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Consistent Conjectural Variations Equilibria: Characterization & Stability for a Class of Continuous Games

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Abstract— **Leveraging tools from the study of linear fractional transformations and algebraic Riccati equations, a local characterization of consistent conjectural variations equilibrium is given for two player games on continuous action spaces with costs approximated by quadratic functions. A discrete time dynamical system in the space of conjectures is derived; a solution method for computing fixed points of these dynamics (equilibria) is given via solving an eigenvalue problem; local stability properties of the dynamics around the equilibria are characterized; and conditions are given that guarantee a unique stable equilibrium.**

I. Introduction

In many multi-agent systems, agents learn about their opponents and the environment through interaction. Moreover, agents often have bounded rationality—e.g., humans are known to not behave rationally [1] and machines inherently have bounded computational capabilities and are limited to making decisions based on their prescribed algorithmic process. Much of the literature on using game theory to model multi-agent systems has focused on static equilibrium notions that assume agents are rational such as Nash or correlated equilibria. These equilibrium concepts do not capture the dynamic nature of learning systems or cases in which agents form opponent models.

To address these issues, several different fields have examined the use of opponent models. The following examples are demonstrative. In machine learning, opponent modeling [2], [3] can empirically improve the performance of reinforcement learning agents in some environments. In game theory, opponent models known as *conjectural variations* [4] have been used to analyze strategic behaviors of firms in oligopoly and electricity markets [5], [6], [7], [8]. At the intersection of these areas, in prior work, we investigated the connection between gradient play and opponent anticipation leveraging conjectural variations [9], empirically validated the use of conjectures in human-machine co-adaptation experiments [10], and showed the relationship to implicit learning algorithms in Stackelberg games [11]. Despite existing work, there still remains several technical challenges in terms of characterizing the dynamic interaction of learning agents who form opponent models.

Motivated by coupled non-cooperative learning systems wherein decision-makers have an opponent model and optimize with respect to this model, we provide a novel characterization of a (consistent) conjectural variations equilibrium ((C)CVE) [12], [13]. A CVE is a non-cooperative equilibrium concept—predating even Nash—in which each agent chooses their most favorable action taking into account that opponent strategies are a conjectured mapping of their own strategy. To gain intuition, a CVE can be thought of as a *double-sided Stackelberg equilibrium.* Indeed in a Stackelberg game, the "leader" best responds to a myopic follower by solving $\min_x\{f(x,y)|\ y\in \operatorname{argmin}_{y'} g(x,y')\}$. When both players act like a leader, we have a double-sided Stackelberg game. This is a special case of a CVE wherein the conjecture is simply the myopic best response model of the follower. Conjectures can be more general mappings, however. Such an equilibrium is *consistent* if each player's strategy in equilibrium is consistent with that which is conjectured by its opponent. Unlike a Nash equilibrium, a (C)CVE handles strategic uncertainty through the use of conjectures, and has the following interpretation in terms of incentives: at a CVE no player has an incentive to deviate according to their own beliefs. Our interest in this equilibrium concept is precisely due to its aptitude for capturing dynamic contexts, or situations of bounded (procedural) rationality. In particular, as we highlight, CCVE can be seen as arising from repeated best response given an opponent model.

Contributions. We leverage tools from the study of linear fractional transformations, and algebraic Riccati equations to provide a novel characterization of consistent conjectural variations equilibria for two-player $d_1\times d_2$ continuous games with quadratic costs; a quadratic game can also be thought of as a local approximation of more general costs. Focusing on conjectures that are affine in player actions, we derive a set of coupled Riccati equations and show that CCVE exist if these equations have solutions. Additionally, we show that these coupled Riccati equations naturally lead to a discrete time dynamical system when they are iterated. We give a general solution method for computing fixed points of these dynamics via solving an eigenvalue problem. We analyze the local stability properties, and give conditions that guarantee a unique, stable CCVE. An expanded version of this paper with more details and numerical examples is given in [14].

II. Preliminaries

Consider the two-player game $\mathcal{G}=(f_1,f_2)$ such that $f_i\in C^2(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2},\mathbb{R})$ for each $i\in\{1,2\}$. The function $f_i:\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\to\mathbb{R}$ is player i's cost, which they seek to minimize by choosing $x_i\in\mathbb{R}^{d_i}$. Let $x=(x_1,x_2)\in\mathbb{R}^d$ where $d=d_1+d_2$ is the dimension of the joint action space.

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Define the set of *conjectures* $\mathcal{C}_1 \times \mathcal{C}_2$ to be the set of mappings

$$\mathcal{C}_1 \times \mathcal{C}_2 = \{(c_1, c_2)|\ c_1 : \mathbb{R}^{d_2} \to \mathbb{R}^{d_1},\ c_2 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}\}.$$

Definition 1: A tuple $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\} \in \mathbb{R}^{d_1 \times d_2} \times \mathcal{C}_1 \times \mathcal{C}_2$ constitute a *consistent conjectural variations equilibrium (CCVE)* if $x_i^{\mathsf{c}} = c_i^{\mathsf{c}}(x_{-i}^{\mathsf{c}})$ for each $i = 1, 2$, and

$$x_i^{\mathsf{c}} = \underset{x_i}{\operatorname{argmin}}\{f_i(x_i, x_{-i})|\ x_{-i} = c_{-i}^{\mathsf{c}}(x_i)\}, \quad \forall\ i = 1, 2.$$

Given an *a priori* fixed set of conjectures $(c_1, c_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, the point $(x_1^{\mathsf{c}}, x_2^{\mathsf{c}})$ is a generalized Nash equilibrium of the constrained game $\{\min_{x_i} f_i(x_i, c_{-i}(x_i))|\ x_i = c_i(c_{-i}(x_i))\}_{i=1}^2$. However, finding a CCVE requires finding the maps $(c_1^{\mathsf{c}}, c_2^{\mathsf{c}})$, so the problem of characterizing CCVE *does not* immediately reduce to a generalized Nash equilibrium problem [15].

