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Constructing Feedback Linearizable Discretizations for Continuous-Time Systems using Retraction Maps.

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***Abstract*—Control laws for continuous-time dynamical systems are most often implemented via digital controllers using a sample-and-hold technique. Numerical discretization of the continuous system is an integral part of subsequent analysis. Feedback linearizability of such sampled systems is dependent upon the choice of discretization map or technique. In this article, for feedback linearizable continuous-time systems, we utilize the idea of retraction maps to construct discretizations that are feedback linearizable as well. We also propose a method to functionally compose discretizations to obtain higher-order integrators that are feedback linearizable.**

***Index Terms*—Numerical algorithms, Feedback Linearization, Sampled-data control**

I. Introduction

Digital controllers facilitate the implementation of continuous-time control systems via discretization. For non-autonomous systems i.e., for systems with inputs this is done via (a) sample and hold technique where the control input is held constant between two sampling intervals and (b) a discretization scheme that solves the evolution of the continuous-time dynamical systems numerically. Different numerical schemes result in different discretizations of the continuous time systems. On Euclidean spaces i.e., for systems evolving on $\mathbb{R}^n$, some of the common numerical integration schemes are Euler Integrations methods, Runge-kutta-based methods, Simpson 1/3 rule, etc. [1]. While these schemes perform well for systems evolving in Euclidean spaces, when implemented for systems evolving on general manifolds, they do not guarantee that the system states stay on the manifold. In order to maintain the non-euclidean structure of the underlying manifold one would like to construct integrators that respect the underlying geometry of the continuous-time dynamical system. Such integrators are called geometric integrators and these result in more accurate long-term behavior. A summary of geometric integrator schemes is given in [1], [2]. Retraction maps are a generalization of Euclidean discretizations on non-euclidean manifolds (see [3], [4]). Retraction maps allow us to construct geometric discretizations that guarantee the system states stay on the manifold.

Feedback linearization allows us to transform a nonlinear control system into a linear system via a coordinate transformation and control feedback. This allows us to utilize the control design methods such as pole placement [5] etc., available for linear systems to synthesize controls for the nonlinear system. A study of feedback linearization for continuous time systems is provided in [6]–[8] and references therein. A discrete-time equivalent of feedback linearization is studied in [9]–[13]. Sampling, in general, does not preserve the feedback linearization, i.e., given a feedback-linearizable continuous time system, under sample and hold method and a particular choice of discretization the resulting discrete-time system need not be feedback linearizable (in discrete time) in general [14]–[17]. Since feedback linearization allows us to utilize the advantages of linear control theory, it is of interest to find (numerical) discretizations that are feedback linearizable.

Contribution: In this article, given a feedback-linearizable continuous-time system we utilize retraction maps to construct discretizations that are feedback linearizable. We also provide a way to compose these discretizations to generate symmetric discretizations that are accurate up to the second order while maintaining feedback linearizability. However, this requires multirate sampling.

II. Retraction and Discretization Maps

Let M be an n dimensional manifold and TM be the associated tangent bundle. Let $TM \ni (x, v) \longmapsto \tau_M(x, v) := x$ be the canonical projection onto the manifold. Further, for each $x \in M$, let 0_x be the zero vector in T_xM.

Definition 1 (Retraction map [3]*):* Let $\mathcal{R}\colon TM \longrightarrow M$ be smooth and $\mathcal{R}_x =: \mathcal{R}|_{T_xM}$, then $\mathcal{R}$ is a retraction if for all $x \in M$, (1) $\mathcal{R}_x(0_x) = x$, and (2) $T_{0_x}\mathcal{R}_x$ is the identity map on T_xM.

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Definition 2 (Discretization map [3]*):* Let $U \subset TM$ be an open neighborhood of the zero section of the tangent bundle TM. $U \ni (x,v) \longmapsto R(x,v) := (R^1(x,v), R^2(x,v)) \in M \times M$ is a discretization map if, for any $x \in M$, it satisfies (1) $(x, 0_x) \longmapsto R(x, 0_x) = (x, x)$, and (2) $T_{(x,0_x)}R^2 - T_{(x,0_x)}R^1 = \mathrm{Id}_{T_xM} \colon T_{(x,0_x)}T_xM \simeq T_xM \longrightarrow T_xM$ is equal to the identity map on T_xM, where $T_{(x,0_x)}R^i$ is the tangent map of R^i, $i \in \{1,2\}$ at $(x, 0_x) \in TM$.

