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Krasovskii and Shifted Passivity Approaches to Mixed Input/Output Consensus

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Abstract— **In this letter, we consider nonlinear network systems under unknown disturbances and address the problem of *mixed input/output consensus*, i.e., consensus among disjoint sets of input and output nodes. We develop two control schemes based on different notions of passivity: 1) Krasovskii passivity and 2) shifted passivity. Furthermore, we propose an *input consensus* controller which is applicable to either Krasovskii or shifted passive systems. Finally, we validate the proposed controllers in simulation by achieving current sharing in a heterogeneous DC microgrid and power sharing in an AC power system, which are Krasovskii and shifted passive, respectively.**

I. Introduction

A key goal in cooperative control is the development of protocols and control algorithms that allow a group of agents to reach an agreement on a specific quantity of interest [1]. In the last decades, agreement problems have been examined in different settings, including consensus [1], distributed optimization [2], and synchronization [3]. Among various solutions in the literature to agreement problems, those based on passivity are of particular interest, e.g., [4]–[10] because passivity is a property possessed by various physical systems.

In this letter, we focus on consensus problems, which can be formulated in various ways depending on the consensus control objective. For instance, motivated by the current sharing problem in DC microgrids [11], [12], an *output consensus* controller for nonlinear systems affected by external disturbances is proposed in [10] based on the notions of Krasovskii passivity [13], [14] and shifted passivity [15]–[17]. As another example, *input consensus* under unknown disturbances naturally appears as a solution to a special case of optimal resource allocation problems in power networks [18], also known as optimal load frequency control or economic dispatch [19], [20], where (weighted) input consensus corresponds to achieving proportional power sharing [18]. However, input consensus control is not yet well investigated for general nonlinear systems under unknown disturbances.

Contribution: To develop a unified approach to handle input and output consensus, we generalize the results on output consensus in [10] by designing Krasovskii and shifted passivity based control schemes that achieve mixed input/output consensus in nonlinear network systems affected by unknown disturbances. More precisely, we consider a general scenario where consensus is achieved among the input of some agents and the output of the other agents. Subcases are input consensus and output consensus. The output consensus subcase recovers the controller in [10]. Furthermore, we develop another controller that is specialized to achieve input consensus and at the same time guarantees a form of output regulation, which is applicable to either Krasovskii or shifted passive systems.

The three proposed controllers and their properties are summarized as follows:

- the Krasovskii passivity based output feedback controller can be implemented in a distributed way by assuming the unknown disturbance to be constant;
- the shifted passivity based output feedback controller can handle unknown time-varying disturbances while the controller generally is not distributed.
- the distributed output feedback controller for input consensus is applicable to Krasovskii or shifted passive systems under unknown constant disturbances.

We test the proposed control schemes in simulation addressing relevant agreement problems in power system applications, i.e., current sharing in DC microgrids [11], [12] and the optimal resource allocation problem [18].

Notation: The set of real numbers and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The n-dimensional vector whose all components are 1 is denoted by $\mathbb{1}_n$. The $n \times n$ identity matrix is denoted by I_n. For $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ ($P \succeq 0$) means that P is symmetric and positive (semi) definite. For $x \in \mathbb{R}^n$, its Euclidean norm weighted by $P \succ 0$ is denoted by $|x|_P := \sqrt{x^\top P x}$. If $P = I_n$, this is simply described by $|x|$.

II. Problem Formulation

Consider a compact form of a nonlinear network system under unknown disturbance $d : \mathbb{R} \to \mathbb{R}^r$:

$$\begin{cases} \dot{x} = f(x, u, d) \\ y = h(x, d), \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m$ are of class C^1 such that $\partial f(x, u, d)/\partial u$ is of full column rank at each $(x, u, d) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$. Note that the disturbance d does not necessarily affect all the outputs or state equations.

In this paper, we study weighted *mixed input/output consensus*, i.e., consensus among disjoint sets of input nodes $u_1, \dots, u_\ell$ and output nodes $y_{\ell+1}, \dots, y_m$, where $0 \le \ell \le m$. Let $u_{\mathrm{I}} := [u_1, \cdots, u_\ell]^\top$, $u_{\mathrm{II}} := [u_{\ell+1}, \cdots, u_m]^\top$, $y_{\mathrm{I}} := [y_1, \cdots, y_\ell]^\top$ and $y_{\mathrm{II}} := [y_{\ell+1}, \cdots, y_m]^\top$. The control objective is stated as follows.

Problem 2.1 (mixed input/output consensus): Given $0 \le \ell \le m$ and weight $M \in \mathbb{R}^{m\times m}$, design a controller for the system (1) such that

$$\lim_{t\to\infty} \left(M \begin{bmatrix} u_{\mathrm{I}}(t) \\ y_{\mathrm{II}}(t) \end{bmatrix} - \alpha_d(t)\,\mathbb{1}_m \right) = 0 \tag{2}$$

for some $\alpha_d : \mathbb{R} \to \mathbb{R}$. ◁

When $\ell = 0$ (resp. $\ell = m$), Problem 2.1 corresponds to weighted output consensus, i.e., $\lim_{t\to\infty}(My(t) - \alpha_d(t)\mathbb{1}_m) = 0$ (resp. input consensus, i.e., $\lim_{t\to\infty}(Mu(t) - \alpha_d(t)\mathbb{1}_m) = 0$). Note that the sets of nodes $\{1, \dots, \ell\}$ and $\{\ell + 1, \dots, n\}$ are disjoint, and thus we exclude consensus between the input u_i and output y_i of the same node.

