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Bipartite Formation over Undirected Signed Networks with Collision Avoidance

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Abstract— **We address the problem of bipartite formation control, with collision avoidance, for double integrators with limited sensing ranges. We assume that the systems are interconnected over an undirected, signed, and structurally balanced network. Then, to ensure that the proximity constraints are satisfied, we design a barrier-Lyapunov-function-based control law that guarantees connectivity maintenance for cooperative agents, and inter-agent collision avoidance for all agents. Relying on the edge-based agreement, we establish asymptotic stability of the bipartite formation control for signed networks. Finally, we illustrate our theoretical results via numerical simulations.**

I. Introduction

Despite the abundant literature on consensus and synchronization of multi-agent systems [1], most of the available works on distributed control pertain to the case of cooperative agents. However, there are scenarios in which not all the agents cooperate, but some compete. This is the case, *e.g.*, in Robotics, in the context of herding control [2], [3]. In these *coopetition networks* [4], competitive interactions are represented by negative weights on the edges and cooperative ones by positive weights. In view of this, the classical consensus goal (stabilizing over a common equilibrium point) is unachievable. Instead, the general attainable goal for this kind of networked systems is *multi-partite consensus*, also called *fragmentation* [4]. More precisely, for structurally balanced undirected networks, the achievable goal is *bipartite consensus* [5], in which all agents converge to the same state in modulus but with opposite signs. There are various studies on bipartite consensus control, *e.g.*, for single or double-integrators [6] and linear high-order dynamics [7].

In addition to the convergence towards a desired position/trajectory, autonomous vehicles' secondary objective is to guarantee inter-agent collision avoidance, while maintaining the information exchange amongst cooperative agents. These objectives are encoded as inter-agent constraints and are often handled by artificial potential (navigation or barrier) functions—see *e.g.*, [8], [9] and [10]. A barrier Lyapunov function [10] is a type of Lyapunov function that is used to ensure that the multi-agent system satisfies certain safety constraints and to make sure that the system does not enter a particular unsafe region. Yet, few works in the literature focus on ensuring that the inter-agent constraints of a system are respected in a scenario where agents may be in competition. The connectivity-constrained multi-swarm herding is solved in [2] using a mixed integer quadratically constrained program, and non-cooperative herding is achieved for single-integrator dynamics in [3] while preventing some agents from escaping from a protected zone using control barrier functions. Yet, the control law in [2], [3] is based on optimization methods, and the system is modeled by a traditional cooperative network. Moreover, in [3], only a two-agents case is considered. In [11], collision avoidance and connectivity maintenance for bipartite flocking is achieved with artificial potential functions. However, in [11], a minimal safety distance between agents is not guaranteed.

In this paper, we study the bipartite formation problem for structurally balanced undirected signed networks of second-order systems under relative distance constraints. We propose a bipartite formation controller that prevents the vehicles both from colliding and separating beyond sensing ranges. Our control design and analysis rely on the edge-based formulation for signed networks [12], which allows us to recast the problem into one of stabilization of the origin in error coordinates. On the other hand, in contrast to all the references mentioned previously, we use barrier Lyapunov functions, and we base our control law on the gradient of a barrier Lyapunov function for signed networks.

Thus, relative to the existing literature, we contribute with a control law that ensures the bipartite formation for structurally balanced signed graphs with guaranteed connectivity maintenance for cooperative agents and inter-agent collision avoidance by respecting a minimal safety distance between any two agents. Our results apply to double integrators, and we establish asymptotic stability of the consensus manifold using Lyapunov's direct method based on the edge-representation. To the best of our knowledge, this has never been done for signed graphs.

The remainder of this paper is organized as follows. Section II presents the model and problem statement adopting the edge-based representation for signed networks. The main results are presented in Section III. In Section IV, we illustrate our numerical simulations and conclude the paper with some closing remarks in Section V.

II. Model and Problem Formulation

A. Coopetition Networks

Consider a group of N dynamical systems modeled by

$$\dot{\bar{x}}_i = v_i, \quad \bar{x}_i,\ v_i \in \mathbb{R} \tag{1a}$$

$$\dot{v}_i = u_i, \quad u_i \in \mathbb{R}, \quad i \leq N, \tag{1b}$$

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with $\bar{x}_i = x_i - d_i$, where $x_i \in \mathbb{R}$ is the position of the ith agent with respect to a global frame[1], and d_i is the relative displacement of the ith agent from the center of a formation.

A commonly pursued initial problem in the literature, whose solution can serve as a basis to tackle more complex missions for such systems, is to gather in formation around a non-specified consensual set-point. In this paper we focus our attention on the case in which neighboring agents may be *cooperative* or *competitive*. More precisely, we assume that the agents interact over an undirected signed graph $\mathcal{G}$ and the set of all vertices is $\mathcal{V} := \{\nu_1, \nu_2, \ldots, \nu_N\}$. Then, the interaction of cooperative agents is expressed by $a_{ij} > 0$ and, if it is competitive, by $a_{ij} < 0$. In this case, formation consensus is impossible, but the systems may achieve *bipartite consensus* [5]. That is, either $\bar{x}_i \to x_c$ or $\bar{x}_i \to -x_c$ and $v_i \to v_c$ or $v_i \to -v_c$ or, more precisely,

$$\lim_{t\to\infty} [\bar{x}_i - \mathrm{sgn}(a_{ij})\bar{x}_j] = 0 \tag{2a}$$

$$\lim_{t\to\infty} [v_i - \mathrm{sgn}(a_{ij})v_j] = 0 \quad \forall i, j \leq N. \tag{2b}$$