Definition 3 (Adjoint discretization): Let R be a discretization on M. Consider the inversion map $(x,y) \ni M \times M \longmapsto \mathcal{I}_M(x,y) := (y,x) \in M \times M$. The adjoint of R is defined by $U \ni (x,v) \longmapsto R^*(x,v) := \mathcal{I}_M(R(x,-v))$.
A discretization is called *symmetric* if $R = R^*$.

Proposition 1: Given $X \in \mathfrak{X}$ a vector field on M and a fixed time discretization map $t \longmapsto (t - \alpha h, t + (1-\alpha)h)$, $\alpha \in [0,1]$, the discretization of X defined by

$$R^{-1}(x_k, x_{k+1}) = hX\underbrace{\left(\tau_M(R^{-1}(x_k, x_{k+1}))\right)}_{\in M}$$

is a first-order discretization of X and second-order if R is symmetric.

Proposition 2: Let M and N be n dimensional manifolds and $M \ni x \longmapsto \phi(x) =: y \in N$ be a diffeomorphism. For a given discretization R on M, $R_\phi := (\phi \times \phi) \circ R \circ T\phi^{-1}$ is a discretization on N (see Figure 1).

Proof: For any given $y \in N$ we have that

$$\begin{aligned} R_\phi(y, 0_y) &= \left((\phi \times \phi) \circ R \circ T\phi^{-1}\right)(y, 0_y) \\ &= \left((\phi \times \phi) \circ R \circ T\phi^{-1}\right)(\phi(x), 0_{\phi(x)}) \\ &= (\phi \times \phi)^{-1} R(x, 0_x) \\ &= (\phi \times \phi)^{-1}(x, x) = (y, y). \end{aligned}$$

This proves the first condition. Now, given a vector $u_y \in T_yN$, we have

$$\begin{aligned} &\left(T_{(y,0_y)}R^2_\phi - T_{(y,0_y)}R^1_\phi\right)(y, u_y) \\ =& \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} [(\phi \circ R^1 \circ T\phi^{-1})(y, su_y) \\ &- (\phi \circ R^2 \circ T\phi^{-1})(y, su_y)] \\ =& T_y\phi \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} [R^1(s(T\phi^{-1})(y, u_y)) \\ - R^2(s(T\phi^{-1})(y, u_y))] \end{pmatrix} \\ =& T_y\phi((T\phi^{-1})(y, u_y)), = (y, u_y) \end{aligned}$$

which proves the second condition. ■

Using the inversion map $\mathcal{I}_M$ one can easily show that $R^*_\phi = (\phi \times \phi) \circ R^* \circ T\phi^{-1}$ is the adjoint discretization of R_ϕ. Further, R_ϕ is symmetric if R is symmetric. From definition of R and R_ϕ, Fig. 1 commutes.

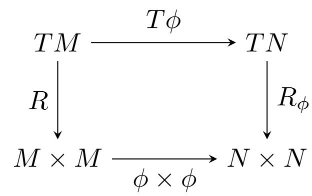

Fig. 1. R and R_ϕ commute as shown above

III. Continuous Time Control System

Let M be an n dimensional manifold and $U \subset \mathbb{R}^m$ be open. For each $u \in U$ let $X(\cdot, u) \in \mathfrak{X}(M)$ be a vector field on M. Then for a fixed $\mathcal{T} > 0$, a continuous-time dynamical system (CS) on M is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x(t), u(t)) \text{ for all } t \in [0, \mathcal{T}], \tag{1}$$

with $t \longmapsto x(t) \in M$ and $t \longmapsto u(t)$ for all $t \in [0, \mathcal{T}]$. A point $(x_0, u_0) \in M \times U$ is said to be an equilibrium point of (1) if $X(x_0, u_0) = 0$.

