2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Data-Driven Exact Pole Placement for Linear Systems

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Abstract— **The exact pole placement problem concerns computing a static feedback law for a linear dynamical system that will assign its poles at a set of pre-specified locations. This is a classic problem in feedback control and numerous methodologies have been proposed in the literature for cases where a model of the system to control is available. In this paper, we study the problem of computing feedback laws for pole placement (and, more generally, eigenstructure assignment) directly from experimental data. Interestingly, we show that the closed-loop poles can be placed *exactly* at arbitrary locations without relying on any model description but by using only finite-length trajectories generated by the open-loop system. In turn, these findings imply that classical control goals, such as feedback stabilization or meeting transient transient performance specifications, can be achieved directly from data without first identifying a system model. Numerical experiments demonstrate the benefits of the data-driven pole-placement approach compared to its model-based counterpart.**

I. Introduction

Data-driven control methods enable the synthesis of controllers directly from data generated by physical systems, and thus elude the need to construct or identify a mathematical model. Data-driven approaches are especially useful in cases where first principles are challenging to apply, models are difficult to identify, or the identification task leads to numerically-unreliable solutions [1], [2]. In these cases, data-driven techniques set out a huge potential as controllers can be synthesized directly from data, and thus uncertainties in the model shall not compromise the controller quality.

Data-driven control is, by now, a well-investigated area of research (see, e.g., the representative works [3]–[5]). Despite the availability of several techniques to synthesize various types of controllers from data, the problem of data-driven pole placement and the (more general) problem of data-driven eigenstructure assignment via static feedback has not been studied until now. The classical problem of pole placement consists of finding a static feedback law such that the poles of the closed-loop system are placed at a set of pre-specified locations; analogously, the problem of eigenstructure assignment is that of finding a feedback law such that the closed-loop system has a pre-specified set of eigenvalues and eigenvectors (hereafter named eigenstructure). Motivated by this, in this paper we study the data-driven pole placement problem and the data-driven eigenstructure assignment problem. Our results show that it is possible to place the closed-loop eigenvalues *exactly* at arbitrary locations (here, “exactly” means that the closed-loop poles can be placed at exact locations, in contrast with cases where they are placed within certain regions) by using formulas that can be applied directly on data. Moreover, our results show that the data-driven eigenstructure assignment problem is feasible under the same conditions required for its model-based counterpart.

Paper contributions. This paper features two main contributions. First, we show that static feedback laws (described by a feedback gain) that place the poles at an arbitrary set of locations can be computed directly from data collected from finite-length open-loop control experiments. We remark that our formulas apply also to cases where the open-loop system is not stable. We provide an explicit formula to compute the feedback gain and we show that the problem is always feasible when the underlying system is controllable. Second, we study the eigenstructure assignment problem and we provide a necessary and sufficient condition to check when such problem is feasible. Moreover, we provide an explicit formula to compute feedback gains that assign a pre-specified eigenstructure. Finally, as a minor contribution, we evaluate via numerical simulations the benefits of the proposed data-driven method as compared to model-based approaches.

Related work. Several techniques have been proposed to synthesize controllers from data while avoiding the need to identify a system model. Solutions for static feedback control are studied in [6], [7], the linear quadratic regulator (LQR) in [3], model predictive control (MPC) in [5], [8], minimum-energy control laws in [4], trajectory tracking problems in [9], distributed control problems in [10], and feedback-optimization controllers are proposed in [11]. Some extensions to the case of nonlinear systems are presented in [12], [13]. Most of these methods exploit the ability to express future trajectories of a linear system in terms of a sufficiently rich past trajectory, as shown by the Fundamental Lemma [14]. With respect to this body of literature, in this work, we focus on the exact pole-placement problem.

The model-based exact pole placement problem has a long history; a non-exhaustive list of references includes [15]–[18]. All these methods derive feedback laws departing from a model description of the system to control, while our focus here is to derive formulas that can be applied directly on data. In line with this work is the recent contribution [19], which studies the problem of placing the closed-loop poles in linear matrix inequality (LMI) regions; in contrast, in this work, we focus on placing the poles at *exact* locations and, in addition, we address the eigenstructure assignment problem.