In the absence of constraints, bipartite consensus for (1) is achieved via the distributed control law

$$\begin{aligned} u_i = &- k_1 \sum_{j=1}^{N} |a_{ij}| [\bar{x}_i - \mathrm{sgn}(a_{ij})\bar{x}_j] \\ &- k_2 \sum_{j=1}^{N} |a_{ij}| [v_i - \mathrm{sgn}(a_{ij})v_j], \end{aligned} \tag{3}$$

where $k_1, k_2 > 0$ if and only if $\mathcal{G}$ contains a spanning tree and is structurally balanced [6]. A signed graph is *structurally balanced* if it may be split into two disjoint sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that for every $i, j \in \mathcal{V}_p, p \in \{1, 2\}$, if $a_{ij} \geq 0$, while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, with $p, q \in \{1, 2\}, p \neq q$, if $a_{ij} \leq 0$. It is *structurally unbalanced*, otherwise [5].

In [6] it is established that the distributed control law in (3) guarantees that the synchronization error $\bar{e}_k := [\bar{e}_{x_k} \ \bar{e}_{v_k}]^\top$, defined as

$$\bar{e}_{x_k} := \bar{x}_i - \mathrm{sgn}(a_{ij})\bar{x}_j, \quad \bar{e}_{v_k} := v_i - \mathrm{sgn}(a_{ij})v_j, \tag{4}$$

where $k \leq M$ denotes the index of the interconnection between the ith and jth agents, converges to zero, thereby making two sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ converge to the same state in module but opposite in signs for a structurally balanced network. However, in [6], neither proximity constraints nor collision avoidance are considered.

In realistic application scenarios, however, agents have limited communication range and must keep a minimum distance from each other to prevent inter-agent collisions. Yet, under these constraints, the control law in (3) does not guarantee bipartite consensus.

Mathematically, the agent i staying in the sensing zone of agent j, where $i, j \in \mathcal{V}_l, \ l = \{1, 2\}$, is expressed as $|x_i - x_j| < R_k$ for all $i, j \in \mathcal{V}_l$, and keeping a minimum distance to avoid collisions between agents means that $|x_i - x_j| > \Delta_k$, where Δ_k denotes the minimum distance to keep between agents i and j, so that there are no collisions. For the multi-agent system, the latter is defined as a set $\mathcal{I}$ consisting of two inter-agent constraint sets. For each pair of nodes communicating with each other, let $\delta_k := x_i - x_j$, with $k \leq M$.The first constraint set pertains to the connectivity constraints among cooperative agents and is defined as

$$\mathcal{I}_r := \{\delta_k \in \mathbb{R} : |\delta_k| < R_k, \ \forall i, j \in \mathcal{V}_l, \ l = \{1, 2\}\}. \tag{5}$$

The second one is defined by the collision-avoidance constraints, *i.e.*,

$$\mathcal{I}_c := \{\delta_k \in \mathbb{R} : \Delta_k < |\delta_k|, \ \forall k \leq M\}. \tag{6}$$

Remark 1: Connectivity constraints for competitive agents are not imposed since they are assumed to have different objectives and, therefore, do not need to stay close to each other. •

Then, for a pair of cooperative agents, the synchronization errors defined in (4) take the form

$$\bar{e}_{x_k} = \bar{x}_i - \bar{x}_j = \delta_k - \bar{d}_k, \quad i, j \in \mathcal{V}_l, \ l = \{1, 2\}, \tag{7}$$

while, for two competitive agents x_i and x_j, the error becomes

$$\bar{e}_{x_k} = \bar{x}_i + \bar{x}_j = \delta_k - \bar{d}_k + 2x_j, \quad i \in \mathcal{V}_1, \ j \in \mathcal{V}_2, \tag{8}$$

with $\bar{d}_k = d_i - \mathrm{sgn}(a_{ij})d_j$.

Thus, the bipartite-consensus objective for (1) is equivalent to making $x_i - \mathrm{sgn}(a_{ij})x_j \to d_i - \mathrm{sgn}(a_{ij})d_j$, or equivalently $\bar{e}_{x_k} \to 0$ and $v_i \to 0$. In that light, we remark that the synchronization errors correspond to edge states [13], which are particularly well-suited to multi-agent systems under state constraints, such as (5) and (6). Hence, in this paper, we study the bipartite consensus problem with constraints in edge-based representation for signed networks [12]. This representation has the advantage of recasting the consensus problem into that of stability of the origin in error coordinates $\bar{e}$. Thus, before presenting our main results, we recall some facts about edge-based signed graphs, according to [5] and [12], and we present some new statements that are useful to establish our main results.

B. The edge-based formulation for signed networks

The elements of the Laplacian of a signed graph, L_s, are

$$\ell_{s_{ij}} = \begin{cases} \sum_{h \leq N} |a_{ih}| & i = j \\ -a_{ij} & i \neq j. \end{cases} \tag{9}$$

The following definition introduces the incidence matrix of a structurally balanced signed graph.