A. Feedback Linearization of Continuous Time Systems

Let M and N be two n-dimensional manifolds and $\phi : M \longrightarrow N$ be a diffeomorphism. Let $X \in \mathfrak{X}(M)$ be a vector field on M. Then $X_\phi := T\phi \circ X \circ \phi^{-1}$ is a vector field on N. Further for the dynamical system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = X_\phi(y(t), u(t)) \text{ for all } t \in [0, \mathcal{T}], \tag{2}$$

with $y(0) = \phi(x(0))$ satisfy $y(t) = \phi(x(t))$, for all $t \in [0, \mathcal{T}]$, where $x(t)$ is a solution of (1).

Definition 4 (Feedback linearization [8]*):* Let $\mathcal{O}(x_0) \ni x_0$ and $\mathcal{O}(u_0) \ni u_0$ be open neighorhoods around x_0 and u_0 of M and U, respectively. Let $\mathcal{O}(x_0) \ni x \longmapsto \phi(x) := y \in N := \mathbb{R}^n$ be a diffeomorphism to its image and $\mathcal{O}(x_0) \times \mathcal{O}(u_0) \ni (x,u) \longmapsto \psi(x,u) := v \in \mathbb{R}^m$ be such that for each fixed x, $\psi(x, \cdot) : U \longrightarrow \mathbb{R}^n$ is invertible. A given continuous time system (1) is said to be (locally) feedback linearizable around (x_0, u_0) on $\mathcal{O}(x_0) \times \mathcal{O}(y_0)$ if there exist matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ such that $X_\phi(y, u) = Ay + Bv$ with $v = \psi(\phi^{-1}y, u)$. The feedback linearized dynamical system is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay(t) + Bv(t) \text{ for all } t \in [0, \mathcal{T}]. \tag{3}$$

For background on feedback linearization, we refer the reader to [6]–[8] and references therein.

Assumption: System (1) is controllable and is feedback equivalent to a controllable linear system (3)

IV. Numerical Discretization of Continuous-Time Systems

Continuous time control systems are implemented via digital controllers using the *sample and hold* method where the control input u is held constant at a fixed value between two successive samples i.e., $u(t) = u_k$ for all $t \in [t_k, t_{k+1}[,\ t_{k+1} = t_k + h,$ for all $k \in \mathbb{N}$, where h is the fixed sampling period. Further, since analytical solutions for (1) are often not available in closed form, the solutions are to be approximated numerically.

Definition 5: Let $U \subset \mathbb{R}^m$ be open and for each $u \in U$, $X(\cdot, u) \in \mathfrak{X}(M)$ is a vector field on x. Let R be a discretization map on M then using Proposition 1 a discretization of $X(\cdot, u)$ is defined by

$$R^{-1}(x_k, x_{k+1}) = hX(\tau_M(R^{-1}(x_k, x_{k+1})), u_k) \tag{4}$$

where the control input u is held constant over the interval $[t_k, t_{k+1}[$ i.e., $u(t) = u_k$ for all $t \in [t_k, t_{k+1}[$.
Different choices for R lead to different numerical discretization schemes. For example, on Euclidean spaces ($M = \mathbb{R}^m$), $R(x, v) = (x, x+v)$ results in the Explicit Euler discretization scheme $x_{k+1} = x_k + hX(x_k, u_k)$. Solving (4) for x_{k+1}, the sampled discrete-time system can be explicitly written as

$$x_{k+1} = F(x_k, u_k; h), \tag{5}$$

where $x_k \in M$ and $u_k \in U$ for all $k \in \mathbb{N}$ and $M \times U \ni (x, u) \longmapsto F(x, u; h) \in M$ is a smooth map (if F is not well defined on entire M one may very well work with a local definition of F, by replacing M with an open neighborhood around x_0 in M). From the properties of retraction maps, one can show that at equilibrium point (x_0, u_0) one has $F(x_0, u_0; h) = x_0$.

A. Feedback Linearization of Discrete-Time Systems

The idea of feedback linearization can be extended to discrete-time systems as well. Consider the discrete-time system given by (5).