For either Krasovskii or shifted passive systems, when $M = I_m$, output consensus can be achieved by the controller $\dot{u} = -Ly$ [10]. However, its direct extension

$$\dot{u} = -L \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} \tag{3}$$

does not guarantee mixed input/output consensus. Thus, in the next sections we modify the controller structure and provide different controllers based on Krasovskii and shifted passivity. In addition, focusing on input consensus only, we develop another controller that ensures a form of output regulation as well as input consensus. This proposed controller is applicable to either Krasovskii or shifted passive systems.

The proposed methods are applicable to differentially (resp. incrementally) passive systems because they are Krasovskii (resp. shifted) passive with respect to the same outputs [13]. However, direct relations between incremental and Krasovskii passivity and between Krasovskii and shifted passivity with respect to the same outputs are not found. Thus, it is worth developing control techniques based on different passivity notions: Krasovskii and shifted passivity.

III. Krasovskii Passivity Based Control Design

Throughout this section, we assume that the unknown disturbance d is constant, i.e., $\dot{d} = 0$. Under this assumption, we propose a Krasovskii passivity based controller for mixed input/output consensus. We first recall the definition of strict Krasovskii passivity and then show the proposed controller.

A. Krasovskii Passivity

Krasovskii passivity is defined as passivity for the extended system of (1), i.e.,

$$\begin{cases} \dot{x} = f(x, u, d) \\ \dfrac{d\dot{x}}{dt} = \dfrac{\partial f(x, u, d)}{\partial x}\dot{x} + \dfrac{\partial f(x, u, d)}{\partial u}\dot{u} \\ \dot{y} = \dfrac{\partial h(x, d)}{\partial x}\dot{x} \end{cases} \tag{4}$$

with the extended state $(x, \dot{x}, u)$, input $\dot{u}$, and output $\dot{y}$. Namely, this is defined as follows [10, Definition 3.1].

Definition 3.1: Given $d \in \mathbb{R}^r$, the system (1) is said to be *strictly Krasovskii passive* on $\mathcal{D}_K \subset \mathbb{R}^n \times \mathbb{R}^m$ if for its extended system (4), there exist $S_K : \mathcal{D}_K \times \mathbb{R}^n \to \mathbb{R}_+$ of class C^1 and $W_K : \mathcal{D}_K \times \mathbb{R}^n \to \mathbb{R}_+$ of class C^0 such that

$$\dot{S}_K(x, u, \dot{x}) \le -W_K(x, u, \dot{x}) + \dot{y}^\top \dot{u} \tag{5a}$$

$$W_K(x, u, \dot{x}) = 0 \iff \dot{x} = 0 \tag{5b}$$

for all $(x, u) \in \mathcal{D}_K$ and $(\dot{x}, \dot{u}) \in \mathbb{R}^n \times \mathbb{R}^m$. ◁

B. Control Design

For strictly Krasovskii passive systems, we design the following controller:

$$\begin{cases} \dot{\rho} = \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} - \rho \\ \begin{bmatrix} K_1 \dot{u}_{\mathrm{I}} \\ \dot{u}_{\mathrm{II}} \end{bmatrix} = -\begin{bmatrix} \dot{y}_{\mathrm{I}} \\ K_2 \dot{y}_{\mathrm{II}} \end{bmatrix} - L_M \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} - K_3 \ddot{\rho} \end{cases} \tag{6}$$

with the controller states $u, \rho \in \mathbb{R}^m$, where $L_M := M^\top L M$, and $0 \preceq L \in \mathbb{R}^{m\times m}$ is such that $\operatorname{rank} L = m - 1$ and $L\mathbb{1}_m = 0$ (e.g., L is the graph Laplacian matrix of a connected undirected graph), and $0 \prec K_1 \in \mathbb{R}^{\ell\times\ell}$, $0 \preceq K_2 \in \mathbb{R}^{(m-\ell)\times(m-\ell)}$, and $0 \preceq K_3 \in \mathbb{R}^{m\times m}$ are tuning parameters (K_2 and K_3 are allowed to be zero). The ρ-dynamics play roles of dumping dynamics to improve transient performances.

