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Phase limitations of multipliers at harmonics via frequency intervals

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***Abstract*— The absolute stability of a Lurye system with a monotone nonlinearity is guaranteed by the existence of a suitable O'Shea-Zames-Falb (OZF) multiplier. A numerically tractable phase condition has recently been proposed under which there can be no suitable OZF multiplier for the transfer function of a given continuous-time plant. The condition has been derived via the so-called duality approach. Here we show that the condition may also be derived from an established frequency interval approach providing an important link between the two hitherto distinct approaches. We show that it leads to significantly improved results compared with the frequency interval approach on a benchmark example.**

I. Introduction

The continuous-time OZF (O'Shea-Zames-Falb) multipliers were discovered by O'Shea [1] and formalised by Zames and Falb [2]. They preserve the positivity of monotone memoryless nonlinearities. Hence they can be used, via loop transformation, to establish the absolute stability of the feedback interconnection between a linear time invariant (LTI) system and any slope-restricted memoryless nonlinearity. An overview is given in [3].

Recent interest is largely driven by their compatability with the integral quadratic constraint (IQC) framework of Megretski and Rantzer [4] and the availability of computational searches [5], [6], [7], [8], [9], [10], [11], [12]. A modification of the search proposed in [8] is used in the Matlab IQC toolbox [13] and analysed in [14].

No single search method outperforms the others, and often a hand-tailored search outperforms an automated search [12]. This motivates the analysis of conditions where a multiplier cannot exist, and hence where *any* search would be fruitless.

Recently we have developed a simple condition [15], derived from an earlier result [16] which is a particular case of a more general analysis based on duality in an optimiztion framework [17], [18], [19]. In [15] we show that the new condition gives better results for the benchmark example in [16], largely because it can be applied systematically. Here we show that the same result may also be derived from the frequency interval approach of [20], [21], providing a first link between the two approaches. Although the two approaches use different mathematical formulations they lead to exactly the same result. We apply the condition to a classic benchmark example [1], [3] and show that it leads to better results than those reported in [21], once again because it can be applied systematically.

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II. Preliminaries

A. Multiplier theory

We are concerned with the input-output stability of the Lurye system (Fig 1) given by

$$y_1 = \boldsymbol{G}u_1,\ y_2 = \boldsymbol{\phi}u_2,\ u_1 = r_1 - y_2 \text{ and } u_2 = y_1 + r_2. \quad (1)$$

Let $\mathcal{L}_2$ be the space of finite energy Lebesgue integrable signals and let $\mathcal{L}_{2e}$ be the corresponding extended space (see for example [22]). The Lurye system is said to be stable if $r_1, r_2 \in \mathcal{L}_2$ implies $u_1, u_2, y_1, y_2 \in \mathcal{L}_2$.

The Lurye system (1) is assumed to be well-posed with $\boldsymbol{G} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ linear time invariant (LTI) causal and stable, and with $\boldsymbol{\phi} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ memoryless and time-invariant. We will use G to denote the transfer function corresponding to $\boldsymbol{G}$. Where appropriate we will consider either $G : j\mathbb{R} \to \mathbb{C}$ (i.e. $G(j\omega)$) or $G : \bar{\mathbb{C}}_+ \to \mathbb{C}$ (i.e. $G(s)$) where $\bar{\mathbb{C}}_+ = \{s \in \mathbb{C} : Re(s) \geq 0\}$. The nonlinearity $\boldsymbol{\phi}$ is assumed to be montone in the sense that $(\boldsymbol{\phi}u)(t_1) \geq (\boldsymbol{\phi}u)(t_2)$ for all $u(t_1) \geq u(t_2)$. It is also assumed to be bounded in the sense that there exists a $C \geq 0$ such that $|(\boldsymbol{\phi}u)(t)| \leq C|u(t)|$ for all $u(t) \in \mathbb{R}$. We say $\boldsymbol{\phi}$ is slope-restricted on $[0, k]$ if $0 \leq (\boldsymbol{\phi}u)(t_1) - (\boldsymbol{\phi}u)(t_2))/(u(t_1) - u(t_2)) \leq k$ for all $u(t_1) \neq u(t_2)$. We say $\boldsymbol{\phi}$ is odd if $(\boldsymbol{\phi}u)(t_1) = -(\boldsymbol{\phi}u)(t_2)$ whenever $u(t_1) = -u(t_2)$.