Definition 6 (Feedback linearization (discrete-time) [9]*):* Let $\mathcal{O}(x_0) \ni x_0$ and $\mathcal{O}(u_0) \ni u_0$ be open neighborhoods around x_0 and u_0. Let $\mathcal{O}(x_0) \ni x \longmapsto y =: \phi(x) \in N := \mathbb{R}^n$ be a diffeomorphism onto its image, and $\mathcal{O}(x_0) \times \mathcal{O}(u_0) \ni (x, u) \longmapsto v =: \psi(x, u) \in \mathbb{R}^m$ be such that for each x, $\psi(x, \cdot)$ is locally invertible. The discrete-time system (5) is said to be feedback linearizable if there exist matrices $A_h \in \mathbb{R}^{n\times n}$ and $B_h \in \mathbb{R}^{n\times m}$ such that

$$\phi(F_h(x, u)) = A_h\phi(x) + B_h\psi(x, u) = A_h y + B_h v.$$

The discrete-time system (5) is linearized to

$$y_{k+1} = A_h y_k + B_h v_k. \tag{6}$$

The feedback linearizability of discrete-time systems has been dealt with in great detail in [9]–[12]. For sampled time continuous time system the feedback linearizability is in general not preserved, i.e., a feedback linearizable continuous time system when implemented with sample and hold may not result in a feedback linearizable discrete-time system. The linearizability is not only dependent upon the underlying continuous-time system but also on the choice of discretization (see [14]). Using this as our motivation we are interested in the following problem – **given a (locally) feedback linearizable continuous time system** (1) **is it possible to construct a numerical discretization** (5) **that is also (locally) feedback linearizable in the sense of Definition 6?**

B. Constructing Feedback Linearizable Discretization Maps

Let M be an n-dimensional manifold. Consider the continuous time system given by (1) on M. Let ϕ and ψ be as in Definition (4). Suppose (1) is feedback linearizable to (3). Keeping $v(t) = v_k$ for all $t \in [t_k, t_{k+1}[$, let R be a discretization map on N, and a discretization scheme for(3) such that it preserves the linearity of (3), i.e., it results in a discrete system $y_{k+1} = A_h y_k + B_h v_k$, where $A_h \in \mathbb{R}^{n\times n}$ and $B_h \in \mathbb{R}^{n\times m}$. Given a discretization map R on N, using Proposition 2, one can construct a discretization map on M

$$R_{\phi^{-1}} = (\phi \times \phi)^{-1} \circ R \circ T\phi, \tag{7}$$

and a discretization scheme for (1)

$$R^{-1}_{\phi^{-1}}(x_k, x_{k+1}) = hX(\tau_M(R^{-1}_{\phi^{-1}}(x_k, x_{k+1}), u_k)), \tag{8}$$

then we have the following result.

Theorem 1: The discretization scheme given by (8), for (1) is feedback linearizable in the discrete-time domain.

Proof: Define $y_k = \phi(x_k)$ and $\psi(x_k, u_k) = v_k$ for all $k \in \mathbb{N}$. From (7), we have

$$\begin{aligned} R^{-1}_{\phi^{-1}}(x_k, x_{k+1}) =&(T\phi^{-1} \circ R^{-1} \circ (\phi \times \phi))(x_k, x_{k+1}) \\ =&(T\phi^{-1} \circ R^{-1})(y_k, y_{k+1}), \end{aligned}$$

and

$$\begin{aligned} & X(\tau_M(R^{-1}_{\phi^{-1}}(x_k, x_{k+1})), u_k) \\ = \;& X\big(\tau_M((T\phi^{-1} \circ R^{-1})(y_k, y_{k+1})), u_k)\big) \\ = \;& X_\phi\big(\tau_N(R^{-1}(y_k, y_{k+1})), u_k)\big). \end{aligned}$$

From (8), we have

$$(x_k, x_{k+1}) = R_{\phi^{-1}}(hX(\tau_M(R^{-1}_{\phi^{-1}}(x_k, x_{k+1})))$$

and therefore,

$$\begin{aligned} & (\phi \times \phi)(x_k, x_{k+1}) \\ = \;& (\phi \times \phi) \circ R_{\phi^{-1}}(hX(\tau_M(R^{-1}_{\phi^{-1}}(x_k, x_{k+1})), u_k) \\ = \;& R\big((hX_\phi(\tau_N(R^{-1}(y_k, y_{k+1})), u_k)\big), \end{aligned}$$

which implies

$$(y_k, y_{k+1}) = R(hX_\phi(\tau_N(R^{-1}(y_k, y_{k+1})), u_k)).$$

Since R preserves linearity and $X_\phi(y, u) = Ay + Bv$ with $v = \psi(\phi^{-1}y, u)$, we have $y_{k+1} = A_h y_k + B_h\psi(x_k, u_k) = A_h y_k + B_h v_k$., thereby linearizing (8).