Although we use the representation (6) for closed-loop analysis, this can be described as an output feedback controller by introducing the intermediate variable $\xi \in \mathbb{R}^N$:

$$\begin{cases} \dot{\rho} = \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} - \rho \\ \dot{\xi} = E^\top M \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} \\ \begin{bmatrix} K_1 u_{\mathrm{I}} \\ u_{\mathrm{II}} \end{bmatrix} = -\begin{bmatrix} y_{\mathrm{I}} \\ K_2 y_{\mathrm{II}} \end{bmatrix} - M^\top E\xi - K_3 \left(\begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix} - \rho \right), \end{cases} \tag{7}$$

where $E \in \mathbb{R}^{m\times N}$ is such that $EE^\top = L$ (e.g., E is the incidence matrix associated with a connected undirected graph). From the last equation, u_{I} can be rewritten as

$$u_{\mathrm{I}} = -(K_1 + \hat{K}_3)^{-1}(y_{\mathrm{I}} + \bar{K}_3 y_{\mathrm{II}} + [I_\ell \quad 0](M^\top E\xi - K_3\rho)),$$

where $\hat{K}_3$ and $\bar{K}_3$ are the left top $\ell \times \ell$ and right top $\ell \times (m - \ell)$ block elements of K_3, respectively. Substituting this into ρ- and ξ-dynamics, we obtain an output feedback controller. Moreover, this is distributed when K_1, K_3, and M are diagonal, and K_2 is a graph Laplacian matrix.

Remark 3.2: When $M = I_m$, $K_1 = I_\ell$, $K_2 = 0$, and $K_3 = 0$, (6) becomes

$$\begin{bmatrix} \dot{u}_{\mathrm{I}} \\ \dot{u}_{\mathrm{II}} \end{bmatrix} = -\begin{bmatrix} \dot{y}_{\mathrm{I}} \\ 0 \end{bmatrix} - L \begin{bmatrix} u_{\mathrm{I}} \\ y_{\mathrm{II}} \end{bmatrix}.$$

Adding $-\dot{y}_{\mathrm{I}}$ is the main modification with respect to (3). For output consensus, i.e., $\ell = 0$, (7) with $K_2 = L_M$ recovers the output consensus controller in [10]. ◁

C. Main Theorem

As the first main result of this paper, we show that the controller (6) achieves mixed input/output consensus for strictly Krasovskii passive systems.

Theorem 3.3: Given $d \in \mathbb{R}^r$, suppose that the closed-loop system consisting of a strictly Krasovskii passive system (1) on $\mathcal{D}_K$ and a controller (6) is positively invariant on a compact set $\Omega_K \subset \mathcal{D}_K \times \mathbb{R}^m$. Then, for each $(x(0), u(0), \rho(0)) \in \Omega_K$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that (2) holds.

Proof: Define

$$
\begin{aligned}
&V_K(x,u,\dot{x},\rho)\\
&:= S_K(x,u,\dot{x}) + \frac{1}{2}\left|L^{1/2}M\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix}\right|^2 + \frac{1}{2}\left|\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} - \rho\right|^2_{K_3}.
\end{aligned}
$$

From (5a) and (6) with $L_M = M^\top LM$, the time-derivative of V_K along the closed-loop trajectory satisfies

$$
\begin{aligned}
&\dot{V}_K(x,u,\dot{x},\rho)\\
&\le -W_K(x,u,\dot{x}) + \dot{y}^\top \dot{u}\\
&\quad - \begin{bmatrix}\dot{u}_{\mathrm{I}}\\ \dot{y}_{\mathrm{II}}\end{bmatrix}^\top \begin{bmatrix}K_1\dot{u}_{\mathrm{I}} + \dot{y}_{\mathrm{I}}\\ \dot{u}_{\mathrm{II}} + K_2\dot{y}_{\mathrm{II}}\end{bmatrix} - \left|\begin{bmatrix}\dot{u}_{\mathrm{I}}\\ \dot{y}_{\mathrm{II}}\end{bmatrix} - \dot{\rho}\right|^2_{K_3}\\
&= -W_K(x,u,\dot{x}) - |\dot{u}_{\mathrm{I}}|^2_{K_1} - |\dot{y}_{\mathrm{II}}|^2_{K_2} - \left|\begin{bmatrix}\dot{u}_{\mathrm{I}}\\ \dot{y}_{\mathrm{II}}\end{bmatrix} - \dot{\rho}\right|^2_{K_3}\\
&=: -\bar{W}_K(x,u,\dot{x},\dot{u},\dot{y},\dot{\rho}) \qquad (8)
\end{aligned}
$$

for all $(x,u,\rho) \in \mathcal{D}_K \times \mathbb{R}^m$. Its time integration yields

$$
\begin{aligned}
&V_K(x(t),u(t),\dot{x}(t),\rho(t))\\
&\quad + \int_0^t \bar{W}_K(x(\tau),u(\tau),\dot{x}(\tau),\dot{u}(\tau),\dot{y}(\tau),\dot{\rho}(\tau))d\tau\\
&\le V_K(x(0),u(0),\dot{x}(0),\rho(0)), \qquad (9)
\end{aligned}
$$

where $\dot{x}(0) = f(x(0),u(0),d)$. Since the closed-loop system is positively invariant on compact Ω_K, the integral term exists for each $(x(0),u(0),\rho(0)) \in \Omega_K$. Also, this is upper bounded and increasing with respect to $t \ge 0$, which implies that its limit at $t \to \infty$ exists and is finite. Moreover, by a similar reasoning as the proof of [10, Theorem 3.2], $(x(\cdot),u(\cdot),\rho(\cdot))$, $(\dot{x}(\cdot),\dot{u}(\cdot),\dot{\rho}(\cdot))$, and $\ddot{x}(\cdot)$ are uniformly continuous for each $(x(0),u(0),\rho(0)) \in \Omega_K$.