As shown in [16], when the costs are *(jointly) strictly convex*, an equivalent characterization of a CCVE in terms of the conjectures is the following: $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\}$ is a CCVE if and only if, for each $i = 1, 2$, we have

$$D_{x_i} f_i(x^{\mathsf{c}}) + D_{x_{-i}} f_i(x^{\mathsf{c}}) D_{x_i} c_{-i}^{\mathsf{c}}(x_i^{\mathsf{c}}) = 0,\ x_i^{\mathsf{c}} = c_i^{\mathsf{c}}(x_{-i}^{\mathsf{c}}), \tag{1}$$

where D_x is the partial derivative operator with respect to a vector x. In the *absence of joint strict convexity*, these are first-order conditions; we call solutions to (1) *first-order CCVE*. A *second-order CCVE* is a solution to (1) with the additional condition $D_{x_i}^2 f_i(x_i^{\mathsf{c}}, c_{-i}^{\mathsf{c}}(x_i^{\mathsf{c}})) \succ 0$. If $f_i(x_i, c_{-i}^{\mathsf{c}}(x_i))$ is strongly convex in x_i, then solutions to (1) are a CCVE.

The focus of this paper is on characterizing CCVE and corresponding conjectures up to first- and second-order using a quadratic approximation of the game around the equilibrium. When the game is quadratic, a second-order CCVE is precisely a CCVE. Even in quadratic games, the existence of CCVE is not guaranteed, and as we show, for affine conjectures the question of existence boils down to finding solutions to coupled asymmetric Riccati equations. This is analogous to the existence of Nash equilibrium in dynamic linear quadratic games (cf. [17], [16, Ch. 6]).

A. Quadratic Game Approximation

The local quadratic approximation of cost f_i is given by

$$f_i(x_i, x_{-i}) = \frac{1}{2}\begin{bmatrix} x_i \\ x_{-i} \end{bmatrix}^\top \begin{bmatrix} A_i & B_i^\top \\ B_i & D_i \end{bmatrix} \begin{bmatrix} x_i \\ x_{-i} \end{bmatrix} + \begin{bmatrix} a_i \\ b_i \end{bmatrix}^\top \begin{bmatrix} x_i \\ x_{-i} \end{bmatrix},$$

where $A_i \in \mathbb{R}^{d_i \times d_i}$, $D_i \in \mathbb{R}^{d_{-i} \times d_{-i}}$, $B_i \in \mathbb{R}^{d_{-i} \times d_i}$, $a_i \in \mathbb{R}^{d_i}$ and $b_i \in \mathbb{R}^{d_{-i}}$ with $A_i = A_i^\top$ and $D_i = D_i^\top$. Further, we assume that $A_i \succ 0$ for each $i = 1, 2$. The D_i matrices penalize player i based solely on x_{-i} and may often be negative or zero. As noted quadratic games are a useful approximation of the behavior of more complex games around an equilibrium.

We consider only the space of affine conjectures; analogous to affine optimal policies in linear quadratic optimization problems, affine conjectures are the most natural class of conjectures for quadratic games as will be illustrated through our analysis. In fact, it is straightforward to show that if a player has an affine conjecture for their opponent, then the best response for that player is itself an affine policy. With this in mind, let player i have an affine conjecture given by $x_{-i} = c_{-i}(x_i) = L_i x_i + \ell_i$. This results in player i facing the following optimization problem:

$$\min_{x_i}\{f_i(x_i, x_{-i})|\ x_{-i} = c_{-i}(x_i) = L_i x_i + \ell_i\}.$$

Conditions for a first-order CCVE in affine conjectures are

$$\begin{aligned} 0 = D_{x_1} f_1(x_1, c_2(x_1)),\ 0 = D_{x_2} f_2(c_1(x_2), x_2), \\ c_2(x_1) = L_1 x_1 + \ell_1, \qquad c_1(x_2) = L_2 x_2 + \ell_2. \end{aligned} \tag{2}$$

Proposition 1: Consider a quadratic game (f_1, f_2), and suppose players are restricted to the class of affine conjectures $c_{-i}(x_i) = L_i x_i + \ell_i$ for $i = 1, 2$. Suppose that there is a solution $\{(L_1^{\mathsf{c}}, \ell_1^{\mathsf{c}}), (L_2^{\mathsf{c}}, \ell_2^{\mathsf{c}})\}$ to the coupled Riccati equations,

$$0 = L_{-i}^\top (A_i + B_i^\top L_i) + (B_i + D_i L_i), \tag{3}$$

$$0 = \ell_{-i}^\top (A_i + B_i^\top L_i) + a_i^\top + b_i^\top L_i,\ \forall\ i \in \{1, 2\}, \tag{4}$$

such that

$$\begin{bmatrix} \ell_2^{\mathsf{c}} \\ \ell_1^{\mathsf{c}} \end{bmatrix} \in \text{range}\,(\mathbf{L}) \quad \text{where } \mathbf{L} := \begin{bmatrix} I & -L_2^{\mathsf{c}} \\ -L_1^{\mathsf{c}} & I \end{bmatrix}. \tag{5}$$

Then $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\}$ such that $x_{-i}^{\mathsf{c}} = c_{-i}^{\mathsf{c}}(x_i^{\mathsf{c}}) = L_i^{\mathsf{c}} x_i^{\mathsf{c}} + \ell_i^{\mathsf{c}}$ for each $i \in \{1, 2\}$ is a first-order CCVE. Moreover $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\}$ is a CCVE if $(L_1^{\mathsf{c}}, L_2^{\mathsf{c}})$ satisfies

$$A_i + (L_i^{\mathsf{c}})^\top B_i + B_i^\top L_i^{\mathsf{c}} + (L_i^{\mathsf{c}})^\top D_i L_i^{\mathsf{c}} \succ 0,\ i = 1, 2. \tag{6}$$

Proof: The first order conditions in (2) plus affine structure of the conjectures lead to the following equations: $0 = x_i^\top (A_i + B_i^\top L_i) + x_{-i}^\top (B_i + D_i L_i) + a_i^\top + b_i^\top L_i$, and $x_i = L_{-i} x_{-i} + \ell_{-i}$ for $i = 1, 2$. Plugging the latter into the former we have that

$$\begin{aligned} 0 = x_{-i}^\top (L_{-i}^\top (A_i + B_i^\top L_i) + (B_i + D_i L_i)) \\ + \ell_{-i}^\top (A_i + B_i^\top L_i) + a_i^\top + b_i^\top L_i,\ \forall i = 1, 2. \end{aligned} \tag{7}$$

Observe that (7) holds if (3) and (4) hold. By assumption there is a solution $\{(L_1^{\mathsf{c}}, \ell_1^{\mathsf{c}}), (L_2^{\mathsf{c}}, \ell_2^{\mathsf{c}})\}$ to (3) and (4) satisfying (5). Hence, solving $\{x_i = L_{-i}^{\mathsf{c}} x_{-i} + \ell_{-i}^{\mathsf{c}},\ i = 1, 2\}$ yields a first order CCVE $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\}$.