Definition 1: Consider a structurally balanced undirected signed network $\mathcal{G}$ that contains N nodes and M edges. The incidence matrix $E_s \in \mathbb{R}^{N \times M}$ of $\mathcal{G}$ is defined as

$$[E_s]_{ik} := \begin{cases} +1, & \text{if } v_i \text{ is the initial node of the edge } e_k; \\ -1, & \text{if } v_i, v_j \in \mathcal{V}_l, l = \{1,2\} \text{ and } v_i \text{ is the terminal node of the edge } e_k; \\ +1, & \text{if } v_i \in \mathcal{V}_1, v_j \in \mathcal{V}_2 \text{ and } v_i \text{ is the terminal node of the edge } e_k; \\ 0, & \text{otherwise,} \end{cases}$$

[1]Application to $x \in \mathbb{R}^n$ is immediate with the Kronecker product.

where $e_k = v_i v_j,\ k \leq M,\ i, j \leq N$ are arbitrarily oriented edges and $\mathcal{V}_1$ and $\mathcal{V}_2$ are the two disjoint sets of vertices. •

Using the incidence matrix, we may express the synchronization errors in (4) in the vector forms

$$\bar{e}_x = E_s^\top \bar{x}, \quad \bar{e}_v = E_s^\top v. \tag{10}$$

As established next, the incidence matrix is also useful to factorize the node and edge Laplacians.

Claim 1: The Laplacian matrix L_s and the edge Laplacian matrix L_{e_s} of a structurally-balanced graph $\mathcal{G}$ satisfy

$$L_s = E_s E_s^\top, \quad L_{e_s} = E_s^\top E_s. \tag{11}$$

Proof of Claim 1: As the signed graph is structurally balanced, we apply the *gauge transformation* on L_s, which consists of pre- and post-multiplying L_s by the matrix $D = \text{diag}(\sigma)$, where $\sigma = [\sigma_1 \ \dots \ \sigma_N]$, with $\sigma_i = \{\pm 1\}$, $i \in \mathcal{I}_N$ to obtain the Laplacian matrix L of an unsigned network. Then again, we apply the *edge* gauge transformation on the incidence matrix E_s of the signed network, using the matrix $D = \text{diag}(\sigma)$ and $D_e = \text{diag}(\sigma_e)$, where $\sigma_e = [\sigma_{e_1} \ \cdots \ \sigma_{eM}]$, with $\sigma_{e_i} = 1$ if $v_i \in \mathcal{V}_1$ and $\sigma_{e_i} = -1$ if $v_i \in \mathcal{V}_2$ with v_i being the initial node of the edge [12, Lemma 4], to obtain the incidence matrix E of an unsigned network.

Now, from the definition of the Laplacian matrix $L = EE^\top$ and the edge Laplacian matrix $L_e = E^\top E$ of an unsigned network, and by applying the edge gauge transformation on the incidence matrix E of an unsigned network from [12, Lemma 4], we have $L = DE_sD_e(DE_sD_e)^\top = DE_sE_s^\top D$ and $L_e = (DE_sD_e)^\top DE_sD_e = D_eE_s^\top E_sD_e$ as $D_eD_e^\top = I_{M\times M}$ and $D^\top D = I_{N\times N}$. Then, the definitions of L_s and L_{e_s} in (11) follow. •

Another utility of the incidence matrix, as defined in Definition 1, is that for structurally balanced signed graphs containing a spanning tree (sufficient and necessary condition for bipartite consensus), it allows to distinguish the state-variables related to an underlying-tree graph $\mathcal{G}_t$, from the rest of states, corresponding to the graph $\mathcal{G}_c := \mathcal{G} \backslash \mathcal{G}_t$. Then, the consensus problem may be addressed as that of the stabilization of the origin for a reduced-order system. To better see this, let the incidence matrix E_s be partitioned as

$$E_s = [E_{t_s} \quad E_{c_s}], \tag{12}$$

where $E_{t_s} \in \mathbb{R}^{N\times N-1}$ is the incidence matrix representing the edges of the spanning tree of and $E_{c_s} \in \mathbb{R}^{N\times M-(N-1)}$ is the incidence matrix representing the remaining edges. In the Proposition 1 below we define E_s in terms of E_{t_s}. Then, after (10) and (12) we have

$$\bar{e}_x = [(E_{t_s}^\top \bar{x})^\top \ (E_{c_s}^\top \bar{x})^\top]^\top, \ \bar{e}_v = [(E_{t_s}^\top v)^\top \ (E_{c_s}^\top v)^\top]^\top,$$

so we may define $\bar{e}_x =: [\bar{e}_{x_t}^\top \quad \bar{e}_{x_c}^\top]^\top$ and $\bar{e}_v =: [\bar{e}_{v_t}^\top \quad \bar{e}_{v_c}^\top]^\top$. The indices $_t$ and $_c$ refer, respectively, to states of the graphs $\mathcal{G}_t$ and $\mathcal{G}_c$. The Proposition 1 also establishes a relation between the consensus errors $\bar{e}_x$ and the spanning-tree errors $\bar{e}_{x_t}$, from which it follows that the objective (2) is attained if $\bar{e}_{x_t} \to 0$ and $\bar{e}_{v_t} \to 0$.