II. Preliminaries

In this section, we recall some useful facts on behavioral system theory from [14]. Given a signal (time-series) $z : \mathbb{Z} \to \mathbb{R}^\sigma$, and scalars $T \in \mathbb{Z}_{\geq 0}$, $i \in \mathbb{Z}_{\geq 0}$, we

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denote the restriction of z to the interval $[i, i+T-1]$ by $z_{[i,i+T-1]} = \{z(i), \ldots, z(i+T-1)\}$. With a slight abuse of notation, we will denote by $z_{[i,i+T-1]} := (z(i), \ldots, z(i+T-1)) \in \mathbb{R}^{\sigma T}$ also the vectorization of $z_{[i,i+T-1]}$, where the distinction will be clear from the context. Given the $T-$long signal $z_{[i,i+T-1]}$, we denote the associated Hankel matrix with $L \geq 1$ (block) rows by:

$$\mathcal{H}_L(z_{[i,i+T-1]}) = \begin{bmatrix} z(i) & z(i+1) & \ldots & z(i+T-L) \\ z(i+1) & z(i+2) & \ldots & z(i+T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ z(i+L-1) & z(i+L) & \ldots & z(i+T-1) \end{bmatrix},$$

Notice that $\mathcal{H}_L(z_{[i,i+T-1]}) \in \mathbb{R}^{L\sigma \times (T-L+1)}$. The following definition is instrumental for our analysis.

Definition 2.1: ***(Persistently exciting signal [14])*** The signal $z_{[i,i+T-1]} \in \mathbb{R}^{\sigma T}$ is persistently exciting of order L if the matrix $\mathcal{H}_L(z_{[i,i+T-1]})$ has full row rank σL. □

We note that persistence of excitation implicitly requires that the number of columns of $\mathcal{H}_L(z_{[i,i+T-1]})$ is non-smaller than the number of rows, thus giving $T-L+1 \geq L\sigma$. We recall the following property of persistently exciting inputs.

Lemma 2.2: ***(Fundamental Lemma [14, Thm 1])*** Assume that the following linear system is controllable: $x(t+1) = Ax(t) + Bu(t)$, where $x : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ and $u : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ and A, B are matrices of suitable dimensions, and let $(u_{[0,T-1]}, x_{[0,T-1]})$ be an input-state trajectory generated by this system. If $u_{[0,T-1]}$ is persistently exciting of order $n+d$:

$$\operatorname{rank} \begin{bmatrix} \mathcal{H}_1(x_{[0,T-1]}) \\ \mathcal{H}_d(u_{[0,T-1]}) \end{bmatrix} = n + dm.$$

□

This condition will play a fundamental role in the sequel.

III. Problem Setting

In this section, we formulate the problem of interest and discuss existing model-based techniques for its solution.

A. Problem formulation

Consider the discrete-time linear time-invariant system:

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ denote, respectively, the system and input matrices, and $x : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ and $u : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ denote, respectively, the state and input signals. We assume that B has full column rank. The behavior of (1) is governed by the *poles* of the system, that is, by the eigenvalues of A. It is often desirable to modify the poles to obtain certain properties, such as system stability or a desired transient performance. This can be achieved using a state-feedback control law of the form $u(t) = -Kx(t) + v(t)$, where $v : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$ is a new free input variable and $K \in \mathbb{R}^{m \times n}$ is the state feedback matrix (also called feedback gain), which should be chosen so that the controlled system

$$x(t+1) = (A - BK)x(t) + v(t), \tag{2}$$

has its poles at desired locations. In line with [15]–[18], we make the following assumption.

Assumption 1 (***Desired set of pole locations***)**:** The set of desired pole locations contains n complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$ and is closed under complex conjugation. □

The data-driven state-feedback pole placement problem can now be made mathematically formal as follows.

Problem 1 (***Pole placement***)**:** Given a set of complex numbers $\mathcal{L}$ satisfying Assumption 1 and historical data $\mathcal{D} = (u, x)$ generated by (1), find, when possible, a state feedback matrix $K \in \mathbb{R}^{m \times n}$ such that the eigenvalues of $A - BK$ are the elements of the set $\mathcal{L}$. □

Conditions for the existence of solutions to the pole placement problem are well known [20]: a solution exists if and only if $\mathcal{L}$ contains all uncontrollable modes of (A, B) [20]. Motivated by this, we will make the following assumption.

Assumption 2 (***Controllability***)**:** All modes of (A, B) are controllable. □

In the single-input case ($m = 1$), the solution to Problem 1, when it exists, is unique [20]. In the multi-input case $m > 1$, the feedback gain K that solves the pole placement problem is in general non-unique. One typical way to select a particular K within the ambiguity set is to choose the one that assigns the closed-loop eigenstructure:

$$(A - BK)X = X\Lambda, \tag{3}$$

where Λ is an $n \times n$ diagonal matrix with spectrum given by $\mathcal{L}$ and X is a non-singular matrix of associated closed-loop eigenvectors, chosen according to some notion of optimality. For instance, [16, Sec. 2.5] shows that choosing a matrix of eigenvectors X that is well-conditioned leads to pole locations that are robust against perturbations of the entries of A. Motivated by this, in addition to Problem 1, we will consider the data-driven state-feedback eigenstructure assignment problem, formulated precisely as follows.

