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# Robust Reference Tracking of Linear Uncertain Systems via Uncertainty Estimation and Composite Control

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***Abstract*— For linear systems with uncertainties and external disturbances, we present an uncertainty estimation and composite control (UECC) to achieve reference tracking and uncertainty estimation simultaneously. In this proposed UECC, we extract the uncertainty information from the error dynamics equation instead of the system dynamics. By reformulating the error dynamics equation as an algebraic equation using auxiliary variables, the need to measure state derivatives in the estimator and controller design is eliminated. Unlike time-delay control (TDC) and uncertainty and disturbance estimator (UDE) methods, we avoid the use of time delay and additional filtering operations to circumvent the noise amplification and oscillations in the control signal. Comparative simulations are provided to verify the effectiveness of the proposed method.**

## I. Introduction

For many practical applications, the mathematical model used in control designs might not be able to precisely describe the actual system behavior due to desired assumptions or imposed linearization. Moreover, systems commonly operate in environments where unpredictable system parameter variations and undesired external disturbances are possible [1]. These uncertainties or unexpected disturbances will affect the stability and control performance of systems. Hence, it is critical to effectively address these uncertainties without relying on an accurate mathematical model, which would significantly reduce the modeling burden while improving the robustness of systems [2], [3].

To address these uncertainties, some advanced control approaches have been developed, such as adaptive control [4] and robust control [5]. However, adaptive control is mainly used for handling linearly parameterized uncertainties, and robust control is derived to address the worst-case control designs at the price of sacrificing the nominal control performance. Recently, the disturbance/uncertainty estimation and attenuation techniques-based control schemes are considered to be another powerful strategy to deal with uncertainties [6], [7], [8]. Unlike the adaptive control that adjusts control gains or identifies system parameters, the disturbance/uncertainty estimation-based methods extract information of uncertainties from the system dynamics and input signals, and then incorporate the extracted information into feedback control to modify the control actions directly. Due to the simple structure, the disturbance/uncertainty estimation-based methods have been widely studied [9] and the references therein.

In the early 1990s, Youcef-Toumi and Ito first proposed a comprehensive control method named time delay control (TDC) that can simultaneously achieve uncertainty compensation and reference tracking [1]. TDC methods use past observation of uncertainties and control inputs to modify control actions directly through a small time delay, which creates a class of control strategies that are computationally inexpensive and easy to implement. Hence, as stated in [10], the algorithm extension of TDC and various applications in different systems are gradually presented. However, TDC methods inherently require all system states and their derivatives to be accessible for feedback. In [11], the authors pointed out that there exist oscillations in the control signal when using TDC methods, and the introduced time delay also brings difficulties into the system analysis. To avoid using the state derivative and get rid of the side effects induced by time delay operation, the uncertainty and disturbance estimator (UDE)-based control was proposed [11], [12], where the time delay operation is replaced by an extra low-pass filter. However, with the development of UDE-based control approaches, some issues still need to be handled. As stated in [13], the additional low-pass filters might lead to windup if the system input is subject to a constraint. Moreover, the time constant in low-pass filters is usually chosen as a small coefficient to guarantee that the bandwidth is wide enough to cover the spectrum of uncertainties. However, this may lead to noise amplification if the measured states contain noise, especially when using a first-order low-pass filter [14].

With the wish to eliminate the side effects induced by time delay or additional filtering operations, a new uncertainty estimation and composite control (UECC) is proposed in this paper, and the asymptotic convergence of reference tracking and uncertainty estimation is achieved simultaneously. In this proposed UECC, we first divide the controller into nominal control, uncertainty compensation, and error feedback term. With the help of nominal control, all unmodeled dynamics and external disturbances can flow to the error dynamics equation. Consequently, we can extract the information of uncertainties from the error dynamics equation, not just from the system dynamics. In this situation, it is more convenient for us to employ existing observers to estimate uncertainties. Furthermore, we reconstruct the error dynamics equation as an algebraic equation instead of a differential equation,

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which allows us not to measure the state derivatives. Based on the reconstructed error dynamics equation, an alternative uncertainty estimator is presented without using time delay or additional filtering operations. Finally, comparative simulation results are provided to verify the effectiveness of the proposed UECC.

*Notation:* $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues of the corresponding matrices. $(\cdot)^T$, $(\cdot)^{-1}$ and $(\cdot)^{+}$ are the transpose, inverse, and generalized inverse of matrices. $\|\cdot\|$ is the 2-norm for vectors and the induced 2-norm for matrices. $I$ denotes the identity matrix with the corresponding dimension. The Laplace transformation of a signal $x(t)$ is denoted by the capital letter $X(s)$, and $s$ is the Laplace operator.

## II. Problem Formulation

The linear systems to be studied are given as

$$\dot{x}(t) = (A + \Delta A)\,x(t) + (B + \Delta B)\,u(t) + d(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $u(t) \in \mathbb{R}^r$ is the control input vector, $A \in \mathbb{R}^{n\times n}$ is the known state matrix, $B \in \mathbb{R}^{n\times r}$ is the control matrix with full column rank $r$, $d(t) \in \mathbb{R}^n$ is the unknown disturbance vector, $\Delta A \in \mathbb{R}^{n\times n}$ and $\Delta B \in \mathbb{R}^{n\times r}$ are unknown system matrices that denote the system parameter variations.