Definition 1: Let $M : j\mathbb{R} \to \mathbb{C}$ and let $G : j\mathbb{R} \to \mathbb{C}$ We say M is suitable for G if there exists $\varepsilon > 0$ such that

$$\text{Re}\left\{M(j\omega)G(j\omega)\right\} > \varepsilon \text{ for all } \omega \in \mathbb{R}. \quad (2)$$

Remark 1: Suppose M is suitable for G and $\angle G(j\omega) \leq -90^o - \theta$ for some ω and θ. Then $\angle M(j\omega) > \theta$. Similarly if $\angle G(j\omega) \geq 90^o + \theta$ then $\angle M(j\omega) < -\theta$.

Definition 2a: Let $\mathcal{M}$ be the class of transfer functions $M : j\mathbb{R} \to \mathbb{C}$ whose (possibly non-causal) impulse response

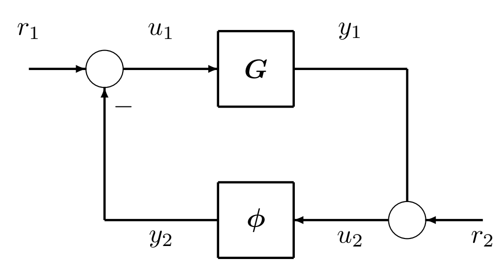

Fig. 1. Lurye system.

is given by

$$m(t) = m_0\delta(t) - h(t) - \sum_{i=1}^{\infty} h_i\delta(t - t_i), \tag{3}$$

with

$$\begin{aligned} &h(t) \geq 0 \text{ for all } t,\ h_i \geq 0 \text{ and } t_i \neq 0 \text{ for all } i, \\ &\quad \text{and } \|h\|_1 + \sum_{i=1}^{\infty} h_i \leq m_0. \end{aligned} \tag{4}$$

We say M is an OZF multiplier if $M \in \mathcal{M}$.

Definition 2b: Let $\mathcal{M}_{\text{odd}}$ be the class of transfer functions $M : j\mathbb{R} \to \mathbb{C}$ whose (possibly non-causal) impulse response is given by (3) with

$$\|h\|_1 + \sum_{i=1}^{\infty} |h_i| \leq m_0. \tag{5}$$

We say M is an OZF multiplier for odd nonlinearities if $M \in \mathcal{M}_{\text{odd}}$.

The Lurye system (1) is said to be absolutely stable for a particular $\boldsymbol{G}$ if it is stable for all $\boldsymbol{\phi}$ in some class Φ. In particular, if there is an $M \in \mathcal{M}$ suitable for G then it is absolutely stable for the class of memoryless time-invariant monotone bounded nonlinearities; if there is an $M \in \mathcal{M}_{\text{odd}}$ suitable for G then it is absolutely stable for the class of memoryless time-invariant odd monotone bounded nonlinearities. Furthermore, if there is an $M \in \mathcal{M}$ suitable for $1/k + G$ then it is absolutely stable for the class of memoryless time-invariant slope-restricted nonlinearities in $[0, k]$; if there is an $M \in \mathcal{M}_{\text{odd}}$ suitable for $1/k + G$ then it is absolutely stable for the class of memoryless time-invariant odd slope-restricted nonlinearities [2], [3].

B. Phase limitations: frequency interval approach

In the frequency interval approach [20], [21] a condition is given on the phase of G such that if it is sufficiently high over one interval and sufficiently low over another then there is no OZF multiplier suitable for G. In [21] we presented the following phase limitation for the frequency intervals $[\alpha, \beta]$ and $[\gamma, \delta]$ (see Fig 2). NB here we follow [20] in our treatment of strict and nonstrict inequalities.

Theorem 1a ([21]): Let $0 < \alpha < \beta < \gamma < \delta$ and define

$$\rho = \sup_{t>0} \frac{|\psi(t)|}{\phi(t)}, \tag{6}$$

with

$$\begin{aligned} \psi(t) &= \frac{\lambda\cos(\alpha t)}{t} - \frac{\lambda\cos(\beta t)}{t} - \frac{\mu\cos(\gamma t)}{t} + \frac{\mu\cos(\delta t)}{t}, \\ \phi(t) &= \lambda(\beta - \alpha) + \kappa\mu(\delta - \gamma) + \phi_1(t), \\ \phi_1(t) &= \frac{\lambda\sin(\alpha t)}{t} - \frac{\lambda\sin(\beta t)}{t} + \frac{\kappa\mu\sin(\gamma t)}{t} - \frac{\kappa\mu\sin(\delta t)}{t}, \end{aligned} \tag{7}$$

and with $\lambda > 0$ and $\mu > 0$ satisfying

$$\frac{\lambda}{\mu} = \frac{\delta^2 - \gamma^2}{\beta^2 - \alpha^2}, \tag{8}$$