Problem 2 (***Eigenstructure assignment***)**:** Given a set of complex numbers $\mathcal{L}$ satisfying Assumption 1, a matrix of linearly independent eigenvectors X, and historical data $\mathcal{D} = (u, x)$ generated by (1), find, when possible, a state feedback matrix $K \in \mathbb{R}^{m \times n}$ such that (3) holds. □

B. Existing model-based pole-placement methods

When A and B are known, several formulas have been proposed in the literature to solve the eigenstructure assignment problem. We next summarize some of the most celebrated. For a matrix M, we denote by $M^\dagger$ its Moore-Penrose inverse; if M is square, $\lambda(M)$ denotes its spectrum.

1) Approach in [16, Thm 3]. Let $B = \begin{bmatrix} U_0, & U_1 \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix}$ with $[U_0, \;\; U_1]$ orthogonal and Z nonsingular. The following choice satisfies (3):
$$K = Z^{-1} U_0^\mathsf{T} (A - X\Lambda X^{-1}). \tag{4}$$
2) Approach in [17, Main Theorem]. Assume that $\lambda(A) \cap \lambda(\Lambda) = \emptyset$ and let G and X satisfy $AX - X\Lambda + BG = 0$. Then, the following choice satisfies (3):
$$K = -GX^{-1}. \tag{5}$$

3) Approach in [18, Thm 1]. Let X be an invertible matrix that satisfies $(I - BB^\dagger)(X\Lambda - AX) = 0$. Then, the following choice satisfies (3):

$$K = B^\dagger(A - X\Lambda X^{-1}). \tag{6}$$

It is evident from (4)-(6) (see also Remark 3.1) that to obtain a numerically-reliable K using these formulas, matrices (A, B) must be known with high precision.

Remark 3.1: ***(Eigenvalue sensitivity with respect to model uncertainty)*** It is possible to quantify the sensitivity of the eigenvalues of $A - BK$ against perturbations of the entries of A or B as follows. Let λ denote a simple eigenvalue of $M := A - BK$ with left and right eigenvectors x and y, respectively. Wilkinson [21] showed that if a perturbation ΔM is made to the entries of M, there exists a simple eigenvalue $\hat{\lambda}$ of $M + \Delta M$ such that

$$|\hat{\lambda} - \lambda| \leq \mathrm{cond}(\lambda, M)\|\Delta M\| + O(\|\Delta M\|^2),$$

where $\mathrm{cond}(\lambda, M) = \frac{\|x\|\|y\|}{|y^* x|}$ denotes the condition number of λ (here, y^* denotes the conjugate transpose of y). Notice that $\mathrm{cond}(\lambda, M) \geq 1$ and $\mathrm{cond}(\lambda, M) = 1$ if and only if M is a normal matrix, that is $M^\mathsf{T} M = M M^\mathsf{T}$. Thus, in a first-order sense, perturbations of the entries of A or B lead to shifts in the eigenvalues of $A - BK$ as amplified by the condition number of the matrix of eigenvectors X. □

Since, in practice, matrices (A, B) must be first identified from (possibly noisy) historical data before (4)-(6) can be applied, a promising way to reduce the sensitivity of the closed-loop pole locations is to bypass the system identification process and to develop methods for determining K directly from data. Motivated by this, the focus of this paper is on deriving direct formulas for pole placement from data, which do not require knowledge of A and B.

IV. Data-Driven pole placement

In this section, we will tackle Problem 1. We will assume the availability of historical data $\mathcal{D} = (u, x)$ generated by (1), and we will use the following representation of the data:

$$\begin{aligned} U_0 &:= \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix} \in \mathbb{R}^{m \times T}, \\ X_0 &:= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} \in \mathbb{R}^{n \times T}, \\ X_1 &:= \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix} \in \mathbb{R}^{n \times T}. \end{aligned} \tag{7}$$

Notice that only the first T samples of u and $T+1$ samples of x are needed to construct U_0, X_0, X_1. In what follows, for a matrix M, we will denote by $\mathscr{R}\{M\}$ the range space generated by its columns and by $\mathscr{N}\{M\}$ their null space.

Theorem 4.1 (***Data-driven pole placement***)**:** Let assumptions 1–2 be satisfied, $\mathcal{L} = \{\lambda_1, \dots, \lambda_n\}$, and $u_{[0,T-1]}$ be persistently exciting of order $n+1$. Then, there exists $M = [m_1, \dots, m_n] \in \mathbb{R}^{T \times n}$, with $\mathrm{rank}(M) = n$, such that:

$$0 = (X_1 - \lambda_i X_0) m_i, \qquad \forall i \in \{1, \dots, n\}. \tag{8}$$

Moreover, for any M that satisfies (8), the matrix

$$K = -U_0 M (X_0 M)^\dagger, \tag{9}$$

satisfies $\det(A - BK - \lambda I) = 0$ for all $\lambda \in \mathcal{L}$. □

Proof: To prove existence of M, notice that

$$(X_1 - \lambda_i X_0) m_i = \begin{bmatrix} A - \lambda_i I, & B \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} m_i. \tag{10}$$