The reference model that generates the desired system behavior is given as

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}} x_{\mathrm{m}}(t) + B_{\mathrm{m}} c(t), \tag{2}$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$ is the reference state vector, $A_{\mathrm{m}} \in \mathbb{R}^{n\times n}$ is the Hurwitz matrix, $B_{\mathrm{m}} \in \mathbb{R}^{n\times r}$ is the reference input matrix, and $c(t) \in \mathbb{R}^r$ is an external bounded command. The error dynamics between (1) and (2) is given as

$$\dot{e} = A_{\mathrm{m}} e + (A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax - Bu - \mathcal{F}_{\mathrm{d}}), \tag{3}$$

where $e(t) = x_{\mathrm{m}}(t) - x(t)$ is the tracking error, and $\mathcal{F}_{\mathrm{d}}(t) = \Delta A x(t) + \Delta B u(t) + d(t)$ is the lumped uncertainty.

If it is possible to determine a controller $u(t)$ to make the term between brackets in (3) be zero at any time, the error $e(t)$ would decay at the rate dictated by $A_{\mathrm{m}}$. However, this decay rate is usually unsatisfied in practice. Therefore, one desired error dynamics with a faster decay rate is written as

$$\dot{e}(t) = (A_{\mathrm{m}} + \mathcal{K})\,e(t), \tag{4}$$

where $\mathcal{K} \in \mathbb{R}^{n\times n}$ is the error feedback matrix such that $A_{\mathrm{m}} + \mathcal{K}$ is Hurwitz. Combining (3) and (4), we have

$$A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax - Bu - \mathcal{F}_{\mathrm{d}} = \mathcal{K} e. \tag{5}$$

Then, the controller $u(t)$ is designed as

$$u(t) = B^{+}\left(A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax - \mathcal{K} e - \hat{\mathcal{F}}_{\mathrm{d}}\right), \tag{6}$$

where $B^{+} = (B^T B)^{-1} B^T$ is the pseudo inverse and can be calculated for $\det(B^T B) \neq 0$, $\hat{\mathcal{F}}_{\mathrm{d}}$ is the estimation of $\mathcal{F}_{\mathrm{d}}$, and $u$ consists of three parts, nominal control $u_{\mathrm{n}} = A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax$, uncertainty compensation $u_{\mathrm{d}} = \hat{\mathcal{F}}_{\mathrm{d}}$, and error feedback term $\mathcal{K} e$. Note that a too large feedback gain $\mathcal{K}$ may lead to oscillations in control signals. Hence, the robustness and the convergence rate should be considered when we set the feedback matrix. Moreover, as reported in [1], [11], the pole placement technology can be employed to choose $\mathcal{K}$.

Substituting (6) into (3), the error dynamics becomes

$$\dot{e} = A_{\mathrm{e}} e - \tilde{\mathcal{F}}_{\mathrm{d}} + \left(I - BB^{+}\right)(A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax - u_{\mathrm{d}} - \mathcal{K} e) \tag{7}$$

where $A_{\mathrm{e}} = A_{\mathrm{m}} + \mathcal{K}$ is the Hurwitz matrix and $\tilde{\mathcal{F}}_{\mathrm{d}} = \mathcal{F}_{\mathrm{d}} - \hat{\mathcal{F}}_{\mathrm{d}}$ is the estimation error. To guarantee that the error (7) vanishes as time goes to infinity, the following structural constraint must be met.

$$\left(I - BB^{+}\right)(A_{\mathrm{m}} x + B_{\mathrm{m}} c - Ax - u_{\mathrm{d}} - \mathcal{K} e) = 0. \tag{8}$$

Obviously, if $B$ is invertible, i.e., $n = r$ and $\operatorname{rank}(B) = n$, the above structural constraint is always met since $I - BB^{+} = I - BB^{-1} = 0$. If not, the choice of $A_{\mathrm{m}}$ and $\mathcal{K}$ would be somewhat restricted due to $\operatorname{rank}(I - BB^{+}) = n - r$. In this situation, as reported in [1], [15], since the canonical form can guarantee that the rest $n - r$ system states are controlled automatically, the system described in canonical form meets such constraint. This is also called the matching condition [14]. Hence, for the system with $n$ states and $r$ inputs, each term in (1) can be partitioned as

$$\begin{aligned} x(t) &= \left[\begin{array}{c} x_{\mathrm{q}}(t) \\ \hline x_{\mathrm{r}}(t) \end{array}\right], A = \left[\begin{array}{c|c} 0 & I_{\mathrm{q}} \\ \hline \multicolumn{2}{c}{A_{\mathrm{r}}} \end{array}\right], B = \left[\begin{array}{c} 0 \\ \hline B_{\mathrm{r}} \end{array}\right], \\ \Delta A &= \left[\begin{array}{c} 0 \\ \hline \Delta A_{\mathrm{r}} \end{array}\right], \Delta B = \left[\begin{array}{c} 0 \\ \hline \Delta B_{\mathrm{r}} \end{array}\right], d(t) = \left[\begin{array}{c} 0 \\ \hline d_r(t) \end{array}\right], \end{aligned} \tag{9}$$

where $x_{\mathrm{q}} \in \mathbb{R}^{n-r}$ and $x_{\mathrm{r}} \in \mathbb{R}^r$ are the system states, $A_{\mathrm{r}} \in \mathbb{R}^{r\times n}$ is the system matrix, $B_{\mathrm{r}} \in \mathbb{R}^{r\times r}$ is a nonsingular matrix, $\Delta A_{\mathrm{r}} \in \mathbb{R}^{r\times n}$ and $\Delta B_{\mathrm{r}} \in \mathbb{R}^{r\times r}$ denote the unmodeled dynamics, and $d_r \in \mathbb{R}^r$ is the unknown disturbance. Then, the lumped uncertainty can be described as $\mathcal{F}_{\mathrm{dr}}(t) = \Delta A_{\mathrm{r}} x(t) + \Delta B_{\mathrm{r}} u(t) + d_{\mathrm{r}}(t)$. The reference model matrices and feedback gain matrix are accordingly given as