Applying Barbalat's lemma [21, Lemma 8.2] to $\int_0^t \bar{W}_K d\tau$ in (9), it follows from (8), the continuity of W_K, and the uniform continuity of (x,u,ρ), $(\dot{x},\dot{u},\dot{\rho})$, and $\ddot{x}$ on Ω_K that

$$
\lim_{t\to\infty} K_3\left(\begin{bmatrix}\dot{u}_{\mathrm{I}}(t)\\ \dot{y}_{\mathrm{II}}(t)\end{bmatrix} - \dot{\rho}(t)\right) = 0, \qquad (10)
$$

and $\lim_{t\to\infty} W_K(x(t),u(t),\dot{x}(t)) = 0$ for each $(x(0),u(0),\rho(0)) \in \Omega_K$. From (5b), we have $\lim_{t\to\infty}\dot{x}(t) = 0$ implying $\lim_{t\to\infty}\ddot{x}(t) = 0$. Thus, the second equation of (4) with full column rank $\partial f/\partial u$ and the third equation respectively lead to $\lim_{t\to\infty}\dot{u}(t) = 0$ and $\lim_{t\to\infty}\dot{y}(t) = 0$. Substituting these and (10) into (6) yields

$$
\lim_{t\to\infty} L_M\begin{bmatrix}u_{\mathrm{I}}(t)\\ y_{\mathrm{II}}(t)\end{bmatrix} = \lim_{t\to\infty} M^\top LM\begin{bmatrix}u_{\mathrm{I}}(t)\\ y_{\mathrm{II}}(t)\end{bmatrix} = 0
$$

for each $(x(0),u(0),\rho(0)) \in \Omega_K$. From the property of L, we have mixed input/output consensus (2). ■

IV. Shifted Passivity Based Control Design

In this section, to deal with the time-varying disturbance $d(\cdot)$, we develop a shifted passivity based control technique for mixed input/output consensus.

A. Shifted Passivity

To define shifted passivity, we assume that the system (1) admits an equilibrium trajectory. Namely, given a bounded continuous $d(\cdot)$, there exists a class C^1 bounded trajectory $(x_d^*(\cdot),u_d^*(\cdot))$ such that $\dot{x}_d^*(\cdot) = f(x_d^*(\cdot),u_d^*(\cdot),d(\cdot))$. Accordingly, we define $y_d^*(\cdot) := h(x_d^*(\cdot),d(\cdot))$. Note that even if some state equations do not depend on d, the overall trajectory x_d^* can depend on d via the network interconnection. Under the assumption for the existence of $(x_d^*(\cdot),u_d^*(\cdot))$, we introduce the dynamics of the error $e_d := x - x_d^*$, i.e.,

$$
\begin{cases}
\dot{e}_d = f(e_d + x_d^*, u, d) - f(x_d^*, u_d^*, d)\\
y = h(e_d + x_d^*, d).
\end{cases} \qquad (11)
$$

Using the error dynamics, shifted passivity is defined for time-varying systems as follows [10, Definition 5.2].

Definition 4.1: The system (1) is said to be *shifted passive* along $(x_d^*(\cdot),u_d^*(\cdot))$ on the error (e_d) space $\mathcal{D}_S \subset \mathbb{R}^n$ if for its error dynamics (11), there exist $S_S : \mathcal{D}_S \to \mathbb{R}_+$ of class C^1 and $W_S : \mathcal{D}_S \to \mathbb{R}_+$ of class C^0 such that

$$
\dot{S}_S(e_d) \le -W_S(e_d) + (y - y_d^*)^\top(u - u_d^*) \qquad (12)
$$

for all $e_d \in \mathcal{D}_S$ and $u \in \mathbb{R}^m$. Also, the system is said to be *strictly shifted passive* when $W_S(e_d) = 0$ implies $e_d = 0$. $\lhd$

B. Control Design

For shifted passive systems, we implement a similar but different controller from (7):

$$
\begin{cases}
\dot{\rho} = \begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} - \rho\\
\dot{\xi} = E^\top M\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix}\\
\begin{bmatrix}y_{\mathrm{I}}\\ u_{\mathrm{II}}\end{bmatrix} = -L_M\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} - M^\top E\xi - K_3\left(\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} - \rho\right)
\end{cases} \qquad (13)
$$

with the state $(\xi,\rho) \in \mathbb{R}^N \times \mathbb{R}^m$, where $E \in \mathbb{R}^{m\times N}$, $L_M \in \mathbb{R}^{n\times n}$, and $0 \preceq K_3 \in \mathbb{R}^{m\times m}$ are as in Section III.