Proposition 1: For a structurally balanced signed graph, there exists a matrix R_s such that

$$E_s = E_{t_s} R_s, \tag{13}$$

where

$$R_s := [I_{N-1} \quad T_s], \quad T_s := (E_{t_s}^\top E_{t_s})^{-1} E_{t_s}^\top E_{c_s}. \tag{14}$$

□

Proof: Applying the edge-gauge transformation and using the partition of E_s as in (12) we express the incidence matrix of an unsigned graph E as

$$E = D[E_{t_s} \quad E_{c_s}]D_e = [DE_{t_s}D_{e_t} \quad DE_{c_s}D_{e_c}], \tag{15}$$

where $D_e = \text{diag}([\sigma_{e_t},\ \sigma_{e_c}])$ with $\sigma_{e_{t_j}} = \{\pm 1\}$ and $\sigma_{e_{cl}} = \{\pm 1\}$ for $j < N-1$ and $l \leq M - N + 1$, $D_{e_t} = \text{diag}(\sigma_{e_t})$ and $D_{e_c} = \text{diag}(\sigma_{e_c})$ are the parts of the edge-gauge transformation matrix corresponding to the spanning tree and the remaining edges, respectively. For an unsigned network, the columns of the incidence matrix representing the remaining edges, E_c, are linearly dependent on the columns of the incidence matrix representing the spanning tree, E_t, and this can be expressed using a matrix T by $E_c = E_tT$ [13, Theorem 4.3]. So, replacing E_c and E_t by the expressions in (15) and left multiplying it by $(DE_{t_s}D_{e_t})^\top$, we obtain

$$T = D_{e_t}(E_{t_s}^\top E_{t_s})^{-1}E_{t_s}^\top E_{c_s}D_{e_c}.$$

Then, we define $T_s := D_{e_t}TD_{e_c}$ and $R_s := [I_{N-1} \quad T_s]$, and the statement in (13) follows. ■

After Proposition 1, we have

$$\bar{e}_x = (E_{t_s}R_s)^\top \bar{x} = R_s^\top \bar{e}_{x_t}, \tag{16a}$$
$$\bar{e}_v = (E_{t_s}R_s)^\top v = R_s^\top \bar{e}_{v_t}, \tag{16b}$$

so the bipartite consensus objective is attained if $\bar{e}_{x_t} \to 0$ and $\bar{e}_{v_t} \to 0$, as stated above. We show how next.

III. Main Results

A. Barrier-Lyapunov-Function-based controller

A *barrier Lyapunov function* (BLF) [10], [13], [14] is defined as follows.

Definition 2: Consider the system $\dot{x} = f(x)$ and let $\mathcal{I}$ be an open set containing the origin. A BLF is a positive definite function $W : \mathcal{I} \to \mathbb{R}_{\geq 0},\ x \mapsto W(x)$, that is $\mathcal{C}^1$, satisfies $\dot{W}(x) \leq 0$, and has the property that $W(x) \to \infty,\ \nabla W(x) \to \infty$ as $x \to \partial\mathcal{I}$. •

Now, from (7) and (8), the constraints in (5) and (6) in terms of the errors in (4) are given by the sets

$$\mathcal{I}_r := \{\bar{e}_{x_k} \in \mathbb{R} : |\bar{e}_{x_k} + \alpha_k| < R_k,\ \forall k \in \mathcal{E}_m\}, \tag{17a}$$
$$\mathcal{I}_c := \{\bar{e}_{x_k} \in \mathbb{R} : \Delta_k < |\bar{e}_{x_k} + \alpha_k|,\ \forall k \leq M\}, \tag{17b}$$

where $\mathcal{I} = \mathcal{I}_r \cup \mathcal{I}_c$ and $\mathcal{E}_m$ consists of the index of m cooperative edges, such that $0 \leq m \leq M$, which are the edges with strictly positive weights, and α_k is defined as

$$\alpha_k := \begin{cases} \bar{d}_k & \text{if} \quad i, j \in \mathcal{V}_l,\ l = \{1, 2\} \\ \delta_k - \bar{e}_{x_k} & \text{if} \quad i \in \mathcal{V}_1,\ j \in \mathcal{V}_2. \end{cases} \tag{18}$$

Then, for each $k \leq M$, we define a BLF $W_k : \mathbb{R} \to \mathbb{R}_{\geq 0}$, as

$$W_k(\bar{e}_{x_k}) = \frac{1}{2}[|\bar{e}_{x_k}|^2 + B_k(\bar{e}_{x_k})], \quad (19)$$

where $B_k(\bar{e}_{x_k})$ is the sum of two functions satisfying Definition 2, each of them encoding the constraints in (17), respectively, *i.e.*,

$$B_k(\bar{e}_{x_k}) = \frac{1}{2}(1+\sigma_k)B_{r_k}(\bar{e}_{x_k}) + B_{c_k}(\bar{e}_{x_k}). \quad (20)$$

In (20) $\sigma_k = 1$ if $k \in \mathcal{E}_m$ if the edge is cooperative, and $\sigma_k = -1$ otherwise. Moreover, $B_k(\bar{e}_{x_k})$ is non-negative and satisfies $B_k(0) = 0$ and $B_k(\bar{e}_{x_k}) \to \infty$ as $|\bar{e}_{x_k}| \to \Delta_k$ for all $k \leq M$ and as $|\bar{e}_{x_k}| \to R_k$ for $k \in \mathcal{E}_m$. However, in view of the constraints defined in (17b), the barrier function has to be modified so that the solution lie in the interior of the constraint sets in (17) and that the convergence to the desired point be ensured. For this, we use the concept of the *gradient recentered barrier function* [15]. Let $\widehat{W}_k : (\Delta_k, \infty) \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ for competitive and $\widehat{W}_k : (\Delta_k, R_k) \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ for cooperative agents defined as