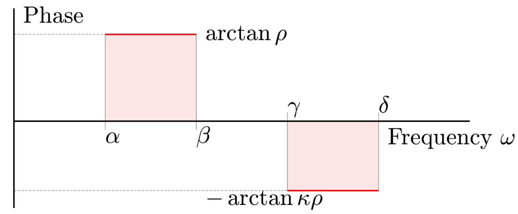

Fig. 2. Illustration of Theorems 1a and 1b. If ρ is sufficiently large then no multiplier can have phase greater than $\arcsin\rho$ on interval $[\alpha, \beta]$ and phase less than $-\arcsin\kappa\rho$ on interval $[\gamma, \delta]$.

and $\kappa > 0$ sufficiently large to ensure $\phi(t) \geq 0$ for all $t > 0$. Then there is no $M \in \mathcal{M}$ with

$$\text{Im}(M(j\omega)) > \rho\text{Re}(M(j\omega)) \text{ for all } \omega \in [\alpha, \beta], \tag{9}$$

and

$$\text{Im}(M(j\omega)) < -\kappa\rho\text{Re}(M(j\omega)) \text{ for all } \omega \in [\gamma, \delta]. \tag{10}$$

The result also holds if we replace (9) and (10) with

$$\text{Im}(M(j\omega)) < -\rho\text{Re}(M(j\omega)) \text{ for all } \omega \in [\alpha, \beta], \tag{11}$$

and

$$\text{Im}(M(j\omega)) > \kappa\rho\text{Re}(M(j\omega)) \text{ for all } \omega \in [\gamma, \delta]. \tag{12}$$

Theorem 1b ([21]): Suppose, with the conditions of Theorem 1a, that we define instead

$$\rho = \sup_{t>0} \max\left\{\frac{|\psi(t)|}{\phi(t)}, \frac{|\psi(t)|}{\tilde{\phi}(t)}\right\}, \tag{13}$$

with

$$\tilde{\phi}(t) = \lambda(\beta - \alpha) + \kappa\mu(\delta - \gamma) - |\phi_1(t)|, \tag{14}$$

and κ is sufficiently large to ensure $\tilde{\phi}(t) \geq 0$ for all $t > 0$. Then there is no $M \in \mathcal{M}_{\text{odd}}$ with (9) and (10) (or with (11) and (12)).

Remark 2: In [21] the ratio κ is restricted to be positive which is sufficient to ensure $\phi(t) \geq 0$ (or $\tilde{\phi} \geq 0$). Here we allow a slight generalisation but the proof is similar.

C. Phase limitations at harmonics

In [15] the following frequency conditions are derived from the duality approach [16], [17], [18], [19]. They can be tested in a systematic manner and are shown to give improved results over those reported in [16].

Theorem 2a ([15]): Let $a, b \in \mathbb{Z}^+$ be coprime and let $M \in \mathcal{M}$. Then

$$\left|\frac{b\angle M(aj\omega) - a\angle M(bj\omega)}{a/2 + b/2 - p}\right| \leq 180^o \tag{15}$$

for all $\omega \in \mathbb{R}$ with $p = 1$.

Theorem 2b ([15]): Let $a, b \in \mathbb{Z}^+$ be coprime and let $M \in \mathcal{M}_{odd}$. Then inequality (15) holds for all $\omega \in \mathbb{R}$ with $p = 1$ when both a and b are odd, but with $p = 1/2$ when either a or b are even.

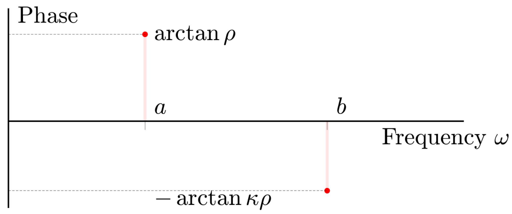

Fig. 3. Illustration of Corollaries 1a and 1b. If ρ is sufficiently large then no multiplier can have phase greater than $\arcsin\rho$ at frequency a and phase less than $-\arcsin\kappa\rho$ at frequency b. Compare Fig 2.

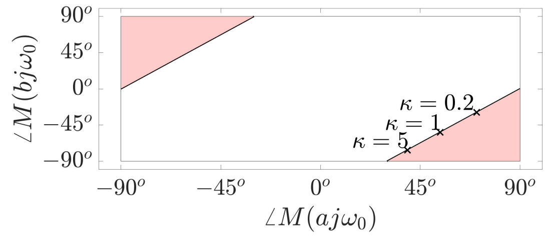

Fig. 4. Phase vs phase plot illustrating Theorem 2a with $a = 2$, $b = 3$. If $M \in \mathcal{M}$ then the pink regions are forbidden. Also shown are the points $(\arctan\rho, -\arctan\kappa\rho)$ when $a = 2$ and $b = 3$, when κ takes the values 0.2, 1 and 5 and when ρ is defined as in Corollary 1a.