The controller equation (13) is described by the implicit form. This admits an explicit form, i.e., can also be represented as an output feedback controller if $K_3 \succ 0$. In fact, the last equation can be decomposed into

$$
\begin{aligned}
&\begin{bmatrix}I_\ell & 0\end{bmatrix}(L_M + K_3)\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} = -y_{\mathrm{I}} - \begin{bmatrix}I_\ell & 0\end{bmatrix}(M^\top E\xi - K_3\rho)\\
&u_{\mathrm{II}} = -\begin{bmatrix}0 & I_{m-\ell}\end{bmatrix}\left((L_M + K_3)\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix} + (M^\top E\xi - K_3\rho)\right).
\end{aligned}
$$

The first equation can be solved with respect to u_{I} if $K_3 \succ 0$. Substituting the obtained u_{I} into the second equation and into the ρ- and ξ-dynamics in (13) results in an output feedback controller. However, this is not distributed in general, since u_{I} is described by using the inverse of the first $\ell \times \ell$ block diagonal element of $L_M + K_3$, and the inverse does not have a distributed structure in general.

Remark 4.2: When $M = I_m$ and $K_3 = 0$, (13) becomes

$$
\begin{bmatrix}\dot{y}_{\mathrm{I}}\\ \dot{u}_{\mathrm{II}}\end{bmatrix} = -L\begin{bmatrix}\dot{u}_{\mathrm{I}}\\ \dot{y}_{\mathrm{II}}\end{bmatrix} - L\begin{bmatrix}u_{\mathrm{I}}\\ y_{\mathrm{II}}\end{bmatrix}.
$$

The essence to achieve mixed input/output consensus in this case is the term $-L[\dot{u}_{\mathrm{I}}^\top \quad \dot{y}_{\mathrm{II}}^\top]^\top$, which is the main difference with respect to (3) and the Krasovskii passivity based controller. For output consensus, i.e., $\ell = 0$, (13) recovers the output consensus controller in [10]. $\lhd$

C. Main Theorem

As the second main result, we show that the controller (13) achieves mixed input/output consensus for shifted passive systems under time-varying disturbance, stated below.

Theorem 4.3: Given bounded continuous $d(\cdot)$, consider the closed-loop system consisting of a system (1) and controller (13). Suppose that

(I) (13) admits an explicit form, e.g., $K_3 \succ 0$;

(II) the closed-loop system admits a class C^1 bounded trajectory $(x_d^*(\cdot), \xi_d^*(\cdot), \rho_d^*(\cdot))$ such that

$$\lim_{t\to\infty} E^\top M \begin{bmatrix} u_{d,\mathrm{I}}^*(t) \\ y_{d,\mathrm{II}}^*(t) \end{bmatrix} = 0;$$

(III) (1) is shifted passive along $(x_d^*(\cdot), u_d^*(\cdot))$ on $\mathcal{D}_S$;

(IV) when rewriting (1) as the error system (11), the closed-loop system is positively invariant on a compact set $\Omega_S \subset \mathcal{D}_S \times \mathbb{R}^N \times \mathbb{R}^m$.

Then, for each $(e_d(t_0), \xi(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that (2) holds.

Proof: Define

$$V_S(e_d, \xi, \rho) := S_S(e_d) + \frac{1}{2}(|\xi - \xi_d^*|^2 + |\rho - \rho_d^*|_{K_3}^2).$$

From (12) and (13), its time-derivative along the closed-loop trajectory satisfies

$$\begin{aligned} &\dot{V}_S(e_d, \xi, \rho) \\ &\le \begin{bmatrix} u_{\mathrm{I}} - u_{d,\mathrm{I}}^* \\ y_{\mathrm{II}} - y_{d,\mathrm{II}}^* \end{bmatrix}^\top \left(\begin{bmatrix} y_{\mathrm{I}} - y_{d,\mathrm{I}}^* \\ u_{\mathrm{II}} - u_{d,\mathrm{II}}^* \end{bmatrix} + M^\top E(\xi - \xi_d^*) \right) \\ &\qquad + (\rho - \rho_d^*)^\top K_3 \left(\begin{bmatrix} u_{\mathrm{I}} - u_{d,\mathrm{I}}^* \\ y_{\mathrm{II}} - y_{d,\mathrm{II}}^* \end{bmatrix} - \rho + \rho_d^* \right) \\ &= - \left| E^\top M \begin{bmatrix} u_{\mathrm{I}} - u_{d,\mathrm{I}}^* \\ y_{\mathrm{II}} - y_{d,\mathrm{II}}^* \end{bmatrix} \right|^2 - \left| \begin{bmatrix} u_{\mathrm{I}} - u_{d,\mathrm{I}}^* \\ y_{\mathrm{II}} - y_{d,\mathrm{II}}^* \end{bmatrix} - \rho + \rho_d^* \right|_{K_3}^2 \\ &=: -\bar{W}_S(u, y, \rho) \end{aligned}$$

for all $(e_d, \xi, \rho) \in \mathcal{D}_S$. Taking the time integration yields

$$\begin{aligned} &V_S(e_d(t), \xi(t), \rho(t)) + \int_{t_0}^t \bar{W}_S(u(\tau), y(\tau), \rho(\tau)) d\tau \\ &\le V_S(e_d(t_0), \xi(t_0), \rho(t_0)). \end{aligned} \tag{14}$$

One can confirm that the limit of the integral term at $t \to \infty$ exists and is finite for each $(e_d(t_0), \xi_d(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$. Similarly to [10, Corollary 3.5], one can show the uniform continuity of $(e_d(\cdot), \xi(\cdot), \rho(\cdot))$ by the positively invariance of the closed-loop system on the compact set Ω_S with the uniform continuity of $d(\cdot)$ and the properties of f and h in (1).