For quadratic games, a second-order CCVE is a CCVE. Expanding out player i's cost, we have that

$$\begin{aligned} f_i(x_i, c_{-i}(x_i)) = \tfrac{1}{2} x_i^\top (A_i + L_i^\top B_i + B_i^\top L_i + L_i^\top D_i L_i) x_i \\ + (a_i^\top + \ell_i^\top B_i + b_i^\top L_i) x_i + \ell_i^\top D_i \ell_i + b_i^\top \ell_i. \end{aligned}$$

Hence, f_i is strongly convex if (6) holds at $(L_1^{\mathsf{c}}, L_2^{\mathsf{c}})$; this is sufficient to guarantee that $\{(x_1^{\mathsf{c}}, x_2^{\mathsf{c}}), (c_1^{\mathsf{c}}, c_2^{\mathsf{c}})\}$ is a CCVE. ■

It is worth pointing out that (3) does not depend on (4) and hence can be solved independently. Additionally, a more restrictive yet simpler-to-check version of (5) is that $\mathbf{L}$ is invertible or, equivalently, $\det\left(I - L_2^{\mathsf{c}} L_1^{\mathsf{c}}\right) \neq 0$. Perhaps more intuitively, supposing the inverse of $(A_i + B_i^\top L_i)$ exists, player i's first order condition is $x_i^\top = -x_{-i}^\top (B_i + D_i L_i)(A_i + B_i^\top L_i)^{-1} - (a_i^\top + b_i^\top L_i)(A_i + B_i^\top L_i)^{-1}$; thus the consistent conjecture conditions are

$$\begin{aligned} L_{-i}^\top = -(B_i + D_i L_i)(A_i + B_i^\top L_i)^{-1}, \\ \ell_{-i}^\top = -(a_i^\top + b_i^\top L_i)(A_i + B_i^\top L_i)^{-1} \quad \forall\ i \in \{1, 2\}. \end{aligned} \tag{8}$$

This shows that if a player has an affine conjecture, then its best response can be written as an affine policy.

We use $(L_i^{\mathrm{c}}, \ell_i^{\mathrm{c}})$ to refer to consistent conjectures—i.e., the solutions to the coupled Riccati equations (3) and the corresponding affine offsets. Solutions may still exist when the inverses in (8) do not, however, as has been shown in special cases in the literature on CCVE such as for scalar Bertrand games, this leads to a multiplicity of solutions and an equilibrium selection problem (see [18], [4] and references therein). Given page constraints, we leave the analysis of these more nuanced cases to a future paper.

For each $i = 1, 2$, define the following linear fractional transformation (LFT) update:

$$L_{-i}^{+} = \texttt{LFT}_{i,-i}(L_i) = -(A_i^\top + L_i^\top B_i)^{-1}(B_i^\top + L_i^\top D_i^\top),$$

where the subscript $(\cdot)_{12}$ can be read as "from 1 to 2". The update for L_i naturally defines discrete-time dynamics in the conjecture parameter space that show how a player should update their conjecture to be consistent with their opponent's current conjecture. It is also useful to think of dynamic updates for each player separately constructed by composing the updates as follows:

$$\begin{aligned} L_i^{+} &= \texttt{LFT}_{-i,i}(\texttt{LFT}_{i,-i}(L_i)) \qquad (9)\\ &= -\left(A_{-i}^\top - (B_i + D_i L_i)(A_i + B_i^\top L_i)^{-1} B_{-i}\right)^{-1} \\ &\quad \cdot (B_{-i}^\top - (B_i + D_i L_i)(A_i + B_i^\top L_i)^{-1} D_{-i}^\top), \quad i = 1, 2. \end{aligned}$$

Remark 1: The first order conditions in (3) guarantee that the players have consistent conjectures. The second order conditions (6) guarantee that player i's cost is convex in x_i. Expounding the first order conditions—characterizing the LFT dynamics, finding fixed points by solving (3), and characterizing their stability—is non-trivial and is the primary focus of this paper. Our results will show that there is a limited number of stable first-order CCVE. The second order conditions (6) can easily be checked. For further discussion, see Section VI and [14].

B. LFT Matrix Representation

We will see in the subsequent section that LFTs can be efficiently represented by matrices and their composition by matrix manipulation. Towards this end, let us define some useful objects that will be used throughout. Define the $d \times d$ symmetric real valued matrices (where $d = d_1 + d_2$)

$$M_1 = \begin{bmatrix} A_1 & B_1^\top \\ B_1 & D_1 \end{bmatrix}, \quad \text{and} \quad M_2 = \begin{bmatrix} D_2 & B_2 \\ B_2^\top & A_2 \end{bmatrix}. \qquad (10)$$

We make the following assumption on M_1 and M_2.

Assumption 1: The matrices M_1, M_2 are invertible.

We will be directly interested in the two products $\mathbf{M}_1 = M_2^{-\top} M_1$ and $\mathbf{M}_2 = M_1^{-\top} M_2$. Note that M_1, M_2 invertible $\iff \mathbf{M}_1, \mathbf{M}_2$ invertible. Let $\operatorname{spec}(\mathbf{M}_1)$ and $\operatorname{spec}(\mathbf{M}_2)$ refer to the spectra of each matrix. A simple argument shows that $\operatorname{spec}(\mathbf{M}_1) = 1/\operatorname{spec}(\mathbf{M}_2)$ where we use $1/(\cdot)$ to mean element-wise inversion. Since M_1, M_2 are symmetric, $\mathbf{M}_1 = \mathbf{M}_2^{-1}$; however, much of the following Riccati analysis works for asymmetric M_1, M_2 as well.