$$A_{\mathrm{m}} = \left[\begin{array}{c|c} 0 & I_{\mathrm{q}} \\ \hline \multicolumn{2}{c}{A_{\mathrm{mr}}} \end{array}\right], B_{\mathrm{m}} = \left[\begin{array}{c} 0 \\ \hline B_{\mathrm{mr}} \end{array}\right], \mathcal{K} = \left[\begin{array}{c} 0 \\ \hline \mathcal{K}_{\mathrm{r}} \end{array}\right], \tag{10}$$

where $A_{\mathrm{mr}} \in \mathbb{R}^{r\times n}$ is the reference model matrix, $B_{\mathrm{mr}} \in \mathbb{R}^{r\times r}$ is the nonsingular matrix, and $\mathcal{K}_{\mathrm{r}} \in \mathbb{R}^{r\times n}$ is the feedback gain.

Combining (9) and (10), one can derive

$$\begin{aligned} I - BB^{+} &= I - \left[\begin{array}{c} 0 \\ \hline B_{\mathrm{r}} \end{array}\right] B_{\mathrm{r}}^{-1}(B_{\mathrm{r}}^T)^{-1} \left[\begin{array}{c|c} 0 & B_{\mathrm{r}}^T \end{array}\right] \\ &= \begin{bmatrix} I & \\ & 0 \end{bmatrix}, \end{aligned} \tag{11}$$

and

$$A_m x + B_m c - Ax - u_d - \mathcal{K} e = \left[\begin{array}{c} 0 \\ \hline \varphi \end{array}\right], \tag{12}$$

where $\varphi = A_{\mathrm{mr}} x + B_{\mathrm{mr}} c - A_{\mathrm{r}} x - u_{\mathrm{dr}} - \mathcal{K}_{\mathrm{r}} e$ and $u_{\mathrm{dr}} = \hat{\mathcal{F}}_{\mathrm{dr}}$. Substituting (11) and (12) into (8), we can verify that the plant (1) described in canonical form (9) always meets the structural constraint (8). Therefore, we assume that the remaining contents satisfy the constraint (8). Then, the error dynamics (7) is rewritten as

$$\dot{e}(t) = A_\mathrm{e}e(t) + u_\mathrm{d}(t) - \mathcal{F}_\mathrm{d}(t), \tag{13}$$

which shows the convergence set of $e$ depends on the estimation error $\tilde{\mathcal{F}}_\mathrm{d}$ only. If we can get a good estimation of $\mathcal{F}_\mathrm{d}$ and incorporate it into $u_\mathrm{d}$, a satisfied reference tracking can be guaranteed simultaneously. Therefore, this paper aims to design an uncertainty estimator and achieve satisfied reference tracking even with large parameter variations in $\Delta A$ and $\Delta B$, and the unexpected disturbance $d(t)$. For this purpose, the following assumptions and lemma are used.

**Assumption 1**. *The state variable $x(t)$ and control signal $u(t)$ in (1) are accessible.*

**Assumption 2**. *The external disturbance $d(t)$ and and its time derivative $\dot{d}(t)$ are available.*

**Assumption 3**. *The solution of (1) is supposed to be uniformly bounded.*

**Lemma 1**. *(see Theorem 4.6 in [16]) If matrix $A_\mathrm{e}$ is Hurwitz, then for any given positive definite symmetric matrix $Q$, there exists a positive definite symmetric matrix $P$ that satisfies the following Lyapunov equation*

$$A_e^T P + P A_e = -Q.$$

Assumptions 1 and 2 are quite general in the community of disturbance/uncertainty observer design. Considering the controller (6) with bounded $c(t)$, the lumped uncertainty $\mathcal{F}_\mathrm{d}(t)$ can be viewed as a function of $x(t)$ and $d(t)$. Note that, for practical applications, hard constraints and energy restrictions make the system state continuous and bounded. Additionally, as reported in [17], since the disturbance/uncertainty observer is mainly designed for control purposes, we can always employ feedback control to guarantee the boundedness of the system state. Hence, based on Assumptions 1-3, we assume without loss of generality that there exist unknown constants $\zeta_\mathrm{d}$ and $\zeta$ to meet the inequalities $\|\mathcal{F}_\mathrm{d}(t)\| \le \zeta_\mathrm{d}$ and $\|\dot{\mathcal{F}}_\mathrm{d}(t)\| \le \zeta$.

## III. Uncertainty estimation

As shown in [1], [10]–[13], [15], the uncertainty information is extracted from the system dynamics. Specifically, time delay [1] or extra filtering operation [11] is applied to each term within (1) to estimate the lumped uncertainties. Note that, with the help of nominal control $u_\mathrm{n}$ and error feedback term $\mathcal{K}e$ in (6), the error dynamics equation (13) contains all unmodeled dynamics and external disturbances. Therefore, we could extract the uncertainty information from the error dynamics equation (13), not just from the system dynamics (1).

