IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Finite time input-to-state stability of discrete time autonomous systems

Yuanqiu Mo, Ran Xing, Huazhou Hou, and Soura Dasgupta

***Abstract*— We consider the finite time input-to-state stability of autonomous discrete time systems, where the state enters a ball around the origin, with a radius determined by the input magnitude, in finite time. This extends the notion of classical input to state stability where this condition is only achieved asymptotically. We provide several types of Lyapunov functions that guarantee finite time input-to-state stability and characterize their equivalence. We also give converse Lyapunov theorems correcting a mistake in [1].**

I. Introduction

Input-to-state stability (ISS) originally introduced in [2], has been studied for both continuous and discrete time systems. A system is ISS if its state trajectory with bounded input remains bounded, and asymptotically drops below a function defined by the input size. Papers on the ISS of continuous time systems include [3], [4], [5]; [6] showed that some ISS results for continuous time systems extend to discrete time. Further, [7] proposed four types of ISS Lyapunov functions, namely, max-form, dissipative-form, implication-form and strong implication-form Lyapunov functions. Discrete time systems in [7] are allowed to have discontinuous dynamics should they possess a property called "$\mathcal{K}$-boundedness". Finite-step Lyapunov functions that decrease in every few steps rather than at each step, are used in [7] and [8]. ISS results for finite and infinite-dimensional nonlinear networks are in [9].

Finite time stability of both continuous and discrete time systems is also well studied. Motivation comes from the fact that in applications like regulating robots to reach a desired position [10], one cannot wait to achieve equilibrium asymptotically, but must rather do so in a finite time. Lyapunov and converse Lyapunov results for finite time stability of continuous time systems with zero inputs are in [11]. Finite time stability of time-varying or stochastic systems are in [12] and [13]. Results on finite time stability of discrete time systems are relatively fewer than those for continuous time ones: [14] proposes Lyapunov and converse Lyapunov theorems for discrete time systems with zero input.

We consider finite time input-to-state stability (FTISS) where the state magnitude drops below a function determined by the input bound and with zero input converges to zero, both in finite time. Several results on FTISS of continuous time systems exist. Thus, [15] gives Lyapunov-based sufficient conditions for the FTISS of continuous time systems, and provides converse Lyapunov theorem under certain assumptions and [16] studies FTISS of impulsive systems. In contrast studies of FTISS of discrete time systems are few. Exceptions are [17] and [1]. Both provide a limited class of Lyapunov functions to check for FTISS. The former has no converse results. The latter purports to provide one, but incorrectly claims that the settling time of FTISS is a valid Lyapunov function (equation (7) on page 2 in [18]). We show here through a counterexample that the time to settle need not be a Lyapunov function.

Thus we study the FTISS of discrete time systems using Lyapunov theory. By extending FTISS Lyapunov functions for continuous time systems proposed in [15], as well as Lyapunov functions for the finite time stability of discrete time systems provided in [14], we provide four types of Lyapunov functions to show the FTISS of discrete time systems. We establish the equivalence between these Lyapunov functions and show that FTISS of a discrete time system implies finite time stability of a special case involving a form of feedback introduced for continuous time systems in [19] and that the settling time of this special case is indeed a valid Lyapunov function. This establishes converse results correctly for the first time.

Section II introduces notations and definitions. Section III proposes four types of FTISS Lyapunov functions and characterizes their equivalence. Section IV provides the converse Lyapunov theorems, and Section V concludes.

II. Notations and definitions

For $\Pi \subseteq \mathbb{R}$, define $\Pi_{\geq c_1} := \{k \in \Pi \mid k \geq c_1\}$ and $\Pi_+ = \Pi_{\geq c_1}$ with $c_1 = 0$. Define $\mathcal{B}_\epsilon(x)$ as the open ball centered at x with radius ϵ. For $x, y \in \mathbb{R}^n_+$, $x < y$ (resp. $x \leq y$) means $x_i < y_i$ (resp. $x_i \leq y_i$) for all $i \in \{1, \cdots, n\}$. We use $|\cdot|$ to denote an arbitrary fixed monotonic norm on $\mathbb{R}^n$, i.e., given $v, w \in \mathbb{R}^n$ with $v \geq w$, then $|v| \geq |w|$: $|\cdot|_\infty$ and $|\cdot|_2$ are the infinity and 2-norm, respectively. For a sequence $\{u(k)\}_{k\in\mathbb{Z}_+}$ with $u(k) \in \mathbb{R}^m$, $||u|| = \sup_{k\in\mathbb{Z}_+}\{|u(k)|\} \leq \infty$ is its sup-norm. Sequences with finite sup-norm are in ℓ^∞; $\lceil a \rceil$ is the smallest integer greater than or equal to a.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is in class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty}\alpha(s) = \infty$. It is in generalized $\mathcal{K}$ ($\mathcal{GK}$) if it is continuous, $\alpha(0) = 0$ and satisfies