D. Other notation

In our discussion of the example (Section IV) phase is expressed in degrees. In the technical proofs (i.e. the Appendix) phase is expressed in radians.

III. Relation to the frequency interval approach

In this Section we show that Theorems 2a and 2b can be derived from Theorems 1a and 1b. Theorems 2a and 2b concern the phase of $M(j\omega)$ at two frequencies, $a\omega$ and $b\omega$, where the ratio a/b is rational, whereas Theorems 1a and 1b concern the phase of $M(j\omega)$ over two frequency intervals $[\alpha, \beta]$ and $[\gamma, \delta]$. Let us begin by considering Theorems 1a and 1b in the limit as the length of the intervals becomes zero and where the ratio of the limiting frequencies is rational (Fig 3).

Corollary 1a: For $0 < t < 2\pi$, define

$$q_+(t) = \frac{b\sin(at) - a\sin(bt)}{b + \kappa a - b\cos(at) - \kappa a\cos(bt)}, \tag{16}$$

where $a, b \in \mathbb{Z}^+$ are coprime and $\kappa > -1/\max(a, b)$ and $\kappa > 0$ if either $a = 1$ or $b = 1$. Define also

$$\rho = \sup_{t\in(0,2\pi)} |q_+(t)|. \tag{17}$$

Then given any $\omega_0 \in \mathbb{R}$ there is no $M \in \mathcal{M}$ that satisfies

$$\angle M(aj\omega_0) > \arctan\rho, \tag{18}$$

and

$$\angle M(bj\omega_0) < -\arctan\kappa\rho. \tag{19}$$

Corollary 1b: Suppose, with the conditions of Corollary 1a and $\kappa > 0$, we define instead

$$\rho = \sup_{t\in(0,2\pi)} \max\left(|q_+(t)|, |q_-(t)|\right), \tag{20}$$

where

$$q_-(t) = \frac{b\sin(at) - a\sin(bt)}{b + \kappa a + b\cos(at) + \kappa a\cos(bt)}. \tag{21}$$

Then given any $\omega_0 \in \mathbb{R}$ there is no $M \in \mathcal{M}_{\text{odd}}$ that satisfies (18) and (19).

Proof: See Appendix. ∎

It turns out that this is equivalent to the phase condition derived via the duality approach in [15]. The inequality boundaries $\angle M(aj\omega_0) = \arctan\rho$ and $\angle M(bj\omega_0) = -\arctan\kappa\rho$ are the same as those for Theorem 2a (or 2b), as illustrated in Fig 4. Specifically we may say:

Theorem 3a: Corollary 1a and Theorem 2a are equivalent results.

Theorem 3b: Corollary 1b and Theorem 2b are equivalent results.

Proof: See Appendix. ∎

IV. Example

Consider the plant

$$G(s) = \frac{s^2}{(s^2 + 2\xi s + 1)^2} \text{ with } \xi > 0. \tag{22}$$

O'Shea [1] shows that there is a suitable multiplier in $\mathcal{M}$ for $1/k + G$ when $\xi > 1/2$ and $k > 0$. By contrast in [21] we showed that there is no suitable multiplier in $\mathcal{M}$ when $\xi = 0.25$ and k is sufficiently large. Specifically the phase of $G(j\omega)$ is above 177.98^o on the interval $\omega \in [0.02249, 0.03511]$ and below -177.98^o on the interval $\omega \in [1/0.03511, 1/0.02249]$. A line search yields that the same condition is true for the phase of $1/k + G(j\omega)$ with $k \geq 269,336.3$ (see Fig 5). Hence there is no suitable multipler $M \in \mathcal{M}$ for $1/k + G$ with $k \geq 269,336.3$. However, in [21] we conclude that the most insightful choice of interval remains open.

By contrast, Theorem 2a with $a = 4$ and $b = 1$ yields there is no suitable multipler $M \in \mathcal{M}$ for $1/k + G$ with $k \geq 32.61$. Specifically the phase $(4\angle(1/k + G(j\omega)) - \angle(1/k + G(4j\omega)))/4$ exceeds 180^o when $k \geq 32.61$ (see Figs 6 and 7). Similarly, Theorem 2b with $a = 3$ and $b = 1$ yields there is no suitable multipler $M \in \mathcal{M}_{odd}$ for $1/k + G$ with $k \geq 39.93$. Specifically the phase $(3\angle(1/k + G(j\omega)) - \angle(1/k + G(3j\omega)))/3$ exceeds 180^o when $k \geq 32.61$.