### A. Related Works

TDC methods [1] assume that all signals remain consistent during a small enough period $\mathcal{L}$. Then, the past observation of uncertainties and control inputs are adopted to update the control actions directly. Hence, by applying a small time delay $\mathcal{L}$ to both sides of (13), $\mathcal{F}_\mathrm{d}$ can be estimated by

$$u_\mathrm{d}(t) = \hat{\mathcal{F}}_\mathrm{d}(t) = A_\mathrm{e}e(t-\mathcal{L}) + u_\mathrm{d}(t-\mathcal{L}) - \dot{e}(t-\mathcal{L}), \tag{14}$$

where $\dot{e}(t-\mathcal{L})$ is the time derivative of $e(t-\mathcal{L})$. Substituting (14) into (6), the control law of TDC is derived as

$$\begin{aligned} u(t) =& B^+[A_\mathrm{m}x + B_\mathrm{m}c - Ax - \mathcal{K}e - A_\mathrm{e}e(t-\mathcal{L}) \\ &+ u_\mathrm{d}(t-\mathcal{L}) - \dot{e}(t-\mathcal{L})]. \end{aligned} \tag{15}$$

From (14) and (15), one can find that TDC inherently requires all states and their derivatives to be accessible for uncertainty estimation and feedback control. In addition, as reported in [11], the introduced time delay operations may excite oscillations in control signals and bring difficulties to the system analysis. To eliminate these side effects induced by time delay, UDE-based control schemes are proposed [11], [12], where a low-pass filter replaces the time delay operation. Furthermore, the whole design of the estimator and controller is carried out in the frequency domain. Hence, the lumped uncertainty $\mathcal{F}_\mathrm{d}$ in (13) is first written in the frequency domain as

$$\mathcal{F}_\mathrm{d}(s) = (A_\mathrm{e} - sI)\,E(s) + U_\mathrm{d}(s). \tag{16}$$

Then, UDE methods apply a low-pass filter to the right hand of (16) to replace the time delay such that we have

$$U_\mathrm{d}(s) = \hat{\mathcal{F}}_\mathrm{d}(s) = [(A_\mathrm{e} - sI)\,E(s) + U_\mathrm{d}(s)]\,\mathcal{G}_f(s), \tag{17}$$

which yields

$$U_\mathrm{d}(s) = \frac{\mathcal{G}_f(s)}{1-\mathcal{G}_f(s)}\,(A_\mathrm{e} - sI)\,E(s), \tag{18}$$

where $\mathcal{G}_f(s)$ is a strictly proper low-pass filter with unity steady-state gain and broad enough bandwidth. Substituting (18) into (6), the control law of UDE in the frequency domain is written as

$$\begin{aligned} U(s) =& B^+[A_\mathrm{m}X(s) + B_\mathrm{m}C(s) - AX(s) \\ &- \mathcal{K}E(s) - \frac{\mathcal{G}_f(s)}{1-\mathcal{G}_f(s)}\,(A_\mathrm{e} - sI)\,E(s)]. \end{aligned} \tag{19}$$

From (18) and (19), since $\mathcal{G}_f(s)$ is strictly proper and $s\mathcal{G}_f(s)$ is physically implementable, the measurement of state derivative is avoidable. In addition, the oscillations in control signals are also avoidable due to the time delays being replaced by additional filtering operations. However, with the wide application of UDE-based control schemes, some issues still need to be resolved. As presented in (19), since $s\mathcal{G}_f(s)$ is commonly chosen as a first-order low-pass filter $\frac{1}{\tau s+1}$ [11], [18], the term $\frac{\mathcal{G}_f(s)}{1-\mathcal{G}_f(s)}$ in (19) becomes $\frac{1}{\tau s}$. Although the resulting integral action is of great importance to achieve satisfactory steady-state reference tracking, it might lead to integral windup if the system input is subject to a constraint [13]. Moreover, the term $\frac{s\mathcal{G}_f(s)}{1-\mathcal{G}_f(s)}$ in (19) would become $\frac{1}{\tau}$. This might lead to noise amplification if the state $x$ contains measurement noise [14], especially when the time constant $\tau$ is chosen as a small parameter.

### B. Proposed Uncertainty Estimator

To address the issues induced by time delay and extra filtering operations, we will introduce another approach to extract the uncertainty information from the error dynamics equation (13) without using time delays and extra filters. Hence, a set of auxiliary variables are first defined as

$$\dot{\omega}_0(t) = A_\mathrm{e}\omega_0(t) + u_\mathrm{d}(t), \tag{20}$$

$$\dot{\omega}_1(t) = A_\mathrm{e}\omega_1(t) - \mathcal{F}_\mathrm{d}(t). \tag{21}$$

Then, the error dynamics equation (13) is rewritten as

$$e(t) = \omega_0(t) + \omega_1(t), \tag{22}$$

where the error dynamics (13) is described as an algebraic equation instead of a differential equation. Furthermore, the information of $u_d(t)$ and $\mathcal{F}_\mathrm{d}(t)$ are implicit in $\omega_0(t)$ and $\omega_1(t)$, respectively. This provides a mapping from the accessible variables $e(t)$ and $u_\mathrm{d}(t)$ to the unknown $\mathcal{F}_\mathrm{d}(t)$. Thus, a feasible uncertainty estimator can be given as

$$u_\mathrm{d}(t) = \hat{\mathcal{F}}_\mathrm{d}(t) = A_\mathrm{e}(e(t) - \omega_0(t)). \tag{23}$$

in which the state derivatives are unnecessary to measure.