Since (A, B) is controllable, $\mathrm{rank}[A - \lambda_i I,\ B] = n$ and thus $[A - \lambda_i I,\ B]$ has a nontrivial (m-dimensional) null space. Thus, it is sufficient to choose the columns of M so that:

$$\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} m_i \in \mathscr{N}\{\begin{bmatrix} A - \lambda_i I, & B \end{bmatrix}\}. \tag{11}$$

Since $u_{[0,T-1]}$ is persistently exciting of order $n+1$, Lemma 2.2 guarantees $\mathrm{rank}[X_0^\mathsf{T}, U_0^\mathsf{T}]^\mathsf{T} = n + m$, and thus m_i can always be chosen so that (11) holds, thus proving existence of M. To show that $\mathrm{rank}(M) = n$, notice that

$$\dim \mathscr{N}\{\begin{bmatrix} X_0 \\ U_0 \end{bmatrix}\} \geq mn, \tag{12}$$

and thus there always exist n linearly independent vectors m_i that satisfy (11). To show (12), notice that $[X_0^\mathsf{T}, U_0^\mathsf{T}]^\mathsf{T}$ is an $(n+m) \times T$ dimensional matrix; since $u_{[0,T-1]}$ is persistently exciting of order $n+1$, we have $T \geq nm + m + n$ (cf. discussion after Definition 2.1), from which (12) follows.

To prove the second part of the claim, notice that

$$\begin{aligned} 0 = (X_1 - \lambda_i X_0) m_i &= (A X_0 + B U_0 - \lambda_i X_0) m_i \\ &= (A - \lambda_i I) X_0 m_i + B U_0 m_i, \end{aligned} \tag{13}$$

where the second identity follows from $X_1 = A X_0 + B U_0$, which holds because X_0, X_1, U_0 are generated by (1). Next, by using (9) we have $-U_0 m_i = K X_0 m_i$. In fact, since $\mathrm{rank}(M) = n$, $\mathrm{rank}(X_0 M) = n$ and thus $(X_0 M)^\dagger$ is a right inverse of $X_0 M$. By substituting this identity into (13):

$$(A - BK - \lambda_i I) X_0 m_i = 0,$$

which proves the claim. ■

The formula (9) provides a direct way to determine feedback gains by performing algebraic operations on the data and without identifying (A, B). The condition (8) specifies a set of linear equations in the unknown M, which can thus be determined using standard linear equation solvers. We refer to Section VI for an illustration of the numerical benefits of using (9) as compared to model-based pole placement.

In our result, persistence of excitation is needed to guarantee existence of M that satisfies (8); persistence of excitation may be relaxed in practice, provided that the data satisfies (11). Finally, we recall that, as a well-known result, the feedback gain K that places the poles at $\mathcal{L}$ is in general not unique. In our formula, non-uniqueness of K is reflected in the non-uniqueness of M. Accordingly, all admissible K that place the poles at $\mathcal{L}$ can be obtained by varying the i-th column of M within the null space of $X_1 - \lambda_i X_0$. We refer to (11) in the proof of the theorem for a precise discussion.

Remark 4.2: ***(Extension to continuous-time systems)*** When the system to control is a continuous-time one, namely, (1) is replaced by $\dot{x}(t) = Ax(t) + Bu(t)$, the formula (9) still holds unchanged, provided that (7) are replaced by

$$
\begin{aligned}
U_0 &:= \begin{bmatrix} u(0) & u(\Delta) & \dots & u((T-1)\Delta) \end{bmatrix} \in \mathbb{R}^{m\times T},\\
X_0 &:= \begin{bmatrix} x(0) & x(\Delta) & \dots & x((T-1)\Delta) \end{bmatrix} \in \mathbb{R}^{n\times T},\\
X_1 &:= \begin{bmatrix} \dot{x}(0) & \dot{x}(\Delta) & \dots & \dot{x}((T-1)\Delta) \end{bmatrix} \in \mathbb{R}^{n\times T},
\end{aligned}
$$

where $\Delta > 0$ is an arbitrary sampling time. □

V. DATA-DRIVEN EIGENSTRUCTURE ASSIGNMENT

In this section, we will tackle the eigenstructure assignment problem. It is natural to begin by asking under what conditions a given nonsingular matrix X can be assigned as eigenvectors. The following result addresses this question.