$$\begin{cases} \alpha(s_1) > \alpha(s_2) & \text{if } \alpha(s_1) > 0 \text{ and } s_1 > s_2 \\ \alpha(s_1) = \alpha(s_2) & \text{if } \alpha(s_1) = 0 \text{ and } s_1 > s_2 \end{cases} \quad (1)$$

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is in class $\mathcal{KL}$ if for a fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{K}$, and for a fixed $r \in \mathbb{R}_+$, $\beta(r, \cdot)$ is decreasing and $\lim_{s\to\infty}\beta(\cdot, s) = 0$. It is in generalized $\mathcal{KL}$ ($\mathcal{GKL}$) if for a fixed $s \in \mathbb{R}_+$, $\beta(\cdot, s) \in \mathcal{GK}$, and for a

Mo and Hou are with the Southeast University, Nanjing 211189, China (email: [yuanqiumo, huazhouhou]@seu.edu.cn). Hou is also with the Purple Mountain Laboratories, Nanjing, China. Xing is with the University of Sydney, NSW 2006, Australia (rxin6338@uni.sydney.edu.au). Dasgupta is with the University of Iowa, Iowa City, Iowa 52242, USA (email: soura-dasgupta@uiowa.edu). This work was supported by the Natural Science Foundation of Zhejiang Province under Grant No. LQ21F030011, the National Natural Science Foundation of China under Grant No. 62203109, Guangdong Basic, Applied Basic Research Foundation under Grant 2020A1515110148 and Natural Science Foundation of Jiangsu Province under grant number BK20220812.

fixed $r \in \mathbb{R}_+$, $\beta(r,\cdot)$ is decreasing and $\lim_{s\to T}\beta(\cdot,s)=0$ for some $T \le \infty$; $\mathrm{id}:\mathbb{R}_+\to\mathbb{R}_+$ obeys $\mathrm{id}(s)=s$ for all $s\in\mathbb{R}_+$.

We consider the discrete time system:

$$x(k+1) = G(x(k),u(k)),\ k\in\mathbb{Z}_+, \tag{2}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u:\mathbb{Z}_+\to\mathbb{R}^m$ is the input in $u(\cdot)\in\ell^\infty(\mathbb{R}^m)$, and $G:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$. Further, $G(0,0)=0$, i.e., $\xi=0$ is an equilibrium of the system with zero input. By $x(k,\xi,u)$ we denote the solution to (2) at time k, with ξ the initial state and u the input. Denote $x(k):=x(k,\xi,u)$.

Define $\mathcal{M}$ as the set of functions from $\mathbb{Z}_+$ to $[-1,1]^m$, i.e.,

$$\mathcal{M} = \{u \mid u:\mathbb{Z}_+\to[-1,1]^m\}. \tag{3}$$

The following, introduced in [19], is a special case of (2)

$$\bar{x}(k+1) = G(\bar{x}(k), d(k)\varphi(\bar{x}(k))) = F(\bar{x}(k),d(k)) \tag{4}$$

where $d\in\mathcal{M}$, the smooth function $\varphi:\mathbb{R}^n\to\mathbb{R}_+$ is given and $F(0,d)=0$ for all $d\in\mathcal{M}$. We call $d(k)\varphi(\bar{x}(k))$ an *admissible feedback law* if φ obeys $\rho_1(|\xi|)\le\varphi(\xi)\le\rho_2(|\xi|)$ for all $\xi\in\mathbb{R}^n$ with $\rho_1,\rho_2\in\mathcal{K}_\infty$. One can view (4) as a perturbed feedback law. *Its importance lies in the fact that as shown in the sequel, its time to converge serves as a natural Lyapunov function for (2).* As before, we denote $\bar{x}(k):=\bar{x}(k,\xi,d)$ as the solution to (4) at time k.

We assume G is $\mathcal{K}$-bounded: for all $\xi\in\mathbb{R}^n$ and $\mu\in\mathbb{R}^m$

$$|G(\xi,\mu)| \le \omega_1(|\xi|)+\omega_2(|\mu|) \tag{5}$$

with $\omega_1,\omega_2\in\mathcal{K}$. Note, $\mathcal{K}$-boundedness implies the continuity of G only at $(0,0)$.