C. Examples

In this section, we present two illustrative quadratic games and comment on CCVE: an open-loop dynamic game and a repeated human vs. machine game.

1) Linear quadratic dynamic game: Consider a two player linear quadratic dynamic game with open loop policies $\mathbf{u}_i = (u_{i,0}, \ldots, u_{i,T-1})$ for $i = 1, 2$:

$$\begin{aligned} f_i(\mathbf{u}_1, \mathbf{u}_2) = \textstyle\sum_{t=0}^{T-1} \; & \tfrac{1}{2} z_t^\top Q_i z_t + \tfrac{1}{2} u_{i,t}^\top R_i u_{i,t} + u_{i,t}^\top R_i^{-i} u_{-i,t} \\ & + \tfrac{1}{2} z_T^\top Q_{i,f} z_T \\ z_{t+1} &= F z_t + G_1 u_{1,t} + G_2 u_{2,t}, \; z_t \in \mathbb{R}^n. \end{aligned}$$

Unfolding the dynamics and letting $Z = [z_0^\top, \ldots, z_T^\top]^\top$, we have that $Z = W_1 \mathbf{u}_1 + W_2 \mathbf{u}_2 + \mathbf{F} z_0$ where

$$W_i = \begin{bmatrix} 0 & \cdots & & & 0 \\ G_i & 0 & \cdots & & 0 \\ FG_i & G_i & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ F^{T-2}G_i & F^{T-3}G_i & \cdots & G_i & 0 \\ F^{T-1}G_i & F^{T-2}G_i & \cdots & FG_i & G_i \end{bmatrix}, \quad i = 1, 2,$$

and $\mathbf{F} = \begin{bmatrix} I & F^\top & \cdots & (F^{T-1})^\top & (F^T)^\top \end{bmatrix}^\top$. Define cost matrices: $\mathbf{Q}_i := \operatorname{diag}(Q_i, \ldots, Q_i, Q_{i,f}) \in \mathbb{R}^{n(T+1)\times n(T+1)}$, $\mathbf{R}_i := \operatorname{diag}(R_i, \ldots, R_i) \in \mathbb{R}^{d_i T \times d_i T}$, and $\mathbf{R}_i^{-i} := \operatorname{diag}(R_i^{-i}, \ldots, R_i^{-i}) \in \mathbb{R}^{d_i T \times d_{-i} T}$. Player i's cost is

$$\begin{aligned} & f_i(\mathbf{u}_i, \mathbf{u}_{-i}) = \tfrac{1}{2} \mathbf{u}_i^\top \mathbf{R}_i \mathbf{u}_i + \mathbf{u}_i^\top \mathbf{R}_i^{-i} \mathbf{u}_{-i} \\ & + \tfrac{1}{2}(W_1 \mathbf{u}_1 + W_2 \mathbf{u}_2 + \mathbf{F} z_0)^\top \mathbf{Q}_i (W_1 \mathbf{u}_1 + W_2 \mathbf{u}_2 + \mathbf{F} z_0). \end{aligned}$$

Expanding and regrouping this cost gives that $A_i = \mathbf{R}_i + W_i^\top \mathbf{Q}_i W_i$, $B_i = \left(\mathbf{R}_{i,-i} + W_i^\top \mathbf{Q}_i W_{-i}\right)^\top$, $D_i = W_{-i}^\top \mathbf{Q}_i W_{-i}$, $a_i^\top = z_0^\top \mathbf{F}^\top \mathbf{Q}_i W_i$, and $b_i^\top = z_0^\top \mathbf{F}^\top \mathbf{Q}_i W_{-i}$. In a typical LQR problem it is assumed that $\mathbf{R}_i \succ 0$ and $Q_i \succeq 0$ in order for solutions to exist (there are conditions that weaken these assumptions), and hence $A_i \succ 0$. In this case A_i is non-degenerate, and hence a sufficient condition for $\mathbf{M}_i$ for $i = 1, 2$ to each be non-degenerate is that the Schur complement of M_i with respect to $(\mathbf{R}_i + W_i^\top \mathbf{Q}_i W_i)$ is non-degenerate; indeed, this follows from the fact that

$$[\det(M_i) \neq 0 \;\forall i \in \{1, 2\}] \iff [\det(\mathbf{M}_i) \neq 0 \;\forall i \in \{1, 2\}].$$

2) Adaptive human-machine interactions: It has recently been shown that CCVE well-model human-machine co-adaptation [10]. In this study the human and the machine have scalar quadratic costs,

$$f_i(x_i, x_{-i}) = \frac{1}{2} \begin{bmatrix} x_i \\ x_{-i} \end{bmatrix}^\top \begin{bmatrix} q_i & r_i \\ r_i & s_i \end{bmatrix} \begin{bmatrix} x_i \\ x_{-i} \end{bmatrix} + \begin{bmatrix} w_i \\ v_i \end{bmatrix}^\top \begin{bmatrix} x_i \\ x_{-i} \end{bmatrix},$$

and series of experiments show convergence of repeated game play to CCVE in a computer-facilitated task. Assumption 1 is satisfied if $\det(M_i) \neq 0 \iff q_i s_i - r_i^2 \neq 0$ for each $i = 1, 2$. This holds for the games studied in [10]; it is shown in the supplement of the same reference that CCVE exist in affine conjectures for the games studied therein.