*Theorem 5.1 (**Feasibility of eigenstructure assignment**)*: Let Assumptions 1–2 hold, $X \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and let $u_{[0,T-1]}$ be persistently exciting of order $n+1$. There exists a solution K to (3) if and only if

$$
\Delta A := A - X\Lambda X^{-1} \in \mathscr{R}\{X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ I_m \end{bmatrix}\}. \tag{14}
$$

□

Proof: Notice that (3) holds if and only if

$$
-BK = X\Lambda X^{-1} - A.
$$

Since K is a free variable, this holds if and only if $X\Lambda X^{-1} - A \in \mathscr{R}\{B\}$. To characterize $\mathscr{R}\{B\}$, let $z \in \mathbb{R}^m$ be arbitrary, and notice that Bz can be expressed as:

$$
Bz = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} g, \tag{15}
$$

for some $g \in \mathbb{R}^{T-1}$. Here, the last identity follows by noting that, because $u_{[0,T-1]}$ is persistently exciting of order $n+1$, by Lemma 2.2 $\operatorname{rank} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} = n+m$, and thus there exists g such that (15) holds. Any g as in (15) can be expressed as:

$$
g = \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ z \end{bmatrix} + w, \tag{16}
$$

where w satisfies $X_0 w = U_0 w = 0$. Next, rewrite (15) as:

$$
Bz = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} g = X_1 g. \tag{17}
$$

By combining (16) with (17), we obtain:

$$
\begin{aligned}
Bz &= X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ z \end{bmatrix} + X_1 w \\
&= X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ z \end{bmatrix} + \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix} w = X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ z \end{bmatrix},
\end{aligned}
$$

where the last identity follows by definition of w. Hence,

$$
\mathscr{R}\{B\} = \mathscr{R}\{X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ I_m \end{bmatrix}\},
$$

which proves the claim. ■

The theorem provides a characterization of all perturbations ΔA of A that can be achieved via static feedback: these are all and only the matrices that belong to the space:

$$
\mathscr{R}\{X_1 \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 \\ I_m \end{bmatrix}\}.
$$

When the open-loop system matrix A is known, the theorem also provides a condition to determine whether the eigenstructure assignment problem admits a solution: the problem is feasible if and only if $A - X\Lambda X^{-1}$ belongs to the range space of the matrix characterized in (14).

Before stating our result, we present the following technical lemma, which is a direct consequence of [16, Cor 1].

Lemma 5.2: Let Assumptions 1–2 hold, $X \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and x_j denote the j-th column of X corresponding to $\lambda_j \in \mathcal{L}$. Then, $x_j \in \mathcal{S}_j$, where

$$
\mathcal{S}_j = \mathscr{N}\{U^{\mathsf{T}}(A - \lambda_j I)\},
$$

and U is such that its columns form a basis for $\mathscr{N}\{B\}$. Moreover, the space $\mathcal{S}_j$ has dimension m. □

We remark that this lemma is of model-based nature, and thus the provided characterization is of little use when A and B are unknown. Despite its nature, in what follows we will use this lemma for technical purposes (to derive necessary conditions for the eigenstructure assignment problem to be feasible and in the proof of the subsequent result). Since the maximum number of independent eigenvectors that can be chosen for each assigned eigenvalue is equal to $\dim(\mathcal{S}_j) = m$, it follows that the algebraic multiplicity of the eigenvalue $\lambda_j \in \mathcal{L}$ to be assigned must be less than or equal to m. The lemma thus motivates the following assumption.

*Assumption 3 (**Eigenstructure properties**)*: The pole locations $\mathcal{L} = \{\lambda_1, \dots, \lambda_n\}$ and eigenvectors X satisfy:

1) $\mathcal{L}$ contains ν complex numbers with associated algebraic multiplicities $\{m_1, \dots, m_\nu\}$ satisfying $m_1 + \dots + m_\nu = n$ and $m_i \le m$ for all $i \in \{1, \dots, \nu\}$,
2) pairs of complex conjugate poles with $\lambda_i = \lambda_j^*$ satisfy $m_i = m_j$,
3) X is such that the desired $(A - BK)$ is non-defective (i.e., it admits n linearly independent eigenvectors). □

With this technical assumption, we now provide the following formula for eigenstructure assignment.