A. Global finite time stability

We first provide the definition of global finite time stability (GFTS) for (2) with zero input, as given in [14].

Definition 1. *(GFTS) The zero input solution $x(k,\xi,0)=0$ of (2) is globally finite time stable if the following hold: (A) Finite time convergence: There exists a settling time function $K(\xi):\mathbb{R}^n\setminus\{0\}\to\mathbb{Z}_{\ge1}$ such that for all $\xi\in\mathbb{R}^n\setminus\{0\}$, $x(k,\xi,0)=0$ for $k\ge K(\xi)$. (B) Lyapunov stability: For every $\epsilon>0$, there exists $\delta>0$ such that for all $\xi\in\mathcal{B}_\delta(0)\setminus\{0\}$, $x(k,\xi,0)\in\mathcal{B}_\epsilon(0)$ for all $k\in\mathbb{Z}_+$.*

The settling time function $K(\xi)$ in Definition 1 obeys

$$K(\xi) = \min\{k\in\mathbb{Z}_{\ge1} \mid x(k,\xi,0)=0\}. \tag{6}$$

As $\xi=0$ is the equilibrium of system (2) with zero input, we must have $K(0)=0$. Even though (4) is time varying, it has a natural definition of GFTS as $F(0,d(k))=0$ for all $d\in\mathcal{M}$. Specifically, it can be viewed as a zero input system with $d(k)$ serving to make it time varying.

Definition 2. *The discrete time system (4) is GFTS if for all $d\in\mathcal{M}$ and all $\xi\in\mathbb{R}^n\setminus\{0\}$ it is Lyapunov stable and finite time convergent with settling time function $\bar{K}_d(\xi)$.*

According to Definitions 1 and 2, if (4) is GFTS, given a $d\in\mathcal{M}$ and a $\xi\in\mathbb{R}^n$, the settling time function $\bar{K}_d(\xi)$ obeys

$$\bar{K}_d(\xi) = \min\{k\in\mathbb{Z}_{\ge1} \mid \bar{x}(k,\xi,d)=0\}, \tag{7}$$

and $\bar{K}_d(0)=0$ by (4). Definition 2 yields the following.

Definition 3. *The discrete time system (2) is said to be weakly robust finite time stable if its special case (4) with an admissible feedback law is GFTS.*

Remark 1. *Note weakly robust finite time stability only requires GFTS of (4) with just one $\phi(\cdot)$, bounded above and below by $\mathcal{K}_\infty$ functions.*

B. Global finite time input-to-state stability

Now we give the formal definition of global finite time input-to-state stability (FTISS).

Definition 4. *(FTISS) The system (2) is globally finite time input-to-state stable if there exist $\beta\in\mathcal{GKL}$ and $\lambda\in\mathcal{K}$ such that for all initial states $\xi\in\mathbb{R}^n$ and all inputs $u\in\ell^\infty(\mathbb{R}^m)$*

$$|x(k)| \le \beta(|\xi|,k)+\lambda(||u||),\ \forall k\in\mathbb{Z}_+. \tag{8}$$

Further, there exists a positive definite $T:\mathbb{R}_+\to\mathbb{Z}_+$ such that: (i) for all $r\in\mathbb{R}_+\setminus\{0\}$, $\beta(r,k)=0$ whenever $k\ge T(r)\in\mathbb{Z}_{\ge1}$ and (ii) $T(0)=0$.

Remark 2. *When β in (8) is in class $\mathcal{KL}$, (2) is globally input-to-state stable (ISS). An equivalent form of (8) is*

$$|x(k)| \le \max\{\bar{\beta}(|\xi|,k),\bar{\lambda}(||u||)\},\ \forall k\in\mathbb{Z}_+, \tag{9}$$

where $\bar{\beta}(r,s)=\beta(2r,s)$ is a $\mathcal{GKL}$ function and $2\mathrm{id}\circ\lambda=\bar{\lambda}\in\mathcal{K}$, with β and λ defined in (8).

Obviously, the discrete time system (2) is GFTS with zero input if it is FTISS. Moreover, the existence of the function $T(r)$ in Definition 4 implies that of the settling function $K(\xi)$ in Definition 1, e.g., we can set $K(\xi)=T(|\xi|)$. The converse also holds if $G(\cdot,\cdot)$ is $\mathcal{K}$-continuous with respect to the input, i.e., $|G(x,u_1)-G(x,u_2)|\le\sigma_u(|u_1-u_2|)$ with $\sigma_u\in\mathcal{K}$, and in this case we can set $T(r)=\sup_{|\xi|_2=r}K(\xi)$.