■

Remark 1: It is important to note that independent of the order of R one can ensure an accuracy of R_ϕ only up to the first order. This is due to the fact that while implementing (1) via the sample and hold, the control input u is to be held constant on the interval $[t_k, t_{k+1}[$. This is in general not possible while simultaneously keeping the linearized control input v constant over $[t_k, t_{k+1}[$ as $v(t) = \psi(x(t), u(t))$. Instead of employing the exact control input $u(t)$ over the interval, we apply the control u_k satisfying $v_k = \psi(x_k, u_k)$ for all $t \in [t_k, t_{k+1}[$, where x_k is the state sampled at $t = t_k$.

C. Linearizability of Adjoint Discretization

Given a discretization map, $R_{\phi^{-1}}$ one can construct an adjoint discretization $R^*_{\phi^{-1}}$ as given by the Definition 3. From proposition 2 we have

$$R^*_{\phi^{-1}} = (\phi \times \phi)^{-1} \circ R^* \circ T\phi. \tag{9}$$

Theorem 2: Let R be a discretization of (3) preserving linearity for h as well as $-h$. Let R^* be the adjoint of R, then $R^*_{\phi^{-1}}$ given by (9) results in a discretization

$$x_{k+1} = F^*(x_k, u_k; h), \tag{10}$$

that is feedback linearizable. Moreover, the linearizing coordinate is given by $x \longmapsto \phi(x) := y$ and the linearized system is given by

$$y_k = A_{-h} y_{k+1} + B_{-h} v_k, \tag{11}$$

with $v_k = \psi(x_{k+1}, u_k)$.
The proof of the above theorem follows a similar process to that of Theorem 1 and hence is omitted. The control input u_k is to be calculated implicitly from the control input v_k. Similar to $R_{\phi^{-1}}$, $R^*_{\phi^{-1}}$ is also accurate upto first order.

V. Constructing Higher Order Discretizations

Definition 7 (Global and truncated error [18]*):* Consider the continuous time system (1), then for a given discretization (5) the *k-step global error* is

$$e_k := x(t_k) - x_k,$$

where $x(t_k)$ is the exact solution of (1) evaluated at $t_k = t_0 + hk$, and the *one-step truncated error* at t_k is

$$\tilde{x}(t_k) = \left(x(t_k + h) - F(x(t_k), u_k; h)\right)/h.$$

Definition 8 (Order of discretization [18]*):* A discretization is of *order* r, if for some fixed $K > 0$, $\|\tilde{x}(t_k)\| \leq K h^r$ for all $t_k \in [0, \mathcal{T}]$ and $h > 0$.
The discretizations R in Definition 1 are in general first order. However, if R is symmetric, it is accurate up to the second order. This serves as our motivation to construct symmetric discretization.

A. Symmetric Discretizations

Let M be an n dimensional manifold and $X \in \mathfrak{X}(M)$ be a vector field on M. Let R be a discretization map on M and R^* be its associated adjoint. Composing R and R^*, we define a discretization scheme as follows :

$$\begin{aligned} R^{-1}(x_k, x_{k+1/2}) &= \frac{h}{2} X(\tau_M(R^{-1}(x_k, x_{k+1/2}))) \\ (R^*)^{-1}(x_{k+1/2}, x_{k+1}) &= \frac{h}{2} X(\tau_M((R^*)^{-1}(x_{k+1/2}, x_{k+1}))) \end{aligned} \tag{12}$$

In the above equation, $x_{k+1/2} \in M$ is to be taken as an intermediate point and is solved implicitly to get a discrete system of type (5).

Proposition 3: The discretization given by (12) is symmetric and is therefore accurate up to the second order.
For proof of Proposition 3, we refer the reader to [19].