These results show a non-trivial improvement over those in [21]. While it should be possible to achieve identical results using either the condition of [16] or that of [21], the conditions of Theorems 2a and 2b can be applied in a systematic manner. For this example we test the criterion for a finite number of coprime integers a and b, and for all $\omega > 0$; we also search over the slope restriction k. Specifically we run a bisection algorithm for k and, for each candidate value of k, a and b, check whether the condition is satisfied for any $\omega > 0$ by gridding. Fig 8 shows the bounds

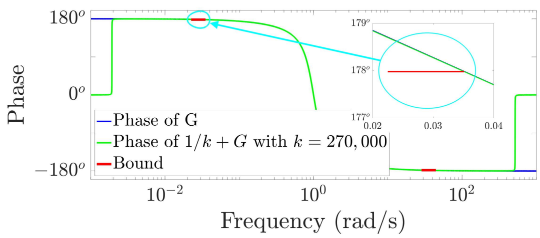

Fig. 5. O'Shea's example with $\zeta = 0.25$. Application of the condition in [21] yields there to be no suitable multiplier $M \in \mathcal{M}$ when $k \geq 270,000$.

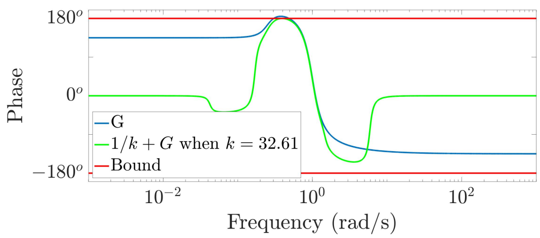

Fig. 6. O'Shea's example with $\zeta = 0.25$. The phase differences $(4\angle G(j\omega) - \angle G(4j\omega))/4$ and $(4\angle(1/k+G(j\omega)) - \angle(1/k+G(4j\omega)))/4$ with $k = 32.61$ are shown. Application of Theorem 2a with $a = 4$ and $b = 1$ yields there to be no suitable multiplier $M \in \mathcal{M}$ when $k \geq 32.61$.

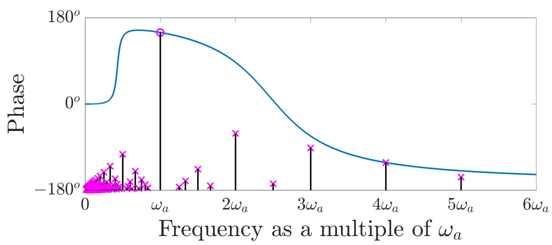

Fig. 7. O'Shea's example with $\zeta = 0.25$. The phase of $1/k + G(j\omega)$ with $k = 32.61$ is shown. The phase of $1/k + G(j\omega_a)$ is 149.42^o at $\omega_a = 0.3938$ and the corresponding forbidden regions are shown. The phase touches the bound at $4\omega_a$.

for several other values of ζ while Fig 9 shows the value of a yielding the lowest bound for each test (the value of b is 1 for each case).

V. Conclusion

In [15] we propose a phase condition that OZF multipliers must satisfy, repeated here as Theorems 2a and 2b. The condition is derived in [15] from the duality framework of [16], [17], [18], [19]. It has the advantage that it can be applied systematically and we show in [15] that it leads to improved results for the benchmark example in [16]. Here we show the condition may also be derived from the frequency interval approach of [21], [20], drawing an important link between the two approaches. We show it leads to significantly improved results over those reported in [21] for the benchmark example of [1].

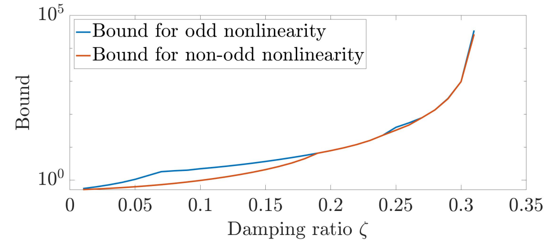

Fig. 8. Bounds on the slope above which Theorem 2a or 2b guarantee there can be no suitable multiplier as damping ratio ζ varies.