III. FTISS Lyapunov functions

In this section, we provide two types of finite time ISS (FTISS) Lyapunov functions, the implication-form FTISS Lyapunov function and the max-form one. For each type we give two FTISS Lyapunov functions.

We call a function $V:\mathbb{R}^n\to\mathbb{R}_+$ proper if there exist functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ such that for all $\xi\in\mathbb{R}^n$

$$\alpha_1(|\xi|)\le V(\xi)\le\alpha_2(|\xi|). \tag{10}$$

Then FTISS Lyapunov functions are given as follows.

Definition 5. *Let $V:\mathbb{R}^n\to\mathbb{R}_+$ be proper. For a given input $u\in\ell^\infty(\mathbb{R}^m)$, V is said to be*

- *an implication-form FTISS Lyapunov function I for (2) if for all $\xi\in\mathbb{R}^n$ and all $k\in\mathbb{Z}_+$, $V(x(k))\ge\phi_{\mathrm{imp1}}(|u(k)|)$ implies, with $c>0$, $0<a<1$ and $\phi_{\mathrm{imp1}}\in\mathcal{K}$,*
$$V(x(k+1))\le\max\{V(x(k))-cV(x(k))^a,0\} \tag{11}$$
- *a max-form FTISS Lyapunov function I for (2) if for all $\xi\in\mathbb{R}^n$ and all $k\in\mathbb{Z}_+$ we have*
$$V(x(k+1))\le\max\{V(x(k))-cV(x(k))^a,\lambda_{\max1}(||u||)\} \tag{12}$$
with $c>0$, $0<a<1$ and $\lambda_{\max1}\in\mathcal{K}$.
- *an implication-form FTISS Lyapunov function II for (2) if for all $\xi\in\mathbb{R}^n$ and all $k\in\mathbb{Z}_+$ we have*
$$V(x(k))\ge\phi_{\mathrm{imp2}}(|u(k)|) \implies V(x(k+1))\le\max\{V(x(k))-b,0\} \tag{13}$$
with $b>0$ and $\phi_{\mathrm{imp2}}\in\mathcal{K}$.
- *a max-form FTISS Lyapunov function II for (2) if for all $\xi\in\mathbb{R}^n$ and all $k\in\mathbb{Z}_+$ we have*
$$V(x(k+1))\le\max\{V(x(k))-b,\lambda_{\max2}(||u||)\} \tag{14}$$

with $b > 0$ and $\lambda_{\max 2} \in \mathcal{K}$.

Remark 3. *The above FTISS Lyapunov functions are an extension of the finite time Lyapunov functions introduced in [14]. In particular, implication-form FTISS Lyapunov functions for continuous time systems and implication-form ISS Lyapunov functions for discrete time systems are in [15] and [8], respectively. They need not be continuous.*

In [1], we do not consider implication-form Lyapunov functions at all. Here we show that in fact the existence of an implication-form FTISS Lyapunov function implies that of an max-form FTISS Lyapunov function.

Lemma 1. *With FTISS Lyapunov functions introduced in Definition 5, there exists a max-form FTISS Lyapunov function I (resp. II) for (2) if it admits an implication-form FTISS Lyapunov function I (resp. II) .*

Proof. If (2) has an implication-form FTISS Lyapunov function I then from (5,10), if $V(x(k)) < \phi_{\text{imp1}}(|u(k)|)$ then

$$\begin{aligned} V(x(k+1)) &\leq \alpha_2(|x(k+1)|) = \alpha_2(|G(x(k), u(k))|) \\ &\leq \alpha_2(\omega_1(|x(k)|) + \omega_2(|u(k)|)) \\ &\leq \alpha_2\big(\omega_1 \circ \alpha_1^{-1} \circ \phi_{\text{imp1}}(|u(k)|) + \omega_2(|u(k)|)\big) \\ &= \lambda_{\max 1}(||u||) \end{aligned} \tag{15}$$

where in (15) $\lambda_{\max 1} = \alpha_2 \circ (\omega_1 \circ \alpha_1^{-1} \circ \phi_{\text{imp1}} + \omega_2) \in \mathcal{K}$. Thus either (15) holds or from (11), $V(x(k+1)) \leq \max\{V(x(k)) - cV(x(k))^a, 0\}$ when $V(x(k)) \geq \phi_{\text{imp1}}(|u(k)|)$. Thus (2) has a max-form FTISS Lyapunov function I by Definition 5.