III. LFT Dynamics: Matrix Form

In this section, we give an extended analysis of the LFT dynamics in (9). Define the blocks of the product matrices $\mathbf{M}_1 = M_2^{-\top} M_1$ and $\mathbf{M}_2 = M_1^{-\top} M_2$ as follows:

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}, \quad \text{and} \quad \mathbf{M}_2 = \begin{bmatrix} \mathbf{D}_2 & \mathbf{C}_2 \\ \mathbf{B}_2 & \mathbf{A}_2 \end{bmatrix}.$$

Theorem 1: The composite LFT update in (9) can be written in the compact form

$$L_i^+ = \big(\mathbf{C}_i + \mathbf{D}_i L_i\big)\big(\mathbf{A}_i + \mathbf{B}_i L_i\big)^{-1}. \tag{11}$$

Proof: We show the proof for $i = 1$ and $-i = 2$ for clarity. Expanding $\mathbf{M}_1 = M_2^{-\top} M_1$ by using block matrix inversion on $M_2^{-\top}$, we deduce that

$$\mathbf{M}_1 = \begin{bmatrix} S_2^{-1}E & S_2^{-1}F \\ A_2^{-\top}(B_1 - B_2^\top S_2^{-1}E) & A_2^{-\top}(D_1 - B_2^\top S_2^{-1}F) \end{bmatrix},$$

with $S_2 = D_2^\top - B_2 A_2^{-\top} B_2^\top$, $E = A_1 - B_2 A_2^{-\top} B_1$, and $F = B_1^\top - B_2 A_2^{-\top} D_1$. We have specifically chosen a block matrix inversion that requires $A_2^\top$ and S_2 to be invertible. This does not require $\det(D_2) \neq 0$. Proceeding from (9),

$$L_1^+ = -\left(A_2^\top - (B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}B_2\right)^{-1} \cdot (B_2^\top - (B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top).$$

Applying the Woodbury matrix identity, we deduce that

$$\begin{aligned} L_1^+ &= -A_2^{-\top}B_2^\top + A_2^{-\top}(B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top \\ &\quad - A_2^{-\top}(B_1 + D_1L_1)(E + FL_1)^{-1}[B_2A_2^{-\top}B_2^\top \\ &\quad - B_2A_2^{-\top}(B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top] \end{aligned}$$

After algebraic manipulation, the term in $[\cdot]$ satisfies

$$\begin{aligned} &B_2A_2^{-\top}B_2^\top - B_2A_2^{-\top}(B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top \\ &= -S_2 + D_2^\top - B_2A_2^{-\top}(B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top \\ &= -S_2 + (E + FL_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top. \end{aligned}$$

Substituting this into the expression for L_1^+, we have that

$$\begin{aligned} L_1^+ &= -A_2^{-\top}B_2^\top + A_2^{-\top}(B_1 + D_1L_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top \\ &\quad - A_2^{-\top}(B_1 + D_1L_1)(E + FL_1)^{-1} \\ &\quad \cdot [-S_2 + (E + FL_1)(A_1 + B_1^\top L_1)^{-1}D_2^\top]. \end{aligned}$$

Distributing the last multiplicative term and canceling out appropriate terms we have that

$$\begin{aligned} L_1^+ &= -A_2^{-\top}B_2^\top + A_2^{-\top}(B_1 + D_1L_1)(E + FL_1)^{-1}S_2 \\ &= (A_2^{-\top}(B_1 + D_1L_1) - A_2^{-\top}B_2^\top S_2^{-1}(E + FL_1)) \\ &\quad \cdot (E + FL_1)^{-1}S_2 \\ &= (\mathbf{C}_1 + \mathbf{D}_1L_1)(\mathbf{A}_1 + \mathbf{B}_1L_1)^{-1}. \end{aligned}$$

which concludes the proof. ■

This update can be written as

$$\begin{bmatrix} I \\ L_1^+ \end{bmatrix} = \mathbf{M}_1 \begin{bmatrix} I \\ L_1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 + \mathbf{B}_1L_1 \end{bmatrix}^{-1}. \tag{12}$$

Starting at $L_1(0)$, iterating (12) for k steps leads to

$$\begin{bmatrix} I \\ L_1(k) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}^k \begin{bmatrix} I \\ L_1(0) \end{bmatrix} \Pi_{t=0}^{k-1}(\mathbf{A}_1 + \mathbf{B}_1L_1(t))^{-1}$$

On the one hand, the evolution of $L_1(k)$ is governed by repeated application of $\mathbf{M}_1$ as in a discrete time linear system. However, the right multiplication by $\Pi_{t=0}^{k-1}(\mathbf{A}_1+\mathbf{B}_1L_1(t))^{-1}$ makes the evolution nonlinear. Some features of the evolution of linear systems do apply, however. Specifically if $[I; L_1(0)]$ initially spans an $\mathbf{M}_1$–invariant subspace, then $[I; L_1(k)]$ will remain within that subspace as well for all k. This fact is at the heart of the equilibrium analysis in the next section.

IV. Equilibrium Analysis via Invariant Subspaces

Equilibrium points for the LFT dynamics can be found using invariant subspaces. The following theorem defines fixed points of the composite LFT dynamics (11) from which first order CCVE can be directly computed.

Theorem 2 (Equilibrium Computation): Let $K_1 = [Y_1; X_1] \in \mathbb{C}^{d \times d_1}$ where $Y_1 \in \mathbb{C}^{d_1 \times d_1}$ and $X_1 \in \mathbb{C}^{d_2 \times d_1}$ define an $\mathbf{M}_1$–invariant subspace where Y_1 is square and nonsingular. It follows that $L_1 = X_1Y_1^{-1} \in \mathbb{C}^{d_2 \times d_1}$ is fixed point of the composite LFT dynamics (11). A completely analogous statement holds for $L_2 = X_2Y_2^{-1}$.

Proof: Select the columns of K_1 to span a right-invariant subspace of $\mathbf{M}_1$, so that $M_2^{-\top}M_1K_1 = K_1\Lambda$. In general, K_1 can be complex leading to complex conjectures. For problems with real parameters, however, K_1 can often be chosen to be real. Even if the invariant subspace contains conjugate pairs of eigenvectors, K_1 can be chosen to be a real basis with vectors spanning any planes of rotation and Λ will simply be block diagonal as opposed to diagonal. The main exception to this is if the $\mathbf{M}_1$-invariant subspace contains only one complex eigenvector from a complex conjugate pair (see Remark 2 below). Since $\mathbf{M}_1$ is invertible, the matrix Λ will be as well. Hence we have that

$$\mathbf{M}_1 \begin{bmatrix} Y_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} Y_1 \\ X_1 \end{bmatrix} \Lambda \implies \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} \begin{bmatrix} I \\ L_1 \end{bmatrix} = \begin{bmatrix} I \\ L_1 \end{bmatrix} \mathbf{H}_1$$

where we have right multiplied by Y_1^{-1} and plugged in L_1 and $\mathbf{H}_1 = Y_1\Lambda Y_1^{-1}$. Note that $\mathbf{H}_1$ is invertible.