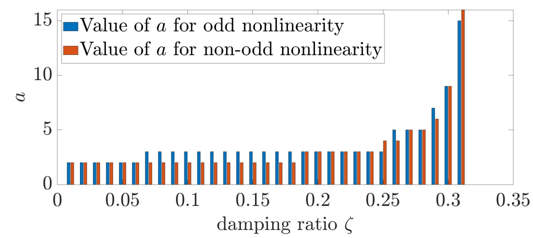

Fig. 9. Values of a used to find the slope bounds shown in Fig 8. The value of b is 1 for all shown results.

Appendix

A. Proofs of Corollaries 1a and 1b

Proof: [Corollary 1a] Without loss of generality let $a < b$. The result follows by setting the intervals

$$\begin{aligned} [\alpha, \beta] &= [a\omega_0 - \varepsilon, a\omega_0 + \varepsilon], \\ \text{and } [\gamma, \delta] &= [b\omega_0 - \varepsilon, b\omega_0 + \varepsilon], \end{aligned} \tag{23}$$

with $\varepsilon > 0$. We find

$$\begin{aligned} \psi(t) &= \frac{2\lambda}{t}\sin(a\omega_0 t)\sin(\varepsilon t) - \frac{2\mu}{t}\sin(b\omega_0 t)\sin(\varepsilon t), \\ \phi(t) &= 2\varepsilon\lambda + 2\varepsilon\kappa\mu + \phi_1(t), \\ \phi_1(t) &= -\frac{2\lambda}{t}\cos(a\omega_0 t)\sin(\varepsilon t) - \frac{2\kappa\mu}{t}\cos(b\omega_0 t)\sin(\varepsilon t), \end{aligned} \tag{24}$$

with $a\lambda = b\mu$. The result follows in the limit as $\varepsilon \to 0$. ■

Proof: [Corollary 1b] In addition

$$\tilde{\phi}(t) = 2\varepsilon\lambda + 2\varepsilon\kappa\mu - |\phi_1(t)|. \tag{25}$$

Once again the result follows in the limit as $\varepsilon \to 0$. ■

B. Proof of Theorems 3a and 3b

Proof: [Theorem 3a] For $0 < t < 2\pi$ define

$$\theta_+(t) = b \arctan q_+(t) + a \arctan \kappa q_+(t). \tag{26}$$

We will show that for each κ all turning points of $\theta_+(t)$ are bounded by $\pm(a+b-2)\frac{\pi}{2}$ and that at least one turning point touches the bounds. This is sufficient to establish the equivalence between Corollary 1a and Theorem 2a.

The turning points of $\theta_+(t)$ occur at the same values of t as the turning points of $q_+(t)$. Specifically

$$\frac{d}{dt}\theta_+(t) = \left(\frac{b}{1+q_+(t)^2} + \frac{a\kappa}{1+\kappa^2 q_+(t)^2} \right) \frac{d}{dt} q_+(t), \tag{27}$$

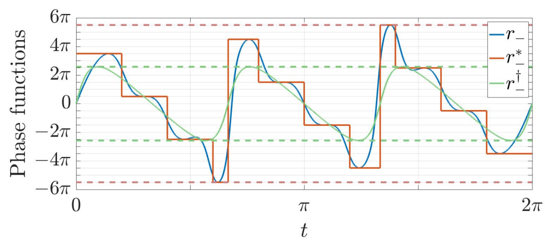

Fig. 10. Phase functions θ_+ (blue), θ_+^* (red) and $\theta_+^\dagger$ (green) with $a = 3$ and $b = 10$. The turning points of θ_+ where $m_+(t) = 0$ take the value $(a+b-2\lambda)\pi/2$ with λ an integer. The function $\theta_+^*(\cdot)$ is piecewise constant and takes these same values. The turning points of θ_+ where $n_+(t) = 0$ take the values of $\theta_+^\dagger$, whose bounds are also shown.

where the derivative of $q_+(t)$ is given by

$$\frac{d}{dt}q_+(t) = ab\frac{m_+(t)n_+(t)}{d_+(t)^2}, \tag{28}$$

with

$$\begin{aligned} m_+(t) &= \sin\frac{at}{2}\cos\frac{bt}{2} + \kappa\sin\frac{bt}{2}\cos\frac{at}{2}, \\ n_+(t) &= a\sin\frac{bt}{2}\cos\frac{at}{2} - b\sin\frac{at}{2}\cos\frac{bt}{2}, \\ d_+(t) &= b\sin^2\frac{at}{2} + \kappa a\sin^2\frac{bt}{2}. \end{aligned} \tag{29}$$