Applying the above procedure, it can also be verified that the existence of an implication-form FTISS Lyapunov function II implies that of a max-form FTISS Lyapunov function II. ∎

The following lemma shows that the existence of an implication-form (resp. max-form) FTISS Lyapunov function I is sufficient for the existence of an implication-form (resp. max-form) FTISS Lyapunov function II.

Lemma 2. *If (2) has an implication-form (resp. max-form) FTISS Lyapunov function I then it has an implication-form (resp. max-form) FTISS Lyapunov function II.*

Proof. We first prove for implication-form FTISS Lyapunov function. Without loss of generality, assume that ϕ_{imp2} defined in (13) obeys $\phi_{\text{imp2}} = \phi_{\text{imp1}}$ with ϕ_{imp1} defined in (11). Then it follows from (11) that when $V(x(k)) \geq \phi_{\text{imp1}}(|u(k)|)$ either

$$V(x(k+1)) = 0 \text{ for } V(x(k)) < c^{\frac{1}{1-a}}, \tag{16}$$

or for $V(x(k)) \geq c^{\frac{1}{1-a}}$,

$$V(x(k+1)) \leq V(x(k)) - cV(x(k))^a \leq V(x(k)) - c^{\frac{1}{1-a}}$$

where (16) uses the fact that $V(x(k)) < cV(x(k))^a$ when $V(x(k)) < c^{\frac{1}{1-a}}$. Then $V(x(k+1)) \leq \max\{V(x(k)) - b, 0\}$ with $b = c^{\frac{1}{1-a}}$ when $V(x(k)) \geq \phi_{\text{imp2}}(|u(k)|)$.

For the implication from max-form FTISS Lyapunov function I to II, without loss of generality, let $v = \lambda_{\max 1}(||u||) = \lambda_{\max 2}(||u||)$. As $V(x(k))$ is positive definite and $cV(x(k))^a \geq 0$ for all $k \in \mathbb{Z}_+$, let $T_1 \leq +\infty$ be the first time step such that $V(x(T_1)) < v$, then $V(x(k)) < v$ for all $k \geq T_1$. When $T_1 \geq 1$, it follows from (12) that for $k < T_1$

$$\begin{aligned} V(x(k)) &\leq V(x(k-1)) - cV(x(k-1))^a \\ &\leq V(x(k-1)) - cv^a \qquad (17) \\ &= V(x(k-1)) - b \qquad (18) \end{aligned}$$

where (17) uses the fact that $V(x(k)) \geq v$ for $k < T_1$ and in (18) $b = cv^a$. Therefore, if there exists a max-form FTISS Lyapunov function I, then for all $k \in \mathbb{Z}_+$ we have

$$V(x(k+1)) \leq \max\{V(x(k)) - b, \lambda_{\max 2}(||u||)\}, \tag{19}$$

thus V is a max-form FTISS Lyapunov function II. ∎

Now we are ready to show that FTISS Lyapunov functions introduced in Definition 5 can conclude the FTISS of (2). As Lemma 1 shows that the existence of implication-form Lyapunov function I (resp. II) implies that of max-form Lyapunov function I (resp. II), proving that (2) is FTISS if it admits max-form Lyapunov function I or II, also proves that it is FTISS if it has implication-form FTISS Lyapunov functions I and II.

Theorem 1. *The discrete time system (2) is FTISS if it admits a max-form FTISS Lyapunov function I.*

Proof. Let $v = \lambda_{\max 1}(||u||)$. As $cV(x(k))^a \geq 0$, if there exists a T_1 such that $V(x(T_1)) \leq v$, then from (12), $V(x(k)) \leq \lambda_{\max 1}(||u||)$ for all $k > T_1$.

Now suppose $x(0) = \xi$ and $V(\xi) > v$. For $k \geq 1$, that if $V(x(k-1)) \leq c^{\frac{1}{1-a}}$, then $V(x(k-1)) - cV(x(k-1))^a \leq 0$, i.e., from (12), $V(x(k)) \leq v$. On the other hand, if $V(x(k-1)) > c^{\frac{1}{1-a}}$, it follows from (12) that for $k \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} &V(x(k-1)) - cV(x(k-1))^a = \\ &V(x(k-1))(1 - cV(x(k-1))^{a-1}) < V(x(k-1)) \end{aligned} \tag{20}$$

where (20) results from $cV(x(k-1))^{a-1} \in (0,1)$. As $0 < a < 1$, if $V(x(k)) > v$, if follows from (12) and (20) that $1 - cV(x(k))^{a-1} < 1 - cV(x(k-1))^{a-1} < \cdots < 1 - cV(\xi)^{a-1}$, together with (20) and $V(x(k-1)) > c^{\frac{1}{1-a}}$, we can obtain

$$\begin{aligned} &V(x(k-1))(1 - cV(x(k-1))^{a-1}) = \\ &V(x(k-2))(1 - cV(x(k-2))^{a-1})(1 - cV(x(k-1))^{a-1}) \\ &\cdots \\ &= V(\xi)(1 - cV(\xi)^{a-1}) \cdots (1 - cV(x(k-1))^{a-1}) \\ &< V(\xi)(1 - cV(\xi)^{a-1})^k \end{aligned} \tag{21}$$