Applying Barbalat's lemma to (14), it follows from the uniform continuity of $(x_d^*(\cdot), \xi_d^*(\cdot), \rho_d^*(\cdot))$ and $(e_d(\cdot), \xi(\cdot), \rho(\cdot))$ on Ω_S that

$$\lim_{t\to\infty} E^\top M \begin{bmatrix} u_{\mathrm{I}}(t) - u_{d,\mathrm{I}}^*(t) \\ y_{\mathrm{II}}(t) - y_{d,\mathrm{II}}^*(t) \end{bmatrix} = 0.$$

Therefore, from item (II), we have

$$\lim_{t\to\infty} E^\top M \begin{bmatrix} u_{\mathrm{I}}(t) \\ y_{\mathrm{II}}(t) \end{bmatrix} = 0$$

for each $(e_d(t_0), \xi(t_0), \rho(t_0)) \in \Omega_S$ and every $t_0 \in \mathbb{R}$. From the property of E, we have weighted mixed input/output consensus (2). ■

Remark 4.4: Theorem 4.3 can be generalized to time-varying systems and can be shown without assuming the full column rank property of $\partial f(x, u, d)/\partial u$. $\lhd$

At the end of this section, we discuss the differences between the two proposed controllers. An advantage of the Krasovskii passivity based controller is to be implemented distributedly without requiring the existence of an equilibrium point, while the unknown disturbance needs to be constant. In contrast, the shifted passivity based controller can handle time-varying disturbances by assuming the existence of an equilibrium trajectory. However, the shifted passivity based controller is not distributed in general.

V. Input Consensus Control

In this section, we focus on input consensus by assuming the unknown disturbance d to be constant. We also aim at achieving a form of output regulation, which is not guaranteed in general by the above controllers. As a modification of (7) (by replacing u with $\dot{u}$ in the last equation), we consider the following dynamic controller:

$$\begin{cases} \dot{\rho} = u - \rho \\ \dot{\xi} = E^\top M u \\ K_1 \dot{u} = (y^* - y) - M^\top E \xi - K_3(u - \rho), \end{cases} \tag{15}$$

where $0 \prec K_1 \in \mathbb{R}^{m\times m}$ and $0 \preceq K_3 \in \mathbb{R}^{m\times m}$ are tuning parameters, and the other free parameter $y^* \in \mathbb{R}^m$ plays the role of specifying the steady value of y.

In fact, this controller achieves input consensus for Krasovskii or shifted passive systems, which is stated without detailed proofs due to the space limitation.

Theorem 5.1: Given $d \in \mathbb{R}^r$, suppose that the closed-loop system consisting of a strictly Krasovskii passive system (1) on $\mathcal{D}_K$ and a controller (15) is positively invariant on a compact set $\bar{\Omega}_K \subset \mathcal{D}_K \times \mathbb{R}^N \times \mathbb{R}^m$. Then, for each $(x(0), u(0), \xi(0), \rho(0)) \in \bar{\Omega}_K$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty}(Mu(t) - \alpha_d(t)\mathbb{1}_m) = 0$, and also it follows that $\lim_{t\to\infty} \mathbb{1}_n^\top M^{-\top}(y(t) - y^*) = 0$ when M is non-singular.

Proof: This can be shown similarly to Theorem 3.3 by using $\bar{V}_K := S_K + (|\dot{u}|_{K_1}^2 + |\dot{\xi}|^2 + |\dot{\rho}|_{K_3}^2)/2$. ■

Theorem 5.2: Given $d \in \mathbb{R}^r$, consider the closed-loop system consisting of a system (1) and a controller (15). Suppose that

(I) the closed-loop system admits an equilibrium point $(x_d^*, u_d^*, \xi_d^*, \rho_d^*)$ such that $y_d^* = y^*$;

(II) (1) is strictly shifted passive at (x_d^*, u_d^*) on $\mathcal{D}_S$;

(III) when rewriting (1) as the error system (11), the closed-loop system is positively invariant on a compact set $\bar{\Omega}_S \subset \mathcal{D}_S \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^m$.

Then, for each $(e_d(t_0), u_d(t_0), \xi(t_0), \rho(t_0)) \in \bar{\Omega}_S$ and every $t_0 \in \mathbb{R}$, there exists $\alpha_d : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to\infty}(Mu(t) - \alpha_d(t)\mathbb{1}_m) = 0$, and also it follows that $\lim_{t\to\infty} y(t) = y^*$.