### C. Convergence Analysis

**Theorem 1**. *For the controlled system (1) with the proposed uncertainty estimator (23), the estimation error $\tilde{\mathcal{F}}_\mathrm{d}$ would exponentially converge to a compact set around zero defined by*

$$\Omega_{\tilde{\mathcal{F}}_\mathrm{d}} := \left\{ \tilde{\mathcal{F}}_\mathrm{d} \mid \|\tilde{\mathcal{F}}_\mathrm{d}\|^2 \leq \frac{\eta^2\zeta^2\lambda_{\max}(P)}{\eta\lambda_{\min}(Q) - \lambda_{\max}(P)} \right\},$$

where $\lambda_{\min}(Q) > \lambda_{\max}(P)/\eta$ and $\eta > 0$ is a tuning parameter.

**Proof** : Based on (13) and (20), the time derivative of $\hat{\mathcal{F}}_\mathrm{d}$ given in (23) is derived as

$$\dot{\hat{\mathcal{F}}}_\mathrm{d} = A_\mathrm{e}(\dot{e}(t) - \dot{\omega}_0(t)) = A_\mathrm{e}(\hat{\mathcal{F}}_\mathrm{d} - \mathcal{F}_\mathrm{d}), \tag{24}$$

with $\tilde{\mathcal{F}}_\mathrm{d} = \mathcal{F}_\mathrm{d} - \hat{\mathcal{F}}_\mathrm{d}$. Then, the time derivative of $\tilde{\mathcal{F}}_\mathrm{d}$ can be further given as

$$\dot{\tilde{\mathcal{F}}}_\mathrm{d} = \dot{\mathcal{F}}_\mathrm{d} - \dot{\hat{\mathcal{F}}}_\mathrm{d} = \dot{\mathcal{F}}_\mathrm{d} + A_\mathrm{e}\tilde{\mathcal{F}}_\mathrm{d}. \tag{25}$$

Select a Lyapunov function as $V_{\tilde{\mathcal{F}}_\mathrm{d}} = \tilde{\mathcal{F}}_\mathrm{d}^T P \tilde{\mathcal{F}}_\mathrm{d}$, and calculate its derivative along (25) as

$$\begin{aligned} \dot{V}_{\tilde{\mathcal{F}}_\mathrm{d}} =& \left(\dot{\mathcal{F}}_\mathrm{d} + A_\mathrm{e}\tilde{\mathcal{F}}_\mathrm{d}\right)^T P\tilde{\mathcal{F}}_\mathrm{d} + \tilde{\mathcal{F}}_\mathrm{d}^T P \left(\dot{\mathcal{F}}_\mathrm{d} + A_e\tilde{\mathcal{F}}_\mathrm{d}\right) \\ =& \tilde{\mathcal{F}}_\mathrm{d}^T \left(A_\mathrm{e}^T P + P A_\mathrm{e}\right)\tilde{\mathcal{F}}_\mathrm{d} + 2\tilde{\mathcal{F}}_\mathrm{d}^T P \dot{\mathcal{F}}_\mathrm{d} \\ \leq& -\lambda_{min}(Q)\|\tilde{\mathcal{F}}_\mathrm{d}\|^2 + 2\lambda_{max}(P)\|\tilde{\mathcal{F}}_\mathrm{d}\|\|\dot{\mathcal{F}}_\mathrm{d}\| \\ \leq& -\left(\lambda_{min}(Q) - \frac{\lambda_{max}(P)}{\eta}\right)\|\tilde{\mathcal{F}}_\mathrm{d}\|^2 + \eta\zeta^2\lambda_{max}(P) \\ \leq& -\alpha_1 V_{\tilde{\mathcal{F}}_\mathrm{d}} + \beta_1, \end{aligned} \tag{26}$$

where $\alpha_1 = \frac{\eta\lambda_{\min}(Q)-\lambda_{\max}(P)}{\eta\lambda_{\max}(P)}$, $\beta_1 = \eta\zeta^2\lambda_{\max}(P)$. $\eta > 0$ is a tuning parameter, and $\lambda_{\min}(Q) > \lambda_{\max}(P)/\eta$ is available by appropriately choosing matrix $Q$. Then, we continue to solve (26) and have

$$0 \leq V_{\tilde{\mathcal{F}}_\mathrm{d}}(\tilde{\mathcal{F}}_\mathrm{d}) \leq V_{\tilde{\mathcal{F}}_\mathrm{d}}(\tilde{\mathcal{F}}_\mathrm{d}(0))e^{-\alpha_1 t} + \frac{\beta_1}{\alpha_1}(1 - e^{-\alpha_1 t}), \tag{27}$$

where $V_{\tilde{\mathcal{F}}_\mathrm{d}}(\tilde{\mathcal{F}}_\mathrm{d}(0))e^{-\alpha_1 t}$ will converge to zero and $\frac{\beta_1}{\alpha_1}(1 - e^{-\alpha_1 t})$ will converge to $\frac{\beta_1}{\alpha_1} = \frac{\eta^2\zeta^2\lambda_{\max}^2(P)}{\eta\lambda_{\min}(Q)-\lambda_{\max}(p)}$ such that $\lim_{t\to+\infty}\|\tilde{\mathcal{F}}_\mathrm{d}\|^2 \leq \frac{\beta_1}{\alpha_1\lambda_{\max}(P)}$ holds. Moreover, if the derivative of $\mathcal{F}_\mathrm{d}$ is zero, i.e., $\zeta = 0$, the estimation error $\tilde{F}_\mathrm{d}$ would exponentially converge to zero. If $\zeta \neq 0$, the compact set $\Omega_{\tilde{F}_\mathrm{d}}$ could be reduced by appropriately choosing matrices $A_\mathrm{e}$ and $Q$. For example, the parameter $\eta$ is selected as $1/\lambda_{\max}(P)$, and the compact set $\Omega_{\tilde{\mathcal{F}}_\mathrm{d}}$ is rewritten as $\frac{\zeta^2}{\lambda_{\min}(Q)-\lambda_{\max}^2(p)}$. In this situation, according to Lemma 1, we can appropriately choose matrices $A_\mathrm{m}, \mathcal{K}$ in (4) and increase the value of $\lambda_{\min}(Q)$ so as to reduce the compact set $\Omega_{\tilde{\mathcal{F}}_\mathrm{d}}$. This completes the proof. □