*Theorem 5.3 (**Data-driven eigenstructure assignment**)*: Let Assumptions 1–3 be satisfied and $u_{[0,T-1]}$ be persistently exciting of order $n+1$. There exists a matrix $M = [m_1, \dots, m_n] \in \mathbb{R}^{T-1\times n}$, with $\operatorname{rank}(M) = n$, that satisfies:

$$
\begin{aligned}
0 &= (X_1 - \lambda_i X_0) m_i, \qquad \forall i \in \{1, \dots, n\},\\
X &= X_0 M,
\end{aligned} \tag{18}
$$

Moreover, given M as in (18), the following K satisfies (3):

$$
K = -U_0 M (X_0 M)^{\dagger}. \tag{19}
$$

□

Proof: We begin by proving the existence of M. By iterating the steps in (10)–(11) for the first condition in (18), we conclude that a matrix M that satisfies (18) exists if and only if the following two conditions hold simultaneously:

$$
\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} m_i \in \mathscr{N}(\begin{bmatrix} A - \lambda_i I, & B \end{bmatrix}), \qquad x_i = X_0 m_i,
$$

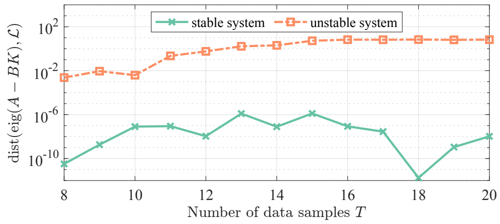

Fig. 1. Accuracy of closed-loop pole locations obtained using (9) when data generated by a stable system (green line) and by an unstable system (orange dashed line). The considered unstable system is the chemical reactor (20); data for the stable system has been generated by pre-stabilizing the reactor model. The simulation illustrates that (9) is more accurate by several orders of magnitude when applied to data generated by a stable system.

where x_i denotes the i-th row of X. Since Lemma 5.2 guarantees that $x_i \in \mathscr{N}\{B\}$ and (12) holds we conclude that there exists at least n linearly independent vectors that satisfy (18). To prove the second part of the claim, notice:

$$\begin{aligned}0 &= (X_1 - \lambda_i X_0)m_i = (AX_0 + BU_0 - \lambda_i X_0)m_i \\ &= (A - \lambda_i I)X_0 m_i + BU_0 m_i,\end{aligned}$$

where the last identity follows from $X_1 = AX_0 + BU_0$, which holds because X_0, X_1, U_0 are generated by (1). Next, by using (9) we have $-U_0 m_i = KX_0 m_i$. In fact, since $\operatorname{rank}(M) = n$, $\operatorname{rank}(X_0 M) = n$ and thus $(X_0 M)^\dagger$ is a right inverse of $X_0 M$. By substituting this identity into (13):

$$(A - BK - \lambda_i I)X_0 m_i = 0,$$

from which λ_i is an eigenvalue of $A - BK$ with eigenvector $X_0 m_i$. The conclusion follows using $X = X_0 M$. ■

The formula (19) provides an explicit way to determine feedback gains that assign the desired eigenstructure by performing algebraic computations on the data. Notice that, with respect to the conditions for pole placement (8), assigning the eigenstructure imposes n^2 additional constraints on M (given by $X = X_0 M$). Similarly to (8), condition (18) specifies a set of linear equations in the unknown M, and thus M can be determined using standard linear equation solvers.

VI. Numerical analysis

In this section, we illustrate the methods via numerical simulations. We first apply the formulas to stabilize the dynamics of a chemical reactor, and then we compare their accuracy with respect to their model-based counterparts.

Consider the following model describing a chemical reactor obtained by discretizing [16, Example 1]:

$$A = \begin{bmatrix} 6.9771 & 2.0379 & 5.0672 & -2.2212 \\ -0.6941 & -0.0434 & -0.4738 & 0.3425 \\ 0.2048 & 0.9081 & 0.3159 & 0.6172 \\ -0.5082 & 0.7106 & -0.2000 & 0.8531 \end{bmatrix},$$

$$B^\mathsf{T} = \begin{bmatrix} 4.8874 & 1.4777 & 5.0448 & 4.6020 \\ -6.5545 & 0.5230 & -1.1389 & -0.1133 \end{bmatrix}. \quad (20)$$

This system is unstable and the open-loop eigenvalues are:

$$\operatorname{eig}(A) = \{7.0162, 1.0798, 0.0002, 0.0065\},$$

and thus state feedback is required to stabilize the system. We move the two unstable modes inside the unitary circle, keeping the original stable modes. We thus assign the set: $\mathcal{L} = \{0.5, 0.3, 0.0002, 0.0065\}$. Historical data is generated by simulating the open-loop system for $T = 10$ time steps by applying i.i.d. Gaussian noise as the input signal and starting from zero initial conditions. It is well-known that this input is persistently exciting of any order. The feedback gain obtained as in (9) using the built-in `fsolve` routine in `Matlab R2022a` to solve (8) is:

$$K = \begin{bmatrix} -0.1758 & -1.3970 & 2.8668 & -2.4679 \\ -0.4441 & 0.2711 & 4.9848 & -4.9424 \end{bmatrix}, \quad (21)$$

leading to the closed-loop eigenvalues:

$$\operatorname{eig}(A - BK) = \{0.4999, 0.3001, 0.0002, 0.0066\}.$$

We interpret the error between the elements of $\mathcal{L}$ and the spectrum of $A - BK$ as numerical error due to the poor conditioning of the regression problem (8) resulting from the use of data generated by an unstable system (whose state is diverging over time). For example, after $t = 10$ time steps, we observed $\|x(10)\| = 1.254 \times 10^5$, which makes the regression matrix in (8) numerically poorly conditioned.