Proof: Consider

$$\bar{V}_S := S_S(e_s) + \frac{1}{2}(|u - u_d^*|^2_{K_1} + |\xi - \xi_d^*|^2 + |\rho - \rho_d^*|^2_{K_3}).$$

Then, it follows from (12) and (15) that

$$\dot{\bar{V}}_S \le -W_S(e_d) - |u - \rho - u_d^* + \rho_d^*|^2_{K_3}.$$

Noting the uniform continuity of each signal on $\bar{\Omega}_S$, Barbalat's lemma leads to $e_d \to 0$, i.e., $x \to x_d^*$ (and thus $y \to y_d^*$) and $K_3(u-\rho) \to K_3(u_d^* - \rho_d^*)$ as $t \to \infty$. Moreover, it is possible to show that $\dot{x} \to 0$ and $\ddot{x} \to 0$ as $t \to \infty$. Thus, (4) with full column rank $\partial f/\partial u$ and constant d implies $\dot{u} \to 0$ as $t \to \infty$. From (15) with $y^* = y_d^*$, it follows that $-M^\top E\xi - K_3(u_d^* - \rho_d^*) \to 0$ as $t \to \infty$. By considering its time derivative, we have $M^\top E E^\top M u \to 0$ as $t \to \infty$, i.e., weighted input consensus. ■

VI. Examples

In this section, we show via numerical simulations the effectiveness of the proposed control approaches on two different applications: a DC microgrid and an AC power system. Due to the page limit, we show numerical simulations only for controllers (7) and (15).

A. Mixed input/output consensus based on Krasovskii passivity: Current sharing in a DC microgrid

In this subsection, we consider a heterogeneous DC microgrid, where each node can be either a voltage or current source supplying a ZIP load, i.e., the parallel combination of constant impedance, current and power loads. Two main control objectives in DC microgrids are current sharing and average voltage regulation [11], [12]. For the sake of clarity of exposition, we consider only two nodes including, respectively, a voltage and a current source (see Fig. 1), whose dynamics can be expressed as follows

$$\begin{aligned} C_1\dot{V}_1 &= I_1 - I_{l1}(V_1) - I_{12} \\ C_2\dot{V}_2 &= u_2 - I_{l2}(V_2) + I_{12} \\ L_1\dot{I}_1 &= -R_1 I_1 - V_1 + u_1 \\ L_{12}\dot{I}_{12} &= V_1 - V_2 - R_{12}I_{12}, \end{aligned} \tag{16}$$

with $I_{li}(V_i) := G_{li}V_i + \bar{I}_{li} + P_{li}/V_i$, $G_{li}, \bar{I}_{li}, P_{li} \in \mathbb{R}_+$, $i = 1, 2$; see Fig. 1 for the meaning of the used symbols. Let $x := [V_1, V_2, I_1, I_{12}]^\top$, $u = [u_1, u_2]^\top$ and $P := \mathrm{diag}(C_1, C_2, L_1, L_{12})$. It can be proven that the system (16) is strictly Krasovskii passive with respect to the storage function $S_K = \dot{x}^\top P\dot{x}/2$ and supply rate $\dot{I}_1\dot{u}_1 + \dot{V}_2\dot{u}_2$, for all $(x, u) \in \mathcal{D}_K \subset \mathbb{R}^4 \times \mathbb{R}^2$ such that $G_{li} - P_{li}/V_i^2 > 0, i = 1, 2$. Then, in order to achieve current sharing (i.e., $I_1(t) = u_2(t)$ when t approaches to infinity), the controller (7) is used, where $u_{\mathrm{I}} = u_2, u_{\mathrm{II}} = u_1, y_{\mathrm{I}} = V_2, y_{\mathrm{II}} = I_1, E = [-1, 1]^\top, M = 100\, I_{2\times 2}, K_1 = 1 \times 10^{-6}, K_2 = 0$, and $K_3 = 0.2\, I_{2\times 2}$. Moreover, in order to regulate the average voltage towards the desired value $V^* = 380\,\mathrm{V}$, the term $R_1 I_1 + 2V^*$ is added to u_{II}. This can be verified by inspecting (7) and the third line of (16) at the steady state, and by selecting K_1 sufficiently small. Figure 2 shows that the voltage average V_{av} converges to the desired value, and mixed input/output (u_2/I_1) consensus is achieved.

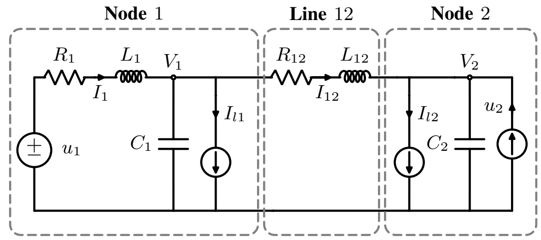

Fig. 1. Electrical scheme of the considered DC microgrid. Parameters: $C_1 = 2.2\,\mathrm{mF}$, $C_2 = 1.9\,\mathrm{mF}$, $L_1 = 1.8\,\mathrm{mH}$, $R_1 = 0.1\,\Omega$, $R_{12} = 50\,\mathrm{m}\Omega$, $L_{12} = 2.1\,\mu\mathrm{H}$, $G_{l1} = 0.08\,\mathrm{S}$, $G_{l1} = 0.04\,\mathrm{S}$, $\bar{I}_{l1} = 12.5\,\mathrm{A}$, $\bar{I}_{l2} = 7.5\,\mathrm{A}$, $P_{l1} = 1\,\mathrm{kW}$ and $P_{l2} = 2.5\,\mathrm{kW}$.