## IV. Composite Control

### A. Composite Control Implementation

In the above section, the related works on TDC and UDE-based control methods are first investigated, then an alternative uncertainty estimator (23) is proposed to circumvent these drawbacks induced by time delays or extra low-pass filters (such as noise amplification and oscillations in control signals). Substituting the proposed estimator (23) into (6), we obtain a UECC method that allows for the asymptotic convergence of reference tracking and uncertainty estimation simultaneously. As presented in (6), the nominal control $u_\mathrm{n}$ attempts to eliminate the unexpected known dynamics $Ax$ and inserts the desired dynamics $A_\mathrm{m}x + B_m c$; the error feedback term $\mathcal{K}e$ attempts to adjust the convergence of the error dynamics; the uncertainty estimator $u_\mathrm{d}$ attempts to eliminate the parameter uncertainties $\Delta Ax(t) + \Delta Bu(t)$ and external disturbances $d(t)$. In addition, from (20) and (23), there is a causality issue when using the proposed estimator $\hat{\mathcal{F}}_\mathrm{d}$. To avoid the causality issue, the initial value of $\hat{\mathcal{F}}_\mathrm{d}(0)$ is set to zero since there is no past information for estimation at the first sampling time.

For the proposed UECC with uncertainty estimator (23) and composite controller (6), the implementation procedure is summarized as follows: 1) Measure the current state information $x(t)$; 2) Calculate the tracking error $e(t)$; 3) Compute the uncertainty estimation vector $\hat{\mathcal{F}}_\mathrm{d}(t)$; 4) Calculate the control action $u(t)$; 5) Repeat step 1.

### B. Stability Analysis

**Theorem 2**. *For the system (1) subject to uncertainties and external disturbances, using the uncertainty estimator (23) and composite controller (6), the closed-loop system is stable for any bounded command $c(t)$, and both the tracking error and estimation error are ultimately bounded by $\Omega_V$.*

**Proof** : With the matrix $P$ derived from Lemma 1, a Lyapunov function is selected as

$$V(e, \tilde{\mathcal{F}}_\mathrm{d}) = e^T P e + \tilde{\mathcal{F}}_\mathrm{d}^T P \tilde{\mathcal{F}}_\mathrm{d}. \tag{28}$$

Calculate its derivative along (13) and (25) as

$$\begin{aligned}\dot{V} =& \dot{e}^T P e + e^T P \dot{e} + \dot{\tilde{\mathcal{F}}}_{\mathrm{d}}^T P \tilde{\mathcal{F}}_{\mathrm{d}} - \tilde{\mathcal{F}}_{\mathrm{d}}^T P \dot{\tilde{\mathcal{F}}}_{\mathrm{d}} \\ =& e^T \left(A_{\mathrm{e}}^T P + P A_{\mathrm{e}}\right) e - 2e^T P \tilde{\mathcal{F}}_{\mathrm{d}} \\ & + \tilde{\mathcal{F}}_{\mathrm{d}}^T \left(A_{\mathrm{e}}^T P + P A_{\mathrm{e}}\right) \tilde{\mathcal{F}}_{\mathrm{d}} + 2\tilde{\mathcal{F}}_{\mathrm{d}}^T P \dot{\mathcal{F}}_{\mathrm{d}} \\ \leq& -\lambda_{\min}(Q)\|e\|^2 - \lambda_{\min}(Q)\|\tilde{\mathcal{F}}_{\mathrm{d}}\|^2 + 2\lambda_{\max}(P)\cdot \\ & \|e\|\|\tilde{\mathcal{F}}_{\mathrm{d}}\| + 2\lambda_{\max}(P)\|\tilde{\mathcal{F}}_{\mathrm{d}}\|\|\dot{\mathcal{F}}_{\mathrm{d}}\| \\ \leq& -\left(\lambda_{\min}(Q) - \frac{\lambda_{\max}(P)}{\vartheta_1}\right)\|e\|^2 - \Bigg(\lambda_{\min}(Q) - \\ & \vartheta_1 \lambda_{\max}(P) + \frac{\lambda_{\max}(P)}{\vartheta_2}\Bigg)\|\tilde{\mathcal{F}}_{\mathrm{d}}\|^2 + \vartheta_2 \lambda_{\max}(P)\zeta^2 \\ \leq& -\alpha_2 V(e, \tilde{\mathcal{F}}_{\mathrm{d}}) + \beta_2, \end{aligned} \tag{29}$$

where $\alpha_2 = \min\left\{\frac{\vartheta_1\lambda_{\min}(Q)-\lambda_{\max}(P)}{\vartheta_1\lambda_{\max}(P)}, \frac{\vartheta_2\lambda_{\min}(Q)-(1+\vartheta_1\vartheta_2)\lambda_{\max}(P)}{\vartheta_2\lambda_{\max}(P)}\right\}$, $\beta_2 = \vartheta_2\lambda_{\max}(P)\zeta^2 > 0$, and $\vartheta_1, \vartheta_2$ are the tuning parameters satisfying