From (12) and (21), we have

$$V(x(k)) \leq \max\{V(\xi)(1 - cV(\xi)^{a-1})^k, v\}. \tag{22}$$

Consider three cases: 1) $c^{\frac{1}{1-a}} \geq V(\xi) > v$; 2) $V(\xi) > v \geq c^{\frac{1}{1-a}}$; and 3) $V(\xi) > c^{\frac{1}{1-a}} > v$. In the first case, it follows from above that $V(x(k)) \leq v$ for all $k \in \mathbb{Z}_{\geq 1}$. In the second case, it follows from (22) that $V(x(k)) \leq v$ for $k \geq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{v}{V(\xi)} \right\rceil$. In the last case, there holds $V(x(k)) \leq v$ for all $k \geq k_1 \in \mathbb{Z}_{\geq 1}$ once $V(x(k_1 - 1)) \leq c^{\frac{1}{1-a}}$. Then $V(x(k)) \leq v$ for $k \geq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{V(\xi)} \right\rceil + 1$.

Therefore, it follows from (10) that $|x(k)| \leq \alpha_1^{-1} \circ$

$\lambda_{\max 1}(||u||)$ for $k \geq T(\xi)$ with $T(\xi)$ obeying

$$T(\xi)\begin{cases} = 0 & V(\xi) \leq v \\ = 1 & c^{\frac{1}{1-a}} \geq V(\xi) > v \\ \leq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{v}{V(\xi)} \right\rceil & V(\xi) > v \geq c^{\frac{1}{1-a}} \\ \leq \left\lceil \log_{[1-cV(\xi)^{a-1}]} \frac{c^{\frac{1}{1-a}}}{V(\xi)} \right\rceil + 1 & V(\xi) > c^{\frac{1}{1-a}} > v \end{cases} \tag{23}$$

From (12) and (23), $V(x(k)) \leq \lambda_{\max 1}(||u||)$ for $k \geq T(\xi)$ and $V(x(k)) \leq V(x(k-1)) - cV(x(k-1))^a$ for $k < T(\xi)$. As V is proper, from (10) there holds $|x(k)| \leq \alpha_1^{-1}(V(x(k))) < \alpha_1^{-1}(V(x(k-1)))$, i.e., the upper bound of $|x(k)|$ decreases at each time step for $k < T(\xi)$, then we show below that we can construct a $\mathcal{GKL}$ function β such that $|x(k)| \leq \beta(|\xi|, k)$ for all $k < \bar{T}(|\xi|)$ and $\beta(|\xi|, k) = 0$ for all $k \geq \bar{T}(|\xi|)$ where $\bar{T}$ obeys $\bar{T}(0) = 0$ and $\bar{T}(|\xi|) \geq \max_{\xi \in \mathbb{R}^n \setminus \{0\}} T(\xi)$ with T satisfying (23). Specifically, such a $\mathcal{GKL}$ function can be constructed as follows: Given the initial state ξ, there exists a linear function such that $|x(k)| \leq |\xi|(-a_\xi k + b_\xi)$ with $a_\xi, b_\xi > 0$ for $k < T(\xi)$. Let $a = \min_{\xi \in \mathbb{R}^n \setminus \{0\}} a_\xi$ and $b = \max_{\xi \in \mathbb{R}^n \setminus \{0\}} b_\xi$. Then the $\mathcal{GKL}$ function β can be constructed as $\beta(r, s) = r(-as + b)$ for $s \leq b/a$ and $\beta(r, s) = 0$ otherwise.

Therefore, for all $k \in \mathbb{Z}_+$ we have $|x(k)| \leq \beta(|\xi|, k) + \lambda(||u||)$, where $\lambda = \alpha_1^{-1} \circ \lambda_{\max 1} \in \mathcal{K}$ with α_1 and $\lambda_{\max 1}$ defined in (10) and (12), respectively. Our claim follows by Definition 4. ■

Similarly, for max-form FTISS Lyapunov function II, we have the following theorem.