For nonautonomous systems, the control input u_k is held constant between $t \in [t_k, t_{k+1}[$, (12) is then modified as

$$\begin{aligned} R^{-1}(x_k, x_{k+1/2}) &= \frac{h}{2} X(\tau_M(R^{-1}(x_k, x_{k+1/2})), u_k) \\ (R^*)^{-1}(x_{k+1/2}, x_{k+1}) & \\ &= \frac{h}{2} X(\tau_M((R^*)^{-1}(x_{k+1/2}, x_{k+1})), u_k). \end{aligned} \tag{13}$$

Under closed-loop performance i.e., applying a feedback control $u_k = u(x_k)$, (12) loses its symmetric nature. This can be overcome by employing multirate sampling methods.

B. Multirate Sampling

Definition 9 (Multirate sampling): Consider a continuous time system given by (1). Let h be the sampling time interval i.e., $x_k = x(t_k)$ and $t_{k+1} = t_k + h$. For a fixed $N \in \{1, 2, \ldots, n\}$, and for each $i \in \{1, 2, \ldots N\}$ let $(x, u) \longmapsto F_i(x, u) =: F_i^u(x)$ be discretizations of (1). The N^{th} step evolution is then given by

$$x_{k+N} = F_N^{u_{k+N-1}} \circ \ldots \circ F_2^{u_{k+1}} \circ F_1^{u_k}(x_k) \tag{14}$$

Sampling states x_k at a rate N times slower than that of control input u_k we get a multistep discretization given by

$$x_{k+N} = \bar{F}(x_k, u_k, \ldots u_{k+N-1}) \tag{15}$$

The control input $u_k, \ldots, u_{k+N-1}$ are to be computed a priori at t_k are functions of the state x_k.
Setting $N = 2$ and F_1 and F_2 as F and F^* from (5) and (7) respectively, under multirate sampling, the discrete system generated by (13) is given by

$$\begin{aligned} x_{k+1/2} =& F(x_k, u_k; h/2) \\ x_{k+1/2} =& F(x_{k+1}, u_{k+1}; -h/2). \end{aligned}$$

Let $u_k = u(x_k)$ be a closed-loop control input for discretization (4). Setting $u_{k+1} = u(x_{k+1})$ renders (13) symmetric and the discretization is given by

$$F(x_k, u(x_k); h/2) = F(x_{k+1}, u(x_{k+1}); -h/2), \tag{16}$$

which is symmetric and therefore is of second order. Corresponding continuous time control input is given by

$$u(t) = \begin{cases} u_k, & t \in [t_k, t_{k+1/2}[\\ u_{k+1}, & t \in [t_{k+1/2}, t_{k+1}[, \end{cases}$$

where $t_{k+1/2} = t_k + \frac{h}{2}$ and $t_{k+1} = t_k + h$.

Theorem 3: Consider the continuous time system given by (1). Let R_ϕ be its discretization as given by (7) and (8) be its associated discretization. Then one can construct a symmetric discretization given by (13), the resulting discrete system given by

$$F(x_k, u_k; h/2) = F(x_{k+1}, u_{k+1}; -h/2) \tag{17}$$

is symmetric and is of second order. Moreover, (17) is feedback linearizable under coordinates $x \longmapsto \phi(x) =: y$ and the modified control input is given by $(x, u) \longmapsto \psi(x, u) =: v$. The linearized system is given by

$$A_{h'} y_k + B_{h'} v_k = A_{-h'} y_{k+1} + B_{-h'} v_{k+1}, \tag{18}$$

where $h' = h/2$.

Remark 2: The control input v_k, v_{k+1} can be computed apriori at $t = t_k$ from y_k. The control input u_k and u_{k+1} are than computed implicitly solving $\psi(x_k, u_k) = v_k$, with $x_k = \phi^{-1}(y_k)$.

Remark 3: Theorem 3 is different from the result in [14] in the sense that here the rate of multi-sampling is fixed apriori while in [14] the sampling rate is chosen such that the resulting scheme is feedback linearizable.