$$\vartheta_1 > \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} > 0, \quad \vartheta_2 > \frac{\lambda_{\min}(Q)\lambda_{\max}(P)}{\lambda_{\min}^2(Q) - \lambda_{\max}^2(P)} > 0,$$

which are available by appropriately choosing the matrices $Q$ and $P$. According to the time derivative of $V(e, \tilde{\mathcal{F}}_{\mathrm{d}})$, we have

$$0 \leq V(e, \tilde{\mathcal{F}}_{\mathrm{d}}) \leq V(e(0), \tilde{\mathcal{F}}_{\mathrm{d}}(0))e^{-\alpha_2 t} + \frac{\beta_2}{\alpha_2}(1 - e^{-\alpha_2 t}),$$

which shows that $V(e, \tilde{\mathcal{F}}_{\mathrm{d}})$ will converge to $\frac{\beta_2}{\alpha_2}$ as time goes to infinite. Therefore, both the error dynamics (13) and estimation dynamics (25) are asymptotic stability. As a result, $e$ and $\tilde{\mathcal{F}}_{\mathrm{d}}$ will converge to a compact set around zero denoted by

$$\Omega_V := \left\{e, \tilde{\mathcal{F}}_{\mathrm{d}} \mid \|e\|^2, \|\tilde{\mathcal{F}}_{\mathrm{d}}\|^2 \leq \frac{\beta_2}{\alpha_2 \lambda_{\max}(P)}\right\}.$$

From the derived compact set $\Omega_V$, we can find that both $e$ and $\tilde{\mathcal{F}}_{\mathrm{d}}$ will exponentially converge to zero if the lumped uncertainty $\mathcal{F}_{\mathrm{d}}$ is time-invariant, i.e., $\zeta = 0$. If $\zeta \neq 0$, $\Omega_V$ can be further reduced by appropriately choosing matrices $A_{\mathrm{e}}$ and $Q$. For example, $\vartheta_1, \vartheta_2$ are selected to satisfy $\vartheta_1 = \vartheta_2 = \frac{1}{\lambda_{\max}(P)}$ such that $\Omega_V$ can be rewritten as $\frac{\zeta^2}{\lambda_{\min}(Q)-(1+\lambda_{\max}^2(P))}$. In this situation, we can increase the value of $\lambda_{\min}(Q)$ through the appropriate design of matrices $A_{\mathrm{m}}$ and $\mathcal{K}$ to reduce the compact set $\Omega_V$ in terms of $\zeta \neq 0$.

Moreover, for the reference model (2), $A_{\mathrm{m}}$ is a Hurwitz matrix, and the given external command $c(t)$ is bounded, then the reference state $x_{\mathrm{m}}$ is bounded. Since the tracking error $e$ is bounded, the system state $x$ is also bounded. For the composite controller (6), the bounded variables $c$, $x$, $e$ and $\hat{\mathcal{F}}_{\mathrm{d}}$ guarantee that the control signal is bounded. Hence, we can conclude that all signals in the related closed-loop system are bounded for any bounded time-varying external command. This completes the proof. □

## V. NUMERICAL EXAMPLE

In this section, a benchmark example of a nonlinear mass spring damper system is used to verify the effectiveness of the proposed UECC, which is given as

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} =& \begin{bmatrix}0 & 1 \\ -0.5 - \Delta a_1 & -1.2 - \Delta a_2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \\ & + \begin{bmatrix}0 \\ 1 + \Delta b\end{bmatrix} u - \begin{bmatrix}0 \\ f(x,t)\end{bmatrix} + \begin{bmatrix}0 \\ d(t)\end{bmatrix},\end{aligned} \tag{30}$$

where $x_1$ and $x_2$ are the displacement and velocity of mass point, respectively. In this case, the lumped uncertainty is denoted as $\mathcal{F}_{\mathrm{d}} = [0, d_{\mathrm{L}}]^T$ and $d_{\mathrm{L}} = -\Delta a_1 x_1 - \Delta a_2 x_2 + \Delta b u - f(x,t) + d(t)$. For the purpose of verification, we assume that $\Delta a_1 = 0.2, \Delta a_2 = 0.5, \Delta b = 0.5, f(x,t) = 1.5x_1^3$. The external disturbance $d(t)$ is chosen as a compound signal $0.1sin(0.2\pi t) + 1(t - 15)$.

The reference model is chosen as

$$\begin{bmatrix}\dot{x}_{\mathrm{m1}} \\ \dot{x}_{\mathrm{m2}}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\omega_n^2 & -2\xi\omega_n\end{bmatrix}\begin{bmatrix}x_{\mathrm{m1}} \\ x_{\mathrm{m2}}\end{bmatrix} + \begin{bmatrix}0 \\ \omega_n^2\end{bmatrix} c(t), \tag{31}$$

where $\omega_{\mathrm{n}} = 5rad/s$, $\xi = 1$ and $c(t)$ is set as a sinusoidal wave with amplitude $1rad$ and frequency $\pi/6\ rad/s$. Considering the controlled system (30) and reference model (31), we can easily check the structural constraint (8) is satisfied. For comparison with TDC and UDE, $\mathcal{L} = 0.005s, \mathcal{K} = 0$ are the same as those in [1], $\tau = 0.005s, \mathcal{K} = 0$ are the same as those in [11], and $\mathcal{K}_{\mathrm{r}}$ used in UECC (10) is set as $[-50, -50]$. The sampling rate is set as $0.001s$.