$$\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) := W_k(\bar{e}_{x_k} + \alpha_k) - W_k(\alpha_k) - \frac{\partial W_k}{\partial s}(\alpha_k)\bar{e}_{x_k} \quad (21)$$

which satisfies $\widehat{W}_k(\alpha_k, 0) = 0$, $\nabla_{\bar{e}_{x_k}}\widehat{W}_k(\alpha_k, 0) = 0$, where $\nabla_{\bar{e}_{x_k}}\widehat{W}_k = \frac{\partial \widehat{W}_k}{\partial \bar{e}_{x_k}}$, and $\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \to \infty$ as $|\delta_k| \to \Delta_k$ for $k \in \mathcal{I}_M$, and as $|\delta_k| \to R_k$ for all $\sigma_k = 1$. Moreover, $\widehat{W}_k(\alpha_k, \bar{e}_{x_k})$ satisfies $\frac{\kappa_1}{2}\bar{e}_{x_k}^2 \leq \widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \leq \kappa_2[\nabla_{\bar{e}_{x_k}}\widehat{W}_k]^2$.

Now we introduce the BLF-gradient-based bipartite-consensus control law for the system (1), given by

$$u_i := -k_1 \sum_{k=1}^{M} [E_s]_{ik} \nabla_{\bar{e}_{x_k}} \widehat{W}_k - k_2 \sum_{k=1}^{M} [E_s]_{ik} \bar{e}_{v_k} - k_3 v_i - k_1 \sum_{k=1}^{M} [\mathbb{B}]_{ik} \nabla_{\alpha_k} \widehat{W}_k, \quad k_3 \geq 0 \quad (22)$$

which, upon defining $\bar{W}(\alpha, \bar{e}_x) = \sum_{k=1}^{M} \widehat{W}_k(\alpha_k, \bar{e}_{x_k})$ and

$$\mathbb{B} = (1+\beta)(\beta I + E_{t_s} L_{e_{ts}}^{-1} E_{t_s}^\top)^{-1}(E - E_s), \quad \beta > 0, \quad (23)$$

we write in the vector form as

$$u = -k_1 E_s \nabla_{\bar{e}_x} \bar{W} - k_2 E_s \bar{e}_v - k_3 v - k_1 \mathbb{B} \nabla_\alpha \bar{W}. \quad (24)$$

Remark 2: In [14], the author proposes a consensus control law for multi-agent systems under proximity constraints but in a simpler scenario where all the agents are cooperative. In [14], the control law corresponds to

$$u_i' = -k_1 \sum_{k=1}^{M} [E]_{ik} \nabla_{\bar{e}_{x_k}} W_k - k_3 v_i, \quad (25)$$

which is a particular case of u_i in (22).

It is also important to stress that the last term on the right-hand side of (22)—see also (23), is added after a Lyapunov control redesign, *i.e.*, to render negative semidefinite the derivative of a Lyapunov function for the closed-loop system—see the proof of Proposition 2. Unfortunately, the definition of $\mathbb{B}$ in (23) requires global knowledge of the topology. •

B. Asymptotic Stability

In accordance with Section II, we analyze the stability of the system (1) in closed loop with the bipartite formation control law (24), expressed in terms of the errors corresponding to the spanning-tree subgraph, $\bar{e}_t$. To that end, we introduce the function $\tilde{W}$ as $\tilde{W}(\alpha, \bar{e}_{x_t}) = \bar{W}(\alpha, R_s^\top \bar{e}_{x_t})$, where $\bar{W}$ is defined above (24). Now, after (16a),

$$\nabla_{\bar{e}_{x_t}} \tilde{W} = \frac{\partial \tilde{W}(\alpha, \bar{e}_{x_t})}{\partial \bar{e}_x}^\top \frac{\partial \bar{e}_x}{\partial \bar{e}_{x_t}} = \nabla_{\bar{e}_x} \bar{W}^\top R_s^\top \quad (26)$$

and, after (13) and (26), the control law in (24) becomes

$$u = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{B} \nabla_\alpha \tilde{W}. \quad (27)$$

Then, differentiating on both sides of (10) and using $\dot{v} = u$ and (27) we obtain, in spanning-tree coordinates

$$\dot{\bar{e}}_{x_t} = \bar{e}_{v_t} \quad (28a)$$

$$\dot{\bar{e}}_{v_t} = -k_1 L_{e_{ts}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{ts}} R_s R_s^\top \bar{e}_{v_t} - k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{B} \nabla_\alpha \tilde{W} \quad (28b)$$

Remark 3: $E_{t_s}^\top E_{t_s} = L_{e_{ts}}$ corresponds to the edge Laplacian of a spanning tree and has $N-1$ edges. With the edge-gauge transformation, we obtain the edge Laplacian matrix corresponding to a spanning tree of an unsigned network, *i.e.*, $L_{e_t} = D_e L_{e_{ts}} D_e$. Also, we have $D_e = D_e^\top = D_e^{-1}$ and $D_e L_{e_{ts}} D_e$ is a similarity transformation, so the spectra of L_{e_t} and $L_{e_{ts}}$ coincide. Consequently, as L_{e_t} has the $N-1$ non-zero eigenvalues of the Laplacian matrix L, so does $L_{e_{ts}}$. •