The top equation gives $(\mathbf{A}_1 + \mathbf{B}_1L_1) = \mathbf{H}_1$. Plugging this result into the bottom equation gives $\mathbf{C}_1 + \mathbf{D}_1L_1 = L_1(\mathbf{A}_1 + \mathbf{B}_1L_1)$ which implies $L_1 = (\mathbf{C}_1 + \mathbf{D}_1L_1)(\mathbf{A}_1 + \mathbf{B}_1L_1)^{-1}$. This verifies that $L_1 = X_1Y_1^{-1}$ is a fixed point of the dynamics as claimed which completes the proof. ■

In the case where Y_1 is not invertible, this method cannot be used and we leave analysis of this case to future work. While the choice of $\mathbf{M}_1$–invariant subspace matters for the computation of the equilibrium, the choice of basis does not.

Proposition 2 (Invariance with respect to basis.): Let $K_1 = [Y_1;\ X_1]$ and $K_1' = [Y_1';\ X_1']$ be two different bases for the same $\mathbf{M}_1$–invariant subspace with Y_1, Y_1' square and non-singular. Then $L_1 = X_1Y_1^{-1} = X_1'Y_1'^{-1}$.

Proof: Since K_1 and K_1' are bases for the same space, there exists square, non-singular W such that $K' = KW$. It follows that $X_1'Y_1'^{-1} = X_1WW^{-1}Y_1^{-1} = X_1Y_1^{-1}$. ■

A. Alternative Computation

The equilibrium solution can be derived from (9) using an alternative method without initially showing that the composite LFT map is given by the formula in Theorem 1. Since the analysis is more direct—and also provides inspiration for Theorem 1 and a useful perspective for proofs later on—we reproduce it here. Expanding and rearranging (9) at equilibrium, we get that $A_2^\top L_1 - (B_1 + D_1 L_1)(A_1 + B_1^\top L_1)^{-1} B_2 L = -B_2^\top + (B_1 + D_1 L_1)(A_1 + B_1^\top L_1)^{-1} D_2^\top$ which implies

$$A_2^\top L_1 + B_2^\top = (B_1 + D_1 L_1)(A_1 + B_1^\top L_1)^{-1}(D_2^\top + B_2 L_1). \tag{13}$$

Using this form of the fixed point equations, we can solve for the equilibrium using a similar invariant subspace argument.

Proposition 3 (Alternative Equilibrium Computation): Let the columns of $K_1 = \begin{bmatrix} Y_1^\top & X_1^\top \end{bmatrix}^\top$ solve the generalized eigenvalue problem $M_1 K_1 = M_2^\top K_1 \Lambda$. Then $L_1 = X_1 Y_1^{-1}$ solves (13).

Proof: The expression $M_1 K_1 = M_2^\top K_1 \Lambda$ gives

$$\begin{bmatrix} A_1 & B_1^\top \\ B_1 & D_1 \end{bmatrix} \begin{bmatrix} I \\ L_1 \end{bmatrix} = \begin{bmatrix} D_2^\top & B_2 \\ B_2^\top & A_2^\top \end{bmatrix} \begin{bmatrix} I \\ L_1 \end{bmatrix} \mathbf{H}_1 \tag{14}$$

where $\mathbf{H}_1 = Y_1 \Lambda Y_1^{-1}$. This expression arises since we have right multiplied by Y_1^{-1} and plugged in $L_1 = X_1 Y_1^{-1}$. Again, since $\mathbf{M}_1$ is non-singular, $\mathbf{H}_1$ will be as well. The top and bottom equation, respectively, can be rearranged to deduce that $(A_1 + B_1^\top L_1)^{-1}(D_2^\top + B_2 L_1) = \mathbf{H}_1^{-1}$ so that $(B_1 + D_1 L_1)\mathbf{H}_1^{-1} = (B_2^\top + A_2^\top L_1)$. Plugging in $\mathbf{H}_1^{-1}$ leads to (13), which concludes the proof. ■

Inspiration for the the composite dynamics can then be seen by noting that for invertible M_2, we see that $M_1 K_1 = M_2^\top K_1 \Lambda \iff M_2^{-\top} M_1 K_1 = K_1 \Lambda$.

At first pass, there are many ways to choose an $\mathbf{M}_1$–invariant subspace to compute L_1. Explicitly, there are $\binom{d}{d_1}$ ways to select a basis of eigenvectors. A further stability analysis (cf. Section V) shows that there is only *one way* to select an invariant subspace that leads to a stable L_1 when the eigenvalues of $\mathbf{M}_1$ have distinct magnitudes.

V. Equilibrium Stability

Next, we characterize the stability properties of fixed points of (3)—which includes the set of CCVE—and show how stability is related to the matrices $\mathbf{M}_i$, $i = 1, 2$. Local stability of a nonlinear system can be characterized by examining the eigenstructure of the local linearization: by the Hartman-Grobman theorem, if the eigenvalues of the local linearization have modulus less than one, then the fixed point is a locally asymptotically stable equilibrium.

Theorem 3 (Perturbation Dynamics): The linearized perturbation dynamics at fixed point (L_1, L_2) are $\Delta L_i^+ = \mathbf{\Omega}_i(\Delta L_i; L_i) = (\mathbf{D}_i - L_i \mathbf{B}_i)\Delta L_i (\mathbf{A}_i + \mathbf{B}_i L_i)^{-1}$.