The simulation results of reference tracking without noises are presented in Fig. 1. One can find that all methods can achieve satisfactory reference tracking and uncertainty estimation. However, it should be noted that all system states and their derivatives are assumed to be accessible when using TDC, while there is no need for $\dot{x}$ when using UDE and UECC. To further test the robustness of these methods under measurement noises, the system states $x_1$ and $x_2$ are simultaneously imposed by white noises with a power of $0.001$ (see Fig. 2(a)). The corresponding simulation results are presented in Fig. 2. From Fig. 2, although all methods can maintain a good tracking performance, there are oscillations of different magnitudes in control signals and uncertainty estimations. Since the noise would be amplified $1/\tau$ times when adopting a first-order low-pass filter in UDE, the magnitude of oscillations is larger than that of UECC, as shown in Fig. 2(b) and 2(c). Moreover, the mean absolute error (MAE), the standard deviation (SD), and the integral square error (ISE) are used to evaluate these simulation results quantitatively. The performance indices of simulation results are presented in Table I, in which all performance indices of UECC are slightly smaller than that of TDC and UDE even in the presence of noises.

## VI. CONCLUSION

In this paper, we propose a UECC scheme to estimate the lumped uncertainty and achieve reference tracking simultaneously. Instead of directly acquiring the uncertainty information from the system dynamics, we extract it from

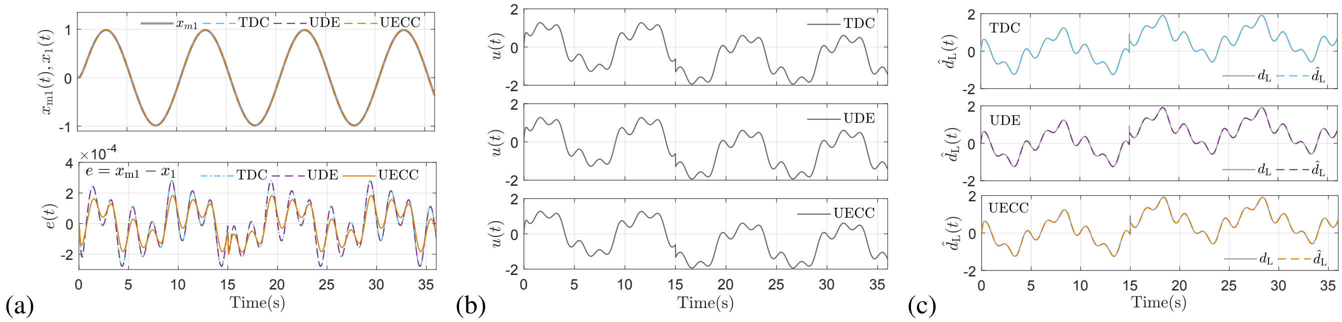

Fig. 1. Simulation results of reference tracking without noises. (a) Tracking responses. (b) Control signals. (c) Estimated values of the lumped uncertainty.

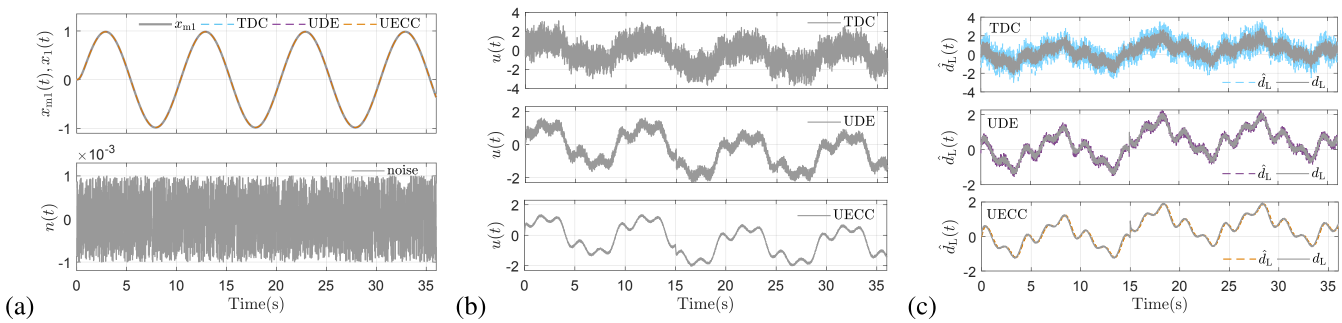

Fig. 2. Simulation results of reference tracking with noises. (a) Tracking responses. (b) Control signals. (c) Estimated values of the lumped uncertainty.

TABLE I

PERFORMANCE INDICES OF SIMULATION RESULTS

| | | **TDC** | **UDE** | **UECC** |
|---|---|---|---|---|
| Without noise | MAE | 1.1631 | 1.1165 | 0.9214 |
| ($\times 10^{-4}$) | SD | 1.8393 | 1.7622 | 1.4192 |
| | ISE | 1.4220 | 1.3081 | 0.8043 |
| With noise | MAE | 5.1315 | 5.0218 | 4.9618 |
| ($\times 10^{-4}$) | SD | 8.0674 | 7.6989 | 7.6053 |
| | ISE | 135 | 123 | 115 |

the error dynamics equation without using time delay or extra filtering operations. Since the error dynamics equation is reconstructed as an algebraic equation through auxiliary variables, the state derivatives are unnecessary to measure. Compared with TDC and UDE, the noise amplification and oscillations in the control signal are avoidable. Simulation results show that the proposed UECC can simultaneously achieve good reference tracking and uncertainty estimation, even with external disturbances or large parameter variations in the system dynamics. Due to the simple structure and intuitiveness, the proposed UECC is easily combined with other advanced control approaches to further enhance the performance of linear uncertain systems.

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