = 0$ and Figs. 2 and 3 show sample paths of $h(x)$ and u when $u = -40x_1 - 40x_2$, respectively. For each figure, the red line describes the average of 10 trials, and for Fig. 1, the blue line denotes the boundary of the safe set χ. Fig. 1 implies that the value of $h(x)$ is sometimes negative, which is considered to reflect a low safety probability (26.3%) when $u = 0$. On the other hand, Fig. 2 implies that the value of $h(x)$ is always positive (at least the results of numerical simulation), which is considered to reflect that the control law $u = -40x_1 - 40x_2$ improves the safety probability (90%).

VII. CONCLUSIONS

In this paper, we proposed a concrete method for calculating the safety probability of a level set of a Lyapunov function and a method for designing compensators for the safety probability. Application to more complex and concrete nonlinear systems is an important issue to demonstrate the practicality of this research.

APPENDIX

In the proofs, we use the notations as follows: $\mathbb{E}[y]$ means the expectation of some random variable y and the minimum of $a, b \in \mathbb{R}$ is described by $a \wedge b := \min(a, b)$.

A. Proof of Theorem 3

This theorem is proven by showing that $h(x)$ is a stochastic ZBF. In the proof, we often use the relationship

$$\mathrm{eigmin}[Y] x^T x \le x^T Y x \le \mathrm{eigmax}[Y] x^T x \tag{39}$$

for a symmetric matrix $Y \in \mathbb{R}^{n\times n}$. If $x \in \mathbb{R}^n_{h\le\mu}$; that is, if

$$-x^T Px + M \le \mu, \tag{40}$$

we obtain

$$x^T x \ge \frac{M-\mu}{\mathrm{eigmax}[P]}. \tag{41}$$

On the other hand, the given assumptions (19) and (20) yield

$$\mathrm{eigmin}[Q] - 2b \cdot \mathrm{eigmax}[PGG^T P] \ge \mathrm{eigmax}[P] \frac{\mathrm{tr}[G^T PG]}{M-\mu}; \tag{42}$$

thus,

$$\frac{M-\mu}{\mathrm{eigmax}[P]} \ge \frac{\mathrm{tr}[G^T PG]}{\mathrm{eigmin}[Q] - 2b \cdot \mathrm{eigmax}[PGG^T P]} \tag{43}$$

is satisfied. Therefore, If (41) holds, (43) results in

$$x^T x \ge \frac{\mathrm{tr}[G^T PG]}{\mathrm{eigmin}[Q] - 2b \cdot \mathrm{eigmax}[PGG^T P]}. \tag{44}$$

Applying (39) to the above inequality, we obtain

$$x^T Qx - \mathrm{tr}[G^T PG] \ge 2bx^T PGG^T Px, \tag{45}$$

which is the same as

$$\mathscr{L}_{Ax,G}(h(x)) \ge bH_G(h(x)). \tag{46}$$

This is a sufficient condition that $h(x)$ is a stochastic ZBF. Consequently, by Theorem 1, the system is safe in $(\chi_{h>\mu}, \chi, 1 - e^{-b\mu})$.

B. Proof of Corollary 1

Because $Q = -PA - A^T P$ is positive definite, $\bar{Q}$ is also positive definite and P is a positive solution for the related Lyapunov equation

$$P(A - BK) + (A - BK)^T P = -\bar{Q}. \tag{47}$$

Therefore, (19) and (20) in Theorem 3 are replaced by

$$\bar{L} := \mathrm{eigmin}[\bar{Q}] - \mathrm{eigmax}[P] \frac{\mathrm{tr}[G^T PG]}{M-\mu} > 0 \tag{48}$$

and

$$\bar{b} \le \frac{L'}{2\mathrm{eigmax}[PGG^T P]}, \tag{49}$$

respectively. Because the given additional assumption is $\mathrm{eigmin}[\bar{Q}] > \mathrm{eigmin}[Q]$, there exists $\bar{b}$ such that $\bar{b} > b$.

C. Proof of Corollary 2

Considering $u = \phi_{po}(x)$ with (24) and $h(x)$ with (18), we obtain

$$\begin{aligned}\mathscr{L}_{Ax,B,G}(\phi_{po}(x), h(x)) = x^T Qx - \mathrm{tr}\left[G^T PG\right] \\ + 2x^T PBR^{-1}B^T Px; \qquad (50)\end{aligned}$$

thus, applying (46), we obtain

$$\mathscr{L}_{Ax,B,G}(\phi_{po}(x), h(x)) \geq 2bx^T PGG^T Px + 2x^T PBR^{-1}B^T Px. \qquad (51)$$

Moreover, we also consider the additional assumption (23), the above inequality results in

$$\mathscr{L}_{Ax,B,G}(\phi_{po}(x), h(x)) \geq 2(b + b^+)x^T PGG^T Px, \qquad (52)$$

which implies (15), provided that b is replaced by $b + b^+$. This completes the proof.

D. Proof of Lemma 2

First, considering (26), $V(x) \geq M - \mu$ is equivalent to $h(x) \geq \mu$. Second, (26) implies

$$\frac{\partial h}{\partial x} = -\frac{\partial V}{\partial x}; \qquad (53)$$

thus, we obtain

$$\mathscr{L}_{f,\sigma}(h(x)) = -\mathscr{L}_{f,\sigma}(V(x)). \qquad (54)$$

Using the above relationship, (14) results in (25). Therefore, $h(x)$ is a stochastic ZBF.

E. Proof of Theorem 4

For $\mathrm{L}_g V \neq 0$, (3) with $u = \phi_N(x)$ implies

$$\begin{aligned}\mathscr{L}_{f,g,\sigma}(\phi_N(x), V(x)) =& \mathrm{L}_f V(x) + L_\sigma^I(V(x)) \\ &- \frac{1}{2}(\mathrm{L}_\sigma V(x))^T R^{-1} \mathrm{L}_\sigma V(x). \qquad (55)\end{aligned}$$

Therefore, if $V(x) \geq M - \mu$, we obtain

$$\mathscr{L}_{f,g,\sigma}(\phi_N(x), V(x)) \leq -(b + \mathrm{eigmin}[R^{-1}])H_\sigma(V(x)). \qquad (56)$$

On the other hand, if $\mathrm{L}_g V = 0$ and $V(x) \geq M - \mu$, (3) with $u = 0$ implies

$$\mathscr{L}_{f,g,\sigma}(\phi_N(x), V(x)) \leq -b' H_\sigma(V(x)), \qquad (57)$$

where $b' \geq b > 0$ because $h(x)$ is a stochastic ZBF for (3) with $u = 0$. Comparing (56) with (57), we obtain

$$\mathscr{L}_{f,g,\sigma}(\phi_N(x), V(x)) \leq -b^+ H_\sigma(V(x)) \qquad (58)$$

for all x satisfying $V(x) \geq M - \mu$. This completes the proof.

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