Proof: Perturbing the equilibrium conjectures gives $L_i^+ + \Delta L_i^+ = (\mathbf{C}_i + \mathbf{D}_i L_i + \mathbf{D}_i \Delta L_i)(\mathbf{A}_i + \mathbf{B}_i L_i + \mathbf{B}_i \Delta L_i)^{-1}$. At equilibrium $L_i = L_i^+$, we have that $(L_i + \Delta L_i^+)(\mathbf{A}_i + \mathbf{B}_i L_i + \mathbf{B}_i \Delta L_i) = (\mathbf{C}_i + \mathbf{D}_i L_i + \mathbf{D}_i \Delta L_i)$. Recall that in equilibrium $L_i(\mathbf{A}_i + \mathbf{B}_i L_i) - (\mathbf{C}_i + \mathbf{D}_i L_i) = 0$. Therefore, we deduce that $\Delta L_i^+ = (\mathbf{D}_i - L_i \mathbf{B}_i)\Delta L_i(\mathbf{A}_i + \mathbf{B}_i L_i + \mathbf{B}_i \Delta L_i)^{-1}$. Applying the Woodbury matrix identity to the inverse and noting limits we further deduce that

$$\begin{aligned} \Delta L_i^+ &= \big(\mathbf{D}_i - L_i \mathbf{B}_i\big)\Delta L_i (\mathbf{A}_i + \mathbf{B}_i L_i)^{-1} \\ &\quad - (\mathbf{D}_i - L_i \mathbf{B}_i)\Delta L_i (\mathbf{A}_i + \mathbf{B}_i L_i)^{-1}\mathbf{B}_i \\ &\quad \cdot [I + \Delta L_i (\mathbf{A}_i + \mathbf{B}_i L_i)^{-1}\mathbf{B}_i]^{-1} \Delta L_i (\mathbf{A}_i + \mathbf{B}_i L_i)^{-1}. \end{aligned}$$

Dropping higher order terms completes the proof. ■

Note that $\mathbf{\Omega}_i(\,\cdot\,; L_i)$ for $i = 1, 2$ are linear operators in the form of a discrete time Lyapunov equation. To understand their stability, we recall a result from discrete time Lyapunov theory given here without proof.

Lemma 1 (DT Lyapunov Operators): For $A, B \in \mathbb{C}^{n \times n}$, the linear operator $\mathcal{A}(X) = AXB$ has eigenvalues of the form $\lambda_j \mu_k$ where $\lambda_j \in \operatorname{spec}(A)$ and $\mu_k \in \operatorname{spec}(B)$.

The following characterization of the spectra of $\mathbf{\Omega}_i(\,\cdot\,; L_i)$ follows immediately.

Theorem 4: The spectrum of the linear operator $\mathbf{\Omega}_i(\,\cdot\,; L_i)$ is given by $\operatorname{spec}(\mathbf{\Omega}_i) = \big\{ \frac{\lambda_j}{\mu_k} \;\big|\; \lambda_j \in \operatorname{spec}(\mathbf{D}_i - L_i \mathbf{B}_i),\ \mu_k \in \operatorname{spec}(\mathbf{A}_i + \mathbf{B}_i L_i) \big\}$.

We now establish equivalent conditions for local stability.

Theorem 5: At a fixed point $(L_1^{\mathrm{c}}, L_2^{\mathrm{c}})$ of (9), without loss of generality (for $i = 1, 2$), suppose $\operatorname{spec}(\mathbf{M}_1)$ can be divided into two sets $\rho_{\mathrm{L}}(\mathbf{M}_1)$ and $\rho_{\mathrm{S}}(\mathbf{M}_1)$ with cardinality d_1 and d_2 respectively, where all elements of $\rho_{\mathrm{L}}(\mathbf{M}_1)$ have strictly larger magnitude than all elements of $\rho_{\mathrm{S}}(\mathbf{M}_1)$. The following are equivalent:

a. The fixed point $(L_1^{\mathrm{c}}, L_2^{\mathrm{c}})$ is locally asymptotically stable with respect to (9) for $i = 1, 2$.
b. Eigenvalues $\xi_j \in \operatorname{spec}(\mathbf{\Omega}_1(\cdot; L_1^{\mathrm{c}}))$ satisfy $|\xi_j| < 1\ \forall\ j$.
c. The matrix $K_1 \in \mathbb{C}^{d \times d_1}$ from Theorem 2 (and Proposition 3) is chosen to span the $\mathbf{M}_1$–invariant subspace corresponding to the eigenvalues in $\rho_{\mathrm{L}}(\mathbf{M}_1)$.

Theorem 5 not only establishes equivalent conditions for stability, but also shows that it is sufficient to establish stability for one player in order to show the combined dynamics (i.e., (9) for $i = 1, 2$) are stable. However, the per player (local) rates of convergence depend on the eigenstructure of their individual dynamics.

Corollary 1: Players locally converge to $(L_1^{\mathrm{c}}, L_2^{\mathrm{c}})$ with iteration complexity $O(\xi_{i,\max}^k)$ where $\xi_{i,\max} := \max_{\xi \in \operatorname{spec}(\Omega_i(\,\cdot\,; L_i^{\mathrm{c}}))} |\xi|$ for player $i = 1, 2$, respectively.

Without loss of generality, the following lemma characterizes the eigenstructure of $\mathbf{M}_1$ and $\mathbf{M}_2$.

Lemma 2: The matrices L_1 computed from Theorem 1 and L_2 from (8) define the following similarity transforms on $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively:

$$\begin{bmatrix} I & 0 \\ -L_1 & I \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ L_1 & I \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{B}_1 \\ 0 & \mathbf{H}_1' \end{bmatrix}, \tag{15}$$

$$\begin{bmatrix} I & -L_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{D}_2 & \mathbf{C}_2 \\ \mathbf{B}_2 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} I & L_2 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbf{H}_2' & 0 \\ \mathbf{B}_2 & \mathbf{H}_2 \end{bmatrix}, \tag{16}$$

where $\mathbf{H}_1 = \mathbf{A}_1 + \mathbf{B}_1 L_1$, $\mathbf{H}_1' = \mathbf{D}_1 - L_1 \mathbf{B}_1$, $\mathbf{H}_2 = \mathbf{A}_2 + \mathbf{B}_2 L_2$, and $\mathbf{H}_2' = \mathbf{D}_2 - L_2 \mathbf{B}_2$. Furthermore, the spectrum of the $\mathbf{M}_1$-invariant subspace spanned by $[I; L_1]$ is $\operatorname{spec}(\mathbf{H}_1)$ and the spectrum of the $\mathbf{M}_2$-invariant subspace spanned by

$[L_2; I]$ is spec$(\mathbf{H}_2)$ and we can also write

$$\mathbf{H}_1 = \mathbf{A}_1 + \mathbf{B}_1 L_1 = \left(D_2^\top + B_2 L_1\right)^{-1}\left(A_1 + B_1^\top L_1\right)$$
$$\mathbf{H}_2' = \mathbf{D}_2 - L_2\mathbf{B}_2 = (A_1 + B_1^\top L_1)^{-\top}(D_2^\top + B_2 L_1)^\top.$$

and $\mathbf{H}_2'$ is similar to $\mathbf{H}_1^{-\top}$.