Theorem 2. *The discrete time system (2) is FTISS if it admits a max-form FTISS Lyapunov function II.*

Proof. The proof follows from that of Theorem 1. In this case, we have $|x(k)| \leq \beta(|\xi|, k) + \lambda(||u||)$ with $\lambda = \alpha_1^{-1} \circ \lambda_{\max 2} \in \mathcal{K}$, where α_1 and $\lambda_{\max 2}$ are defined in (10) and (14), respectively. β is a $\mathcal{GKL}$ function and $\beta(|\xi|, s) = 0$ when $s \geq \bar{T}(|\xi|)$ where $\bar{T}(0) = 0$, $\bar{T}(|\xi|) \geq T(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ with T obeying $T(\xi) = 0$ when $V(\xi) \leq v$ and $T(\xi) = \left\lceil \frac{v - V(\xi)}{b} \right\rceil$ otherwise, where $v = \lambda_{\max 2}(||u||)$ and b is defined in (14). ■

Theorems 1 and 2 show that max-form FTISS Lyapunov functions I and II both lead to the FTISS of (2) without assuming its $\mathcal{K}$-boundedness defined in (5). Theorems 1, 2 and Lemma 1 guarantee the FTISS of (2) if it admits implication-form FTISS Lyapunov functions I or II, however, in this case $\mathcal{K}$-boundedness is needed. Such a pattern can also be found in ISS Lyapunov functions [7] (See Theorem 10). While existence of a max-form ISS Lyapunov function can lead to the ISS of discrete time autonomous systems, $\mathcal{K}$-boundedness is needed for implication-form Lyapunov function.

IV. Converse Lyapunov theorems

We give two converse FTISS Lyapunov theorems. We note that [1] claims the settling time function for (2) to be a Lyapunov function. This is however, a mistake as the settling time function may fail to satisfy (10). To see this, consider the Adaptive Bellman-Ford (ABF) algorithm in [20], which estimates distances of nodes in a network from a source set S. With $\hat{d}_i(k)$ and $d_i(k)$ the estimated and true distance of node i from S, respectively, ABF has a natural state vector with elements $x_i(k) = \hat{d}_i(k) - d_i(k)$. Suppose $x_i(0) > 0$ for all i. Then for an n-node network, [20] shows that the settling time is upper bounded by n. Thus with $\xi = [x_1(0), \cdots, x_n(0)]^T$, $T(\xi) \leq n$ for all ξ in the positive orthant. Thus, the settling time $T(\xi)$ is upper bounded by a number that is independent of the initial state magnitude, and thus cannot be bounded from below by a $\mathcal{K}_\infty$ function. Thus $T(\xi)$ fails to meet a minimal requirement for a valid Lyapunov function.

Instead we show here that the settling time of (4) satisfies (10) and is the right Lyapunov function. We first state the following result from [19].

Lemma 3. *For any $\mathcal{K}_\infty$ function ρ, there exist a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}_+$ and a $\mathcal{K}_\infty$ function $\hat{\rho}$ such that $\hat{\rho}(|\xi|) \leq \phi(\xi) \leq \rho(|\xi|)$ for all $\xi \in \mathbb{R}^n$.*

The following theorem shows that FTISS of (2) implies its weakly robust finite time stability or equivalently GFTS of (4).

Theorem 3. *The discrete time system (2) is weakly robust finite time stable if it is FTISS.*

Proof. From Remark 2, if (2) is FTISS, there exist a $\mathcal{GKL}$ function β and a $\mathcal{K}$ function λ such that

$$|x(k)| \leq \max\{\beta(|\xi|, k), \lambda(||u||)\}, \ \forall k \in \mathbb{Z}_+, \tag{24}$$

where $\beta(|\xi|, k) = 0$ when $k \geq T(|\xi|)$ with $T(r) \in \mathbb{Z}_+$ for all $r \in \mathbb{R}_+ \setminus \{0\}$ and $T(0) = 0$.

Without loss of generality, λ in (24) can be assumed to be a $\mathcal{K}_\infty$ function. Further, it follows from Theorem 1 in [15] and Lemma 2.12 in [19] that there exists a $\mathcal{GKL}$ function $\hat{\beta}$ such that the following holds

- $\hat{\beta}(r, 0) \geq r$ for all $r \in \mathbb{R}_+$;
- $\hat{\beta}(r, s) \geq \beta(r, s)$ for all $r \in \mathbb{R}_+$, with $\hat{\beta}(r, s) = 0$ for $s \geq T(r)$;
- $\hat{\beta}(r/2, s) = 0$ when $s \geq \hat{T}/2$ if $\hat{\beta}(r, s) = 0$ when $s \geq \hat{T}$ for any $r, \hat{T} \in \mathbb{R}_+$.