On the interval $0 < t < 2\pi$ the derivatives of both $q_+(t)$ and $\theta_+(t)$ are zero when either $m_+(t) = 0$ or $n_+(t) = 0$. We consider the two cases separately. In both cases we use the identity

$$q_+(t) = \frac{b\tan\frac{at}{2}\left(1+\tan^2\frac{bt}{2}\right) - a\tan\frac{bt}{2}\left(1+\tan^2\frac{at}{2}\right)}{b\tan^2\frac{at}{2}\left(1+\tan^2\frac{bt}{2}\right) + \kappa a\tan^2\frac{bt}{2}\left(1+\tan^2\frac{at}{2}\right)}. \tag{30}$$

Case 1: Suppose t_1 satisfies $m_+(t_1) = 0$. At these values

$$q_+(t_1) = \cot\frac{at_1}{2}, \tag{31}$$

and

$$\kappa q_+(t_1) = -\cot\frac{bt_1}{2}. \tag{32}$$

Hence if we define

$$\theta_+^*(t) = b\left[\frac{\pi}{2} - \frac{at}{2}\right]_{[-\pi/2,\pi/2]} + a\left[-\frac{\pi}{2} + \frac{bt}{2}\right]_{[-\pi/2,\pi/2]}, \tag{33}$$

for $t \in [0, 2\pi]$ we find $\theta_+(t_1) = \theta_+^*(t_1)$ for all t_1 satisfying $m_+(t_1) = 0$. The function $\theta_+^*(\cdot)$ is piecewise constant, taking values $(-a-b+2\lambda)\pi/2$ with $\lambda = 1, \ldots, a+b-1$. On each piecewise constant interval there is a t_1 satisfying $m_+(t_1) = 0$. Hence these turning points of $\theta_+(t)$ lie within the bounds $\pm(a+b-2)\frac{\pi}{2}$ with at least one on the bound.

Case 2: Suppose t_2 satisfies $n_+(t_2) = 0$. Define

$$q_+^\dagger(t) = \frac{(b^2-a^2)\sin at}{a^2+b^2+2\kappa ab - (b^2-a^2)\cos at}, \tag{34}$$

and

$$\theta_+^\dagger(t) = b\arctan q_+^\dagger(t) + a\arctan\kappa q_+^\dagger(t). \tag{35}$$

Then $q_+(t_2) = q_+^\dagger(t_2)$ and $\theta_+(t_2) = \theta_+^\dagger(t_2)$ for all t_2 satisfying $n_+(t_2) = 0$. It follows that $|\theta_+(t_2)| \leq |\bar{\theta}^\dagger|$ for all such t_2 where

$$\bar{\theta}^\dagger = b\arctan\bar{q}^\dagger + a\arctan\kappa\bar{q}^\dagger, \tag{36}$$

and

$$\bar{q}^\dagger = \frac{b^2-a^2}{2\sqrt{ab(a+\kappa b)(b+\kappa a)}}. \tag{37}$$

With some abuse of notation write $\bar{\theta}^\dagger = \bar{\theta}^\dagger(\kappa)$; i.e. consider $\bar{\theta}^\dagger$ as a function of κ and observe that $\bar{\theta}^\dagger(\kappa) \to 0$ as $\kappa \to 0$. We find

$$\begin{aligned} \frac{d}{d\kappa}\bar{\theta}^\dagger(\kappa) =& \frac{-(a+b\kappa)(a^2-b^2)^2}{(2ab+(a^2+b^2)\kappa)(2ab\kappa+a^2+b^2)} \\ &\times\sqrt{\frac{ab}{(a+b\kappa)(a\kappa+b)}}, \\ &\leq 0 \text{ for the given range of } \kappa. \end{aligned} \tag{38}$$

Without loss of generality assume $b > a$. If $a > 1$ the interval is $\kappa > 0$ and

$$\begin{aligned} |\bar{\theta}^\dagger(\kappa)| &\leq \bar{\theta}^\dagger(0), \\ &= b\arctan\left(\frac{b^2-a^2}{2ab}\right), \\ &< (a+b-2)\frac{\pi}{2}. \end{aligned} \tag{39}$$

If $a = 1$ the interval is $\kappa > -1/b$ and

$$\begin{aligned} |\bar{\theta}^\dagger(\kappa)| &\leq \bar{\theta}^\dagger(-1/b), \\ &= (b-a)\frac{\pi}{2}, \\ &= (a+b-2)\frac{\pi}{2}. \end{aligned} \tag{40}$$