Proof: Each block of (15) is immediate with the zero block coming from (11). Observe that (8) can be rewritten as $(A_1 + B_1^\top L_1)^{-\top}[I\ L_1^\top]M_1^\top = [I\ -L_2]$. Expanding $[I\ -L_2]\mathbf{M}_2$, we get

$$\begin{aligned}\left[I\ -L_2\right] M_1^{-\top} M_2 &= (A_1 + B_1^\top L_1)^{-\top}\left[I\ \ L_1^\top\right] M_2 \\ &= (A_1 + B_1^\top L_1)^{-\top}\mathbf{H}_1^{-\top}\left[I\ L_1^\top\right] M_1^\top \\ &= \mathbf{H}_2'\left[I\ -L_2\right]\end{aligned}$$

where the second line comes from (14) and $\mathbf{H}_2' = [A_1 + B_1^\top L_1]^{-\top}\mathbf{H}_1^{-\top}[A_1 + B_1^\top L_1]^\top$. Note that $\mathbf{H}_2'$ and $\mathbf{H}_1^{-\top}$ are similar. The above gives us the top row of the following:

$$\begin{bmatrix} I & -L_2 \\ 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{D}_2 & \mathbf{C}_2 \\ \mathbf{B}_2 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_2' & 0 \\ \mathbf{B}_2 & \mathbf{A}_2 + \mathbf{B}_2 L_2 \end{bmatrix}\begin{bmatrix} I & -L_2 \\ 0 & I \end{bmatrix}$$

and the bottom row is then immediate. Right multiplying by $[I\ L_2; 0\ I]$ gives (16). The characterization of the invariant subspaces spanned by $[I; L_1]$ and $[L_2; I]$ follows immediately from the block diagonal structure. The alternate characterizations of $\mathbf{H}_1$ and $\mathbf{H}_2'$ follow from the characterization of $\mathbf{H}_1$ given in Proposition 3 and the definition of $\mathbf{H}_2'$ above which concludes the proof. ■

Proof: [Proof of Theorem 5] We use the fact that spec$(\mathbf{M}_i)$ = spec$(\mathbf{H}_i) \sqcup$ spec$(\mathbf{H}_i')$ which follows from the block diagonal structures in Lemma 2. The perturbation dynamics at equilibrium for each player are $\Delta L_i^+ = \mathbf{\Omega}_i(\Delta L_i; L_i) = \mathbf{H}_i'\Delta L_i \mathbf{H}_i^{-1}$. b. holds (by Thm. 4) iff spec$(\mathbf{H}_1) = \rho_{\mathrm{L}}(\mathbf{M}_1)$ and spec$(\mathbf{H}_1') = \rho_{\mathrm{S}}(\mathbf{M}_1)$ which in turn holds if and only if K_1 is chosen corresponding to $\rho_{\mathrm{L}}(\mathbf{M}_1)$. Since $\mathrm{spec}(\mathbf{M}_1) = 1/\,\mathrm{spec}(\mathbf{M}_2)$ and $\mathbf{H}_2'$ and $\mathbf{H}_1^{-\top}$ are similar (cf. Lemma 2), the above holds if and only if spec$(\mathbf{H}_2') = 1/\rho_{\mathrm{L}}(\mathbf{M}_1)$ and spec$(\mathbf{H}_2) = 1/\rho_{\mathrm{S}}(\mathbf{M}_1)$ and by Theorem 4 this is equivalent to $|\xi_j| < 1$ for all $\xi_j \in$ spec$(\mathbf{\Omega}_2(\cdot; L_2^{\mathrm{c}}))$. Moreover, b. and the equivalent statement for $\mathbf{\Omega_2}$ are equivalent to a. by Hartman-Grobman [19]. ■

Remark 2: Theorem 5 implies that if the eigenvalues of $\mathbf{M}_1$ (and $\mathbf{M}_2$) distinctly divide into "large" and "small" sets, then there is a unique way to choose an asymptotically stable fixed point of the composite dynamics (9). When the eigenvalues cannot clearly be divided this way, there may be multiple ways to construct marginally stable points. Two interesting cases are when there are eigenvalues from the same Jordan subspace or complex eigenvalues from the same conjugate pair in each set. In the latter, the only marginally stable conjectures are complex and any associated real conjectures exhibit oscillatory behavior analogous to elliptic Möbius transformations. These cases will be examined in a subsequent paper.

VI. Comments on Second Order Conditions

For stable first order CCVE, the second order conditions (6) can be checked to see if each player's optimization is convex. This is not guaranteed and will depend on the relative magnitudes of the parameters A_i, B_i, and D_i. Simple analysis shows that $M_1, M_2 \succ 0$ is sufficient to guarantee (6); however, this is often not true since D_1, D_2 might be zero, low-rank, indefinite, or even negative-definite. Simple numerical experiments also show that $M_1, M_2 \succ 0$ is far too conservative a condition and often not necessary for (6) to hold. For further discussion and numerics, see [14].

VII. Discussion & Open Questions

This paper introduces a novel analysis of CCVE by drawing on tools from the analysis of coupled Riccati equations. There are a number of interesting open questions including how players might adapt their conjectural variations in dynamic games by interacting with opponents, as well as how players might adopt policy gradient like procedures to learn their policies contingent on conjectures adapted over time.

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