Proposition 2: Consider the system (1) in closed loop with the control law (27), under the conditions of Proposition 1 and assume that the resulting network contains an underlying spanning tree. Let $L_{e_{ts}}$ denote the edge Laplacian corresponding to that spanning-tree. Then, the origin for the closed-loop system is asymptotically stable for any initial conditions respecting the constraints in (17), and $|\alpha_k(0)| > \Delta_k$ for all $k \in \mathcal{I}_M$. In addition, the constraints hold for all t. Furthermore, if $k_3 > 0$, $v \to 0$. □

Remark 4: The condition on $\alpha_k(0)$ is not restrictive. For cooperative agents, $\alpha_k = \bar{d}_k$, so $|\alpha_k(0)| > \Delta_k$ means that the formation must respect the collision-avoidance constraints (the formation must be "safe"). For competitive agents, $|\alpha_k(0)| > \Delta_k$ means that $|-2x_j(0) + \bar{d}_k| > \Delta_k$, which restricts the initial conditions in *absolute* coordinates, *i.e.*, with respect to a fixed frame. However, in the scenario considered in this paper, the measurements are relative (edge coordinates). That is, absolute positions are irrelevant. So, $x_j(0)$ may be conveniently defined by replacing the origin of the fixed frame if needed. •

Proof: First, we express the constraints in (17) in terms of the spanning-tree coordinates. Let $\mathcal{I}_t = \mathcal{I}_{r_t} \cup \mathcal{I}_{c_t}$, with

$$\mathcal{I}_{r_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R} : |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k| < R_k, \ \forall k \in \mathcal{I}_m\}, \quad (29)$$

$$\mathcal{I}_{c_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R} : \Delta_k < |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k|, \ \forall k \in \mathcal{I}_M\}. \quad (30)$$

where r_{s_k} is the kth column of R_s.

Next, consider the Lyapunov function candidate

$$V(\alpha, \bar{e}_t, v) = (1+\beta)k_1 \tilde{W}(\alpha, \bar{e}_{x_t}) + \frac{1}{2}\Big[\bar{e}_{v_t}^\top L_{e_{ts}}^{-1} \bar{e}_{v_t} + \beta |v|^2\Big]. \quad (31)$$

We note that V in (31) is positive definite in the state variables $\bar{e}_{x_t}$ and $\bar{e}_{v_t}$, uniformly in α and radially unbounded in $\bar{e}_{v_t}$. Also, by construction, $V(\alpha, \bar{e}_t, v) \to \infty$, for any fixed $\bar{e}_{v_t}$ and v, as $\bar{e}_{xt} \to \partial \mathcal{I}_t$, where $\partial \mathcal{I}_t$ denotes the boundary of $\mathcal{I}_t$.

Next, we compute the total derivative of V. To that end, we note that from the definition of α in (18), we have $\dot{\alpha} = (E - E_s)^\top v$ and for (1b) and (27), we have

$$\dot{v} = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{B} \nabla_\alpha \tilde{W}. \quad (32)$$

Then, we have

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) = &(1+\beta)k_1 \nabla_{e_{x_t}} \tilde{W}^\top \bar{e}_{v_t} + (1+\beta)k_1 \nabla_\alpha \tilde{W}^\top \dot{\alpha} \\ &+ \bar{e}_{vt}^\top L_{e_{ts}}^{-1} \Big[-k_1 L_{e_{ts}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{ts}} R_s R_s^\top \bar{e}_{v_t} \\ &- k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{B} \nabla_\alpha \tilde{W} \Big] \\ &+ \beta v^\top \Big[-k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} \\ &- k_3 v - k_1 \mathbb{B} \nabla_\alpha \tilde{W} \Big]. \end{aligned} \quad (33)$$

Hence,

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) = &(1+\beta)k_1 \nabla_\alpha \tilde{W}^\top (E - E_s)^\top v - k_3 \beta |v|^2 \\ &- k_1 v^\top \big[E_{t_s} L_{e_{ts}}^{-1} E_{t_s}^\top + \beta I\big] \mathbb{B} \nabla_\alpha \tilde{W} \\ &- \bar{e}_{v_t}^\top \big[(1+\beta) k_2 R_s R_s^\top + k_3 L_{e_{ts}}^{-1}\big] \bar{e}_{v_t}. \end{aligned} \quad (34)$$

However, in view of (23), we have $\big[E_{t_s} L_{e_{ts}}^{-1} E_{t_s}^\top + \beta I\big]\mathbb{B} = (1+\beta)(E - E_s)^\top$. Thus,

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) = &- \bar{e}_{v_t}^\top \Big[(1+\beta) k_2 R_s R_s^\top + k_3 L_{e_{ts}}^{-1}\Big] \bar{e}_{v_t} \\ &- k_3 \beta |v|^2, \end{aligned}$$

which is negative semidefinite.