To further illustrate the challenges in dealing with unstable systems, Fig. 1 compares the accuracy of the closed-loop eigenvalues when (9) is applied to data generated by an unstable system (orange lines) and when it is applied to data generated by a stable system (green lines). The latter is obtained by first stabilizing (20) with $u(t) = -Kx(t) + v(t)$ using K given as in (21) and, subsequently, by using the input $v(t) = -K_2 x(t)$ (see (2)) with K_2 obtained by applying (9) to data generated by $A - BK$. The figure illustrates that the accuracy of the resulting pole locations deteriorates as the size of the data set T increases when the data is generated by an unstable system and, on the other hand, the pole accuracy remains high when the data is generated by a stable system. These findings suggest that a tradeoff must be found for T so that there is enough data to satisfy the persistence of excitation conditions and, at the same time, limited data is used to avoid numerical issues.

Remark 6.1: ***(Challenges in controlling unstable dynamics)*** Challenges related to identifying and controlling unstable dynamics have previously been observed in both the system identification and data-driven control literature; we point out that a promising approach to handle these cases is that of combining multiple short trajectories [22]. □

Next, we compare the accuracy of pole locations obtained using (9) and using a model-based formula, applied to an identified model. In both cases, the methods are applied to noisy data, and we conducted Montecarlo simulations over 100 experiments. We generated the data by simulating:

$$x(t+1) = A_0 x(t) + B_0 u(t) + e(t),$$

with $x(0) \sim \mathcal{N}(0, I_n)$, $u(t) \sim \mathcal{N}(0, I_m)$, $e(t) \sim \mathcal{N}(0, \sigma_e^2 I_n)$, and the matrices A_0 and B_0 have been chosen randomly and such that the modulus of all the eigenvalues of A_0 is inside the unit circle and (A_0, B_0) is controllable, with $m = \lfloor n/2 \rfloor$. We identified (A_0, B_0) from noisy data by

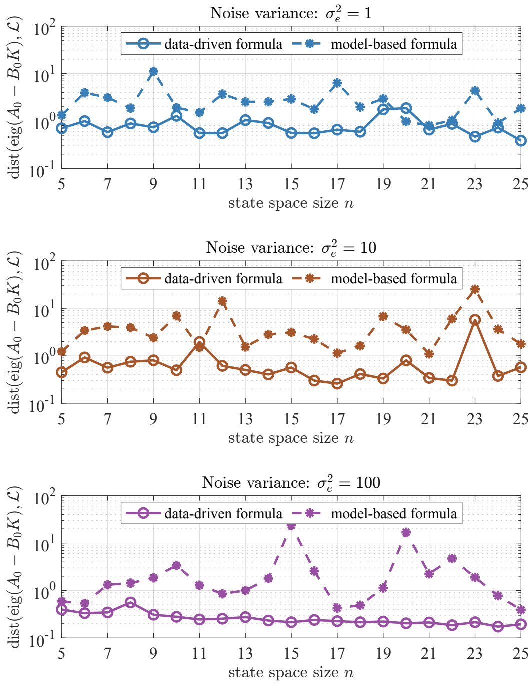

Fig. 2. Montecarlo simulation comparing the accuracy of (9) with that of a model-based formula applied to an identified model. The methods have been applied to noisy data with Gaussian distribution and three different levels of variance: $\sigma_e^2 = 1$ (top), $\sigma_e^2 = 10$ (middle), and $\sigma_e^2 = 100$ (bottom). The results suggest that the higher the noise variance, the more the data-driven formula becomes preferable over a model-based pole placement approach.

solving the least-squares problem:

$$\begin{bmatrix} A & B \end{bmatrix} \in \arg\min_{[A,B]} \|X_1 - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_0 \\ U_0 \end{bmatrix}\|_{\mathrm{F}}.$$

The set $\mathcal{L}$ has been chosen so that its entries are uniformly distributed in $[-n, n]$; for the model-based pole placement, we determined K using the built-in `place` routine in `Matlab R2022a`. Fig. 2 compares the accuracy of the pole locations obtained using the model-based placement routine and the formula (9), for increasing values of the state space size n. The figure illustrates that the pole locations obtained through (9) are more accurate by about one order of magnitude for all considered values of n. Moreover, by comparing the results for three different choices of the noise variance: $\sigma_e^2 = 1$ (top), $\sigma_e^2 = 10$ (middle), and $\sigma_e^2 = 100$ (bottom), the numerics suggest the higher the noise variance, the more the data-driven approach is preferable over the model-based one. We interpret this result by noting that errors in the identified (A, B) propagates nonlinearly through the `place` routine, thus compromising the controller quality.