VI. EXAMPLE

In order to demonstrate the ideas discussed we consider the following example. Consider the following example from [20, Chapter 4, Example 4.2.5]. Let $x \in \mathbb{R}^3$ and $u \in \mathbb{R}$, the continuous time equations (CS) are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3(1+x_2) \\ x_1 \\ x_2(1+x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix} u, \qquad \text{(CS)}$$

which compactly written is $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)u$. System (CS) is feedback linearizable locally around $x = 0$. Setting

$$\phi(x) = \big(x_1, x_3(1+x_2), x_3 x_1 + (1+x_1)(1+x_2)x_2\big) =: y$$

and

$$\psi(x,u) = \big((1+x_1)(1+x_2)(1+2x_2) - x_1 x_3\big)u$$
$$+ x_3^2(1+x_2) + x_2 x_3 (1+x_2)^2 + x_1(1+x_1)(1+2x_2) =: v,$$

(CS) is (feedback) linearizable about $x = 0$ and the linearized system (CLS) is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay + Bv, \qquad \text{(CLS)}$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Note that the standard Euler scheme for (CS), $x_{k+1} = x_k + h(f(x_k) + g(x_k)u_k)$ is not feedback linearizable [10, Theorem 6].

Choosing the Explicit Euler Scheme (EES) $\mathbb{R}^3 \times \mathbb{R}^3 \ni (y, w) \longmapsto R(y, w) := (y, y+w)$ for the linearized continuous system (CLS), this results in

$$y_{k+1} = y_k + h(Ay_k + Bv_k). \qquad (19)$$

Lifting R via ϕ^{-1}, we get a discretization $R_{\phi^{-1}}$ such that for the continuous time system (CS) it induces following discretization scheme

$$\begin{aligned} x_{k+1} &= \phi^{-1}\big((\mathbf{I} + hA)\phi(x_k) + hB\psi(x_k, u_k)\big) \\ &= F(x_k, u_k; h). \end{aligned} \qquad \text{(EES)}$$

where $\mathbf{I}$ is the identity matrix of appropriate order.

Remark 4: We call this scheme (EES) as it is induced by an Euler integration of (CLS), however, this is different from the standard Explicit Euler Scheme for (CS) ($x_{k+1} = x_k + h(f(x_k) + g(x_k)u)$.
One can immediately see that (EES) is feedback linearizable around $x = 0$.

The associated adjoint scheme $R^*(y, w) = (y - w, y)$ defines the Implicit Euler Scheme (IES) for (CLS), which can be lifted via ϕ to define $R^*_{\phi^{-1}}$. The associated discretization scheme is given by

$$x_k = F(x_{k+1}, u_k; h). \qquad \text{(IES)}$$

Discretization	Associated Control
(EES)	$v_k = Ky_k$
(IES)	$v_k = Ky_{k+1}$
(SES)	$v_k = Ky_k, \quad v_{k+1} = Ky_{k+1}$

TABLE I
CONTROL INPUT FOR VARIOUS DISCRETIZATION SCHEMES FOR EXAMPLE (CS).

	Order of error magnitude		
Stepsize	(EES)	(IES)	(SES)
$h = 10^{-1}$	10^{-1}	10^{-1}	10^{-2}
$h = 10^{-2}$	10^{-2}	10^{-2}	10^{-4}
$h = 10^{-3}$	10^{-3}	10^{-3}	10^{-6}

TABLE II
ORDER OF ONE STEP TRUNCATION ERROR $\|\tilde{x}_k\|$ FOR VARIOUS STEP SIZES h FOR EXAMPLE (CS).

Composing $R_{\phi^{-1}}$ and $R^*_{\phi^{-1}}$, and using multirate sampling, the Symmetric Euler Scheme (SES) for (CS) is given by

$$F(x_{k+1}, u_{k+1}; -h) = F(x_k, u_k; h). \qquad \text{(SES)}$$

where u_{k+1} is dependent only on x_k and can be computed apriori. One can see that (IES) and (SES) are feedback linearizable. Moreover, (SES) is symmetric and therefore is accurate up to second-order. The three schemes were implemented to stabilize the system (CS) to the origin. For this purpose, the corresponding control laws were chosen as given by Table I. The schemes were simulated in MATLAB for various initial conditions $x(0)$, feedback gain K, and stepsize h. The error was compared with a standard ODE solver (ODE45). The order of the one-step truncation error for various step sizes is given in Table II. We present simulation plots for one such instance for (SES). The initial condition was fixed at $x(0) = (0.5, 0.25, -0.5)$ and the control gain was set at $K = -\begin{pmatrix} 4.8 & 12 & 4.8 \end{pmatrix}$, the step size was chosen as $h = 10^{-2}$ and the control input was sampled twice for each interval. The system states and control inputs are plotted in Fig. 2 and 3 respectively. In Fig. 4, we plot the global error of the three schemes for $h = 10^{-2}$. One can see that (SES) has less error as compared to (EES) and (IES). From Table II one can see that the (SES) is of second order in nature while the other two are of first order. In Fig. 5 we plot the one-step truncation error for (SES) for various step size $h \in \{10^{-1}, 10^{-2}, 10^{-3}\}$, one can see that truncation error is proportional to h^2, which confirms that the method is of second order.