Next, we use Barbashin-Krasovskiĭ's theorem. To that end, we note that on the set $\{(\bar{e}, v) : \dot{V} = 0\}$ we have $v = 0$ and $\bar{e}_{vt} = 0$. Therefore, $\dot{\alpha} = 0$, which means that α is constant. In turn, after (28b), we have

$$k_1 L_{e_{ts}} \nabla_{\bar{e}_{x_t}} \tilde{W} = -k_1 E_{t_s}^\top \mathbb{B} \nabla_\alpha \tilde{W}. \quad (35)$$

On the other hand, after (21), we have

$$\nabla_\alpha \tilde{W} = \nabla_{\bar{e}_{x_t}} \tilde{W} - \frac{\partial}{\partial \alpha}\left\{\frac{\partial W}{\partial \alpha}(\alpha)\right\} \bar{e}_{x_t}, \quad (36)$$

but since we have $\alpha \equiv$ const on $\{\dot{V} = 0\}$, it follows that the last term of the right-hand-side of (36) equals to zero, so (35) holds if and only if $-k_1[L_{e_{ts}} + E_{t_s}^\top \mathbb{B}] \nabla_{\bar{e}_{x_t}} \tilde{W} = 0$. The latter holds since $[L_{e_{ts}} + E_{t_s}^\top \mathbb{B}]$ is full rank and $\nabla_{\bar{e}_x} \tilde{W}$ vanishes only at zero. It follows that the only solution of (28) that may remain in $\{(\bar{e}, v) : \dot{V} = 0\}$ for all t, is the origin. Asymptotic stability in the large, on the domain of definition of V follows.

Next, we demonstrate inter-agent collision avoidance. From (29), we remark that $\bar{e}_{x_t} \in \mathcal{I}_t$ implies $\bar{e}_x \in \mathcal{I}$. Then, we proceed by contradiction to show that $\mathcal{I}_t$ is forward invariant. Assume that there exist a $T > 0$ such that $\bar{e}_{x_t}(T) \notin \mathcal{I}_t$. It means that $|\bar{e}_{x_{t_k}} + \alpha_k| \to \Delta_k, \ k \in \mathcal{I}_M$ or $|\bar{e}_{x_{t_k}} + \alpha_k| \to R_k, \ k \in \mathcal{I}_m$ for at least one $k \le M$, which makes $\tilde{W}_k(\alpha_k, \bar{e}_{x_{t_k}}) \to \infty$, so $V(\alpha, \bar{e}_t, v) \to \infty$ as $t \to T$. However, the latter contradicts the fact that $\dot{V}(\alpha, \bar{e}_t, v) \le 0$. Inter-agent collision avoidance follows.

We now show that the set $\mathcal{I}$ corresponds to the domain of attraction for the closed-loop system (28) by showing that all solutions starting in $\mathcal{I}_t$ converge to the origin. For any $\epsilon_1 \in (0, R_k)$ and $\epsilon_2 \in (0, \Delta_k)$, consider subsets $\mathcal{I}_{\epsilon_{r_t}} \subset \mathcal{I}_{r_t}$ and $\mathcal{I}_{\epsilon_{c_t}} \subset \mathcal{I}_{c_t}$ defined as $\mathcal{I}_{\epsilon_{r_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : |\bar{e}_{x_k} + \alpha_k| < R_k - \epsilon_1, \ \forall k \in \mathcal{I}_m\}$ and $\mathcal{I}_{\epsilon_{c_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : \Delta_k + \epsilon_2 < |\bar{e}_{x_k} + \alpha_k|, \ \forall k \in \mathcal{I}_M\}$ with $\mathcal{I}_{\epsilon_{r_t}} \cup \mathcal{I}_{\epsilon_{c_t}} = \mathcal{I}_{\epsilon_t}$. From the definition of the $\tilde{W}(\alpha, \bar{e}_{x_t})$, $V(\alpha, \bar{e}_t, v)$ is positive definite for all $\bar{e}_{x_t} \in \mathcal{I}_{\epsilon_t}$ and $\bar{e}_{v_t} \in \mathbb{R}$ and satisfies $a|\bar{e}_{x_t}|^2 + b|\bar{e}_{v_t}|^2 + c|v|^2 \le V(\alpha, \bar{e}_t, v) \le h(|\bar{e}_{x_t}|) + d|\bar{e}_{v_t}|^2 + f|v|^2$ with $a, b, c, d, f > 0$ and h is a strictly increasing everywhere in $\mathcal{I}_{\epsilon_t}$. This means $V(\alpha, \bar{e}_t, v) \to 0$ as $\bar{e}_t \to 0$ and $v \to 0$. Therefore, we have that for all trajectories for the closed-loop system starting in $\mathcal{I}_{\epsilon_t}$, the origin is asymptotically stable. As ϵ_1 and ϵ_2 are arbitrarily small, taking $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, we establish asymptotic stability of the origin of (28) for all trajectories starting in $\mathcal{I}_t$. Thus, bipartite consensus is achieved with inter-agent collision avoidance and connectivity. ■

IV. Simulation Results

We provide some numerical examples to show the performance of our control law in (24) with $k_1 = 1$, $k_2 = 1.2$, $k_3 = 1$ and the barrier Lyapunov function in (21), with $B_{r_k}(s) = \ln\left(\frac{R_k^2}{R_k^2 - |s|^2}\right)$, $B_{c_k}(s) = \ln\left(\frac{|s|^2}{|s|^2 - \Delta_k^2}\right)$. The simulation consists of an undirected signed network of 6 agents and 7 edges as the one depicted in Figure 1, subject to inter-agent collision avoidance and connectivity maintenance restrictions. We define the orientation of the edges as follows: $e_1 = x_1 + x_2$, $e_2 = x_1 - x_3$, $e_3 = x_1 + x_4$, $e_4 = x_2 + x_5$, $e_5 = x_2 - x_6$, $e_6 = x_3 + x_4$ and $e_7 = x_5 + x_6$. According to Definition 1, the incidence matrix corresponding to the graph is

$$E_s = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The set of agents may be split into two disjoint subgroups, such as $\mathcal{V}_1 = \{x_1, \ x_3, \ x_5\}$ and $\mathcal{V}_2 = \{x_2, \ x_4, \ x_6\}$ so the network is structurally balanced. From the decomposition in (12), edges $e_i, \ i \le 5$ correspond to $\mathcal{G}_t$ and the remaining edges e_6 and e_7 correspond to $\mathcal{G}_c$. The respective agents' initial states are $x(0) = [3.5, \ 3.6, \ -4.5, \ 5.3, \ -2, \ 0]^\top$,