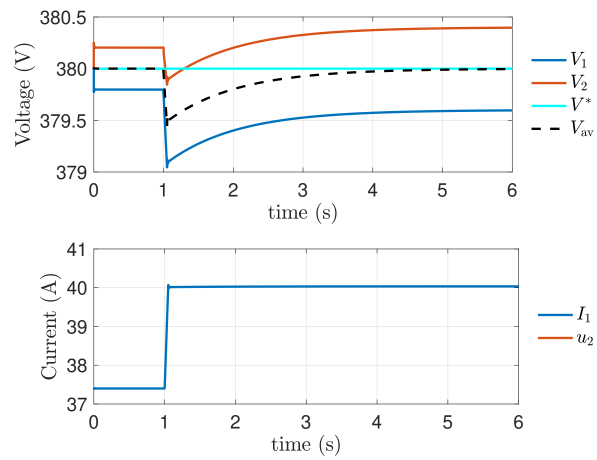

Fig. 2. Mixed input/output consensus (7): Current sharing in a DC microgrid, where at $t = 1\,\mathrm{s}$ the loads become $P_{l1} = 5\,\mathrm{kW}$ and $P_{l2} = 0.5\,\mathrm{kW}$. **(Top)** Time evolution of the voltages and their average (dashed line) together with the corresponding reference (cyan line). **(Bottom)** Time evolution of generated currents.

B. Input consensus based on shifted passivity: Power sharing in an AC power system

In this subsection, we consider a power network with four control areas connected in a ring topology and described by the well-known second-order swing equations, i.e.,

$$\begin{aligned} \dot{\eta} &= B^\top\omega \\ T\dot{\omega} &= -A\omega - B\Gamma\sin\eta - P_l + u, \end{aligned} \tag{17}$$

where η and ω represent the voltage angle difference and the frequency deviation, respectively, P_l is the power demand, and u is the generated power; we refer the interested reader

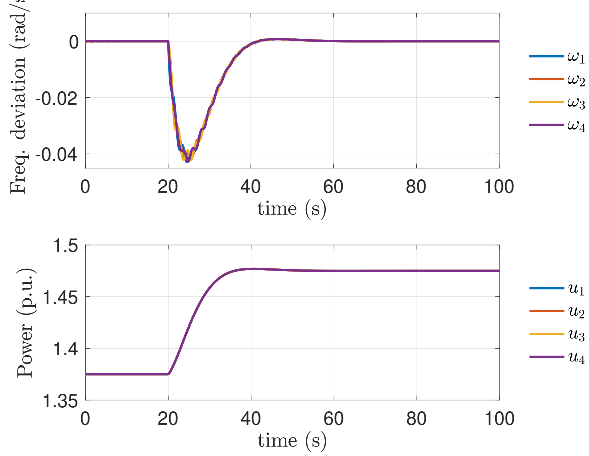

Fig. 3. Input consensus (15): Power sharing in an AC power system; see [19, Section 7] for the simulation settings and system parameters. **(Top)** Time evolution of the frequency deviations. **(Bottom)** Time evolution of the control inputs (generated powers).

to [19] for further details about system (17) and the meaning of its parameters. It has been proven that (17) is output strictly shifted passive with respect to input u and output ω at the equilibrium $(\bar{\eta}, \mathbf{0})$, which fulfills $\bar{\eta}_i \in (-\pi/2, \pi/2)$, $i = 1, \ldots, 4$; see [19, Theorem 1]. Then, in order to solve the optimal resource allocation problem [18], also known as power sharing (i.e., $u_1(t) = \cdots = u_j(t)$ when t approaches to infinity), the controller (15) is used, where $y = \omega, y^* = 0\,\mathrm{rad/s}, M = 100\, I_{4\times 4}, K_1 = 1 \times 10^{-1}$ and $K_3 = 4\, I_{4\times 4}$, while the incidence matrix E is selected such that only the Areas 1, 2 and 3 can exchange information. Moreover, we note that, although (17) is output strictly shifted passive, input consensus can be proven by following similar arguments as in the proof of Theorem 5.2. Figure 3 shows that the frequency deviations converge to zero and input consensus, i.e., power sharing, is achieved.

VII. Conclusion

In this paper, we have studied mixed input/output consensus for Krasovskii or shifted passive systems under unknown disturbances. When the disturbance is constant, the Krasovskii passivity based approach is applicable to design distributed output feedback controllers. The shifted passivity based approach can handle time-varying disturbances while the designed controllers are not distributed in general. We also have developed another controller for input consensus, which is applicable to either Krasovskii or shifted passive systems under unknown constant disturbances. Interesting future work includes considering time-varying disturbances also for Krasovskii passive systems and extending the proposed approaches to more general systems, e.g., dissipative systems with quadratic supply rates [17].

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