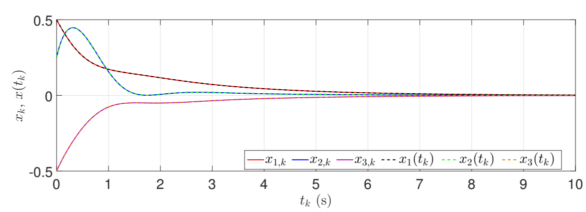

Fig. 2. System state x_k for (SES) plotted against exact discretization $x(t_k)$ for $h = 10^{-2}$ and $t_k \in [0, 10]$.

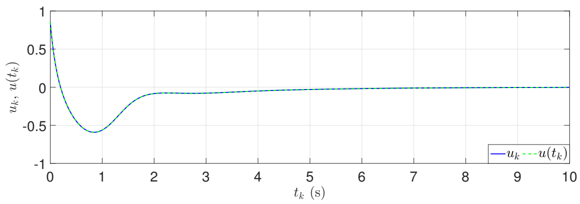

Fig. 3. Control input u_k for (SES) plotted against exact discretization $u(t_k)$ for $h = 10^{-2}$ and $t_k \in [0, 10]$.

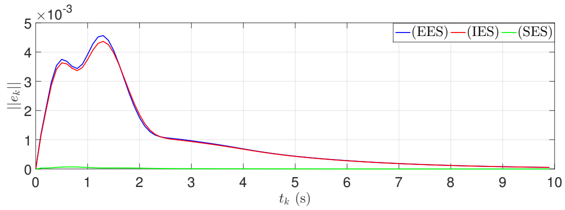

Fig. 4. Global error $\|e_k\|$ for (EES), (IES), and (SES) for $h = 10^{-2}$ and $t_k \in [0, 10]$.

VII. Conclusions

In this article, we have utilized the idea of retraction maps and their lifts under diffeomorphism to construct feedback linearizable discretization. Given a continuous-time feedback linearizable system, we show that one can build first-order discretization that preserves feedback linearizability. This is done by lifting a discretization of the linearized continuous time system. We have also shown a way to functionally compose two first-order discretizations to design second-order discretizations that are feedback linearizable. It is observed that symmetric methods are of second order and therefore have higher accuracy for larger step sizes, however, this comes at the cost of multi-rate sampling.

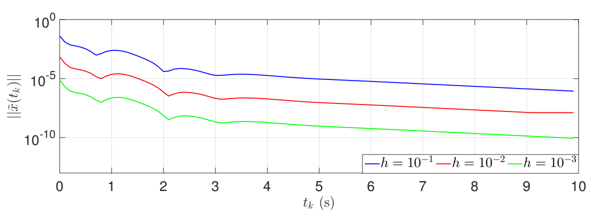

Fig. 5. One-step truncated error $\|\tilde{x}(t_k)\|$ for (SES) for $h \in \{10^{-1}, 10^{-2}, 10^{-3}\}$ and $t_k \in [0, 10]$.

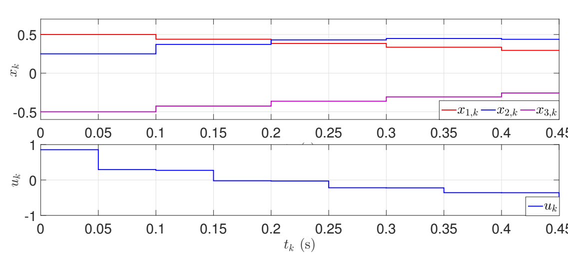

Fig. 6. Zoomed in control for (SES), showing multisampling. The control input u_k is applied twice over each sampling interval $h = 10^{-1}$.

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