■

Proof: [Theorem 3b] The proof is similar to that for Theorem 3a. We have already established appropriate bounds for $\theta_+(t)$. If we define

$$\theta_-(t) = b\arctan q_-(t) + a\arctan\kappa q_-(t), \tag{41}$$

then we need to show it is also bounded appropriately. Similar to the previous case, the turning points of $\theta_-(t)$ occur at the same values of t as the turning points of $q_-(t)$. On the interval $t \in (0, 2\pi)$ the derivative of $q_-(t)$ is given by

$$\frac{d}{dt}q_-(t) = ab\frac{m_-(t)n_-(t)}{d_-(t)^2}, \tag{42}$$

with

$$\begin{aligned} m_-(t) &= \kappa\sin\frac{at}{2}\cos\frac{bt}{2} + \sin\frac{bt}{2}\cos\frac{at}{2}, \\ n_-(t) &= b\sin\frac{bt}{2}\cos\frac{at}{2} - a\sin\frac{at}{2}\cos\frac{bt}{2}, \\ d_-(t) &= b\cos^2\frac{at}{2} + \kappa a\cos^2\frac{bt}{2}. \end{aligned} \tag{43}$$

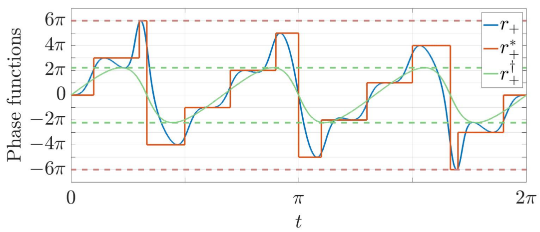

Fig. 11. Phase functions θ_- (blue), θ_-^* (red) and $\theta_-^\dagger$ (green) with $a = 3$ and $b = 10$. The turning points of θ_- where $m_-(t) = 0$ take the value $(a + b + 1 - 2\lambda)\pi/2$ with λ an integer. The function $\theta_-^*(\cdot)$ is piecewise constant and takes these same values. The turning points of θ_- where $n_-(t) = 0$ take the values of $\theta_-^\dagger$, whose bounds are also shown.

We will consider the cases $m_-(t) = 0$ and $n_-(t) = 0$ separately. This time we use the identity

$$q_-(t) = \frac{b \tan \frac{at}{2} \left(1 + \tan^2 \frac{bt}{2}\right) - a \tan \frac{bt}{2} \left(1 + \tan^2 \frac{at}{2}\right)}{b \left(1 + \tan^2 \frac{bt}{2}\right) + \kappa a \left(1 + \tan^2 \frac{at}{2}\right)}. \tag{44}$$

Case 1: Suppose t_1 satisfies $m_-(t_1) = 0$. Then

$$q_-(t_1) = \tan \frac{at_1}{2}, \tag{45}$$

and

$$\kappa q_-(t_1) = -\tan \frac{bt_1}{2}. \tag{46}$$

Hence if we define

$$\theta_-^*(t) = b \left[\frac{at}{2}\right]_{[-\pi/2,\pi/2]} - a \left[\frac{bt}{2}\right]_{[-\pi/2,\pi/2]}, \tag{47}$$

for $t \in [0, 2\pi]$ we find $\theta_-(t_1) = \theta_-^*(t_1)$ for all t_1 satisfying $m_-(t_1) = 0$. The function $\theta_-^*(\cdot)$ is piecewise constant, taking values $(-a - b - 1 + 2\lambda)\pi/2$ with $\lambda = 1, \ldots, a + b$ when either a or b are even, and values $(-a - b + 2\lambda)\pi/2$ with $\lambda = 1, \ldots, a+b-1$ when a and b are both odd. On each piecewise constant interval there is a t_1 satisfying $m_-(t_1) = 0$. Hence these turning points of $\theta_-(t)$ lie within the bounds $\pm(a + b - 1)\frac{\pi}{2}$ (if either a or b even) or $\pm(a + b - 2)\frac{\pi}{2}$ (if a and b both odd) with at least one on the bound.

Case 2: Suppose t_2 satisfies $n_-(t_2) = 0$. Define

$$q_-^\dagger(t) = \frac{(b^2 - a^2) \sin at}{a^2 + b^2 + 2\kappa ab + (b^2 - a^2) \cos at}, \tag{48}$$

and

$$\theta_-^\dagger(t) = b \arctan q_-^\dagger(t) + a \arctan \kappa q_-^\dagger(t). \tag{49}$$

Then $q_-(t_2) = q_-^\dagger(t_2)$ and $\theta_-(t_2) = \theta_-^\dagger(t_2)$ for all t_2 satisfying $n_-(t_2) = 0$. It follows that $|\theta_-(t_2)| \leq |\bar{\theta}^\dagger|$ for all such t_2 where $\bar{\theta}^\dagger$ is given by (36). This time we only consider $\kappa > 0$ but the previous analysis establishes that these turning points lie within the bounds.

■

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