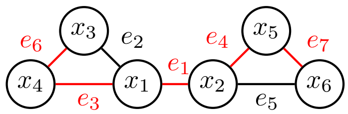

Fig. 1. An undirected signed network of 6 agents. The black lines represent cooperative edges, and the red line represents the competitive edge.

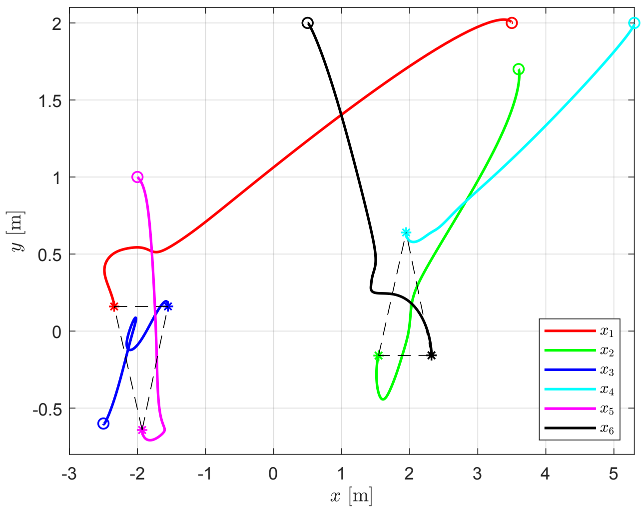

Fig. 2. Bipartite formation of system (1) with control input (24).

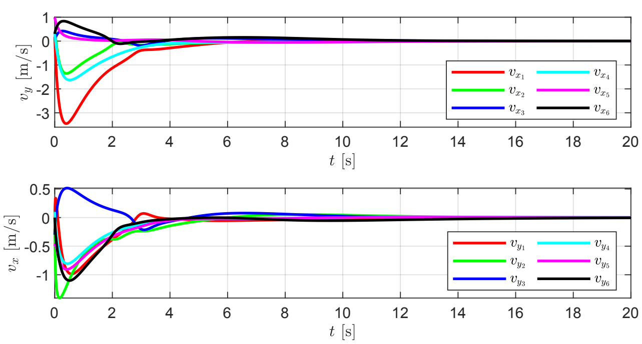

Fig. 3. Bipartite formation of system (1) with control input (24) on velocity, where $k_3 > 0$. The velocities of all agents converge to zero.

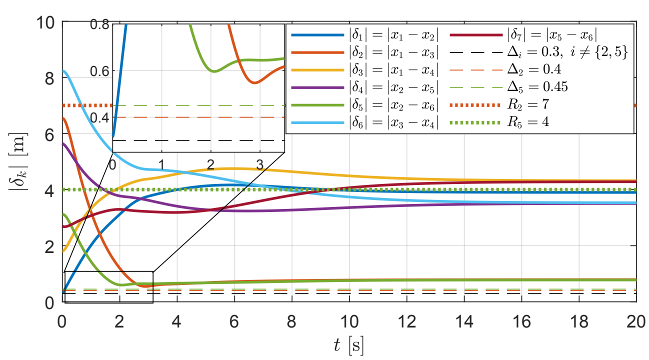

Fig. 4. Trajectories of the norm of the inter-agent distances with control input (24). The dashed and dotted lines are the distance constraints for agents. All inter-agent safety constraints are satisfied.

$y(0) = [2,\ 1.7,\ -2.6,\ 2,\ 1,\ 2]^\top$, $v_x(0) = [0.2,\ 0.4,\ 0.1,\ 0.3,\ 1,\ 0.3]^\top$, $v_y(0) = [0,\ -0.2,\ -0.3,\ 0.1,\ -0.5,\ 0]^\top$ and the relative displacements are $d_x = [-0.4,\ -0.4,\ 0.4,\ 0,\ 0,\ 0.4]^\top$ and $d_y = [0.4,\ -0.4,\ 0.4,\ 0.4,\ -0.4,\ -0.4]^\top$. The constraints sets are $\Delta = [0.3,\ 0.4,\ 0.3,\ 0.3,\ 0.45,\ 0.3,\ 0.3]^\top$ and $R = [7,\ 4]^\top$. The paths of each agent up to bipartite formation are depicted in Figure 2. The agents reach the desired formation around two symmetric consensus points. Furthermore, it is clear from Figure 4 that the inter-agent constraints in (5)-(6) are always respected.

V. Conclusions

We presented a BLF-based control law for structurally balanced undirected signed networks to address the problem of bipartite formation control for double integrators. Our control law ensures inter-agent collision avoidance for any two agents and connectivity maintenance for cooperative agents. Via a change of coordinates, we established the asymptotic stability of the system using Lyapunov's direct method. Further research is focused on extending these results to structurally unbalanced and directed signed networks and rendering the controller fully distributed.

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