VII. Conclusions

In this paper, we derived data-driven formulas to compute static feedback matrices that assign arbitrarily the eigenstructure of a linear dynamical system. By leveraging the linearity of the dynamics and a persistence of excitation condition, we showed for the first time that the closed-loop eigenstructure can be assigned *exactly*. Further, we illustrated the benefits of the data-driven methods, as compared to the model-based counterpart, through a set of numerical simulations, which showcase the numerical robustness of the approach, especially in the presence of noise in the measured data. This paper also opens several directions for future research, including an analytic investigation of the sensitivity of the closed-loop pole locations in the presence of noise, and the derivation of methods to handle uncontrollable modes.

References

[1] V. Krishnan and F. Pasqualetti, "On direct vs indirect data-driven predictive control," in *IEEE Conf. on Decision and Control*, Austin, TX, Dec. 2021, pp. 736–741.

[2] F. Dörfler, J. Coulson, and I. Markovsky, "Bridging direct & indirect data-driven control formulations via regularizations and relaxations," *IEEE Transactions on Automatic Control*, 2022, (Early access).

[3] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2019.

[4] G. Baggio, V. Katewa, and F. Pasqualetti, "Data-driven minimum-energy controls for linear systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 589–594, 2019.

[5] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: In the shallows of the DeePC," in *European Control Conference*, 2019, pp. 307–312.

[6] T. M. Maupong and P. Rapisarda, "Data-driven control: A behavioral approach," *Systems & Control Letters*, vol. 101, pp. 37–43, 2017.

[7] S. Talebi, S. Alemzadeh, N. Rahimi, and M. Mesbahi, "Online regulation of unstable LTI systems from a single trajectory," *arXiv preprint*, 2020, arXiv:2006.00125.

[8] J. Berberich, J. Koehler, M. A. Müller, and F. Allgöwer, "Data-driven model predictive control with stability and robustness guarantees," *IEEE Transactions on Automatic Control*, 2020, to appear.

[9] L. Xu, M. Turan Sahin, B. Guo, and G. Ferrari-Trecate, "A data-driven convex programming approach to worst-case robust tracking controller design," *arXiv preprint*, 2021, arXiv:2102.11918.

[10] A. Allibhoy and J. Cortés, "Data-based receding horizon control of linear network systems," *IEEE Control Systems Letters*, vol. 5, no. 4, pp. 1207–1212, 2020.

[11] G. Bianchin, M. Vaquero, J. Cortés, and E. Dall'Anese, "Online stochastic optimization for unknown linear systems: Data-driven synthesis and controller analysis," *arXiv preprint*, Aug. 2021, arXiv:2108.13040.

[12] J. Berberich and F. Allgöwer, "A trajectory-based framework for data-driven system analysis and control," in *European Control Conference*, 2020, pp. 1365–1370.

[13] M. Guo, C. De Persis, and P. Tesi, "Data-driven stabilization of nonlinear polynomial systems with noisy data," *IEEE Transactions on Automatic Control*, vol. 67, no. 8, pp. 4210–4217, 2021.

[14] J. C. Willems, P. Rapisarda, I. Markovsky, and B. D. Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[15] A. Pandey, R. Schmid, and T. Nguyen, "Performance survey of minimum gain exact pole placement methods," in *European Control Conference*, 2015, pp. 1808–1812.

[16] J. Kautsky, N. K. Nichols, and P. Van Dooren, "Robust pole assignment in linear state feedback," *International Journal of control*, vol. 41, no. 5, pp. 1129–1155, 1985.

[17] S. P. Bhattacharyya and E. De Souza, "Pole assignment via Sylvester's equation," *Systems & Control Letters*, vol. 1, no. 4, pp. 261–263, 1982.

[18] M. A. Rami, S. El Faiz, A. Benzaouia, and F. Tadeo, "Robust exact pole placement via an LMI-based algorithm," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 394–398, 2009.

[19] S. Mukherjee and R. R. Hossain, "Data-driven pole placement in LMI regions with robustness guarantees," in *IEEE Conf. on Decision and Control*, 2022, pp. 4010–4015.

[20] C.-T. Chen, *Linear System: Theory and Design*. Holt, Rinehart, and Winston, 1984.

[21] J. H. Wilkinson, *The algebraic eigenvalue problem*. London: Oxford University Press, 1988.

[22] H. J. van Waarde, C. De Persis, M. K. Camlibel, and P. Tesi, "Willems' fundamental lemma for state-space systems and its extension to multiple datasets," *IEEE Control Systems Letters*, vol. 4, no. 3, pp. 602–607, 2020.