Then (24) still holds with β being replaced by $\hat{\beta}$, i.e.,

$$|x(k)| \leq \max\{\hat{\beta}(|\xi|, k), \lambda(||u||)\}, \ \forall k \in \mathbb{Z}_+, \tag{25}$$

Let $\rho(r) = \hat{\beta}(r, 0) \in \mathcal{K}_\infty$. Without loss of generality, λ in (25) can be chosen as a $\mathcal{K}_\infty$ function. It follows from Lemma 3 that there exists a function φ (smooth everywhere except possibly at the origin) and a $\mathcal{K}_\infty$ function $\hat{\rho}$ such that

$$\hat{\rho}(|\xi|) \leq \varphi(\xi) \leq \lambda^{-1}\left(\frac{\rho^{-1}(|\xi|)}{4}\right), \ \forall \xi \in \mathbb{R}^n. \tag{26}$$

Consider (4) the special case of (2):

$$\bar{x}(k+1) = G(\bar{x}(k), d(k)\varphi(\bar{x}(k))) \tag{27}$$

with $d \in \mathcal{M}$ and φ defined in (26). By Definition 3 our claim follows if the discrete time system (27) is GFTS. We first prove that

$$\lambda\Big(\big|d(k)\varphi\big(\bar{x}(k, \xi, d)\big)\big|\Big) \leq \frac{|\xi|}{2} \tag{28}$$

with a given $d \in \mathcal{M}$, $\xi \in \mathbb{R}^n$ the initial state of (4), and $\bar{x}(k) := \bar{x}(k, \xi, d)$ the solution to (4) at time k. By (3), (26) and the monotonicity of λ, there holds

$$\lambda\big(\big|d(0)\varphi(\bar{x}(0))\big|\big) \leq \lambda\big(\varphi\big(\bar{x}(0)\big)\big) \leq \frac{\rho^{-1}(|\xi|)}{4} < \frac{|\xi|}{4} \tag{29}$$

where (29) uses the fact that $\rho = \hat{\beta}(\cdot, 0) > \text{id}$. Let

$$k_1 = \inf\Big\{k_1 \in \mathbb{Z}_{\geq 1} \mid \lambda\big(\varphi(\bar{x}(k_1))\big) > \frac{|\xi|}{2}\Big\}. \quad (30)$$

Suppose $k_1 < \infty$. However, it follows from (25) that $|\bar{x}(k)| \leq \hat{\beta}(|\xi|, 0) = \rho(|\xi|)$ for all $k \in \mathbb{Z}_+$, leading to $\lambda\big(\varphi(\bar{x}(k_1))\big) \leq \rho^{-1}(\bar{x}(k_1))/4 \leq |\xi|/4$ by (29), contradicting the definition of k_1 as in (30). Thus, $k_1 = \infty$ and (28) holds.

For any $r \in \mathbb{R}_+$, there exists a $T_r \in \mathbb{Z}_+$ such that $\hat{\beta}(r, s) \leq r/2$ for all $s \geq T_r$. Together with (25) and (28), we have

$$|\bar{x}(k, \xi, d)| \leq \frac{r}{2}, \ \forall |\xi| \leq r, \forall k \geq T_r \text{ and } \forall d \in \mathcal{M}. \quad (31)$$

Proceeding in this way, there exists a sequence of time steps $0 = T_0 \leq T_1 \leq \cdots \leq T_i \leq \cdots$ with $i \in \mathbb{Z}_{\geq 1}$ such that,

$$|\bar{x}(k, \xi, d)| = |\bar{x}(k - T_{i-1}, \bar{x}(T_{i-1}), d|, \ |\bar{x}(T_{i-1})| \leq \frac{r}{2^{i-1}},$$

$$\forall |\xi| \leq r, \forall k \geq T_i = \sum_{j=0}^{i-1} T_{r/2^j} \text{ and } \forall d \in \mathcal{M}, \quad (32)$$

with $\bar{x}(T_{i-1}) := \bar{x}(T_{i-1}, \xi, d)$. Further, with (25), (27) and (28), for all $k \in \mathbb{Z}_+$ there holds

$$|x(k, \bar{x}(T_{i-1}), d)| \leq \max\left\{\hat{\beta}(|\bar{x}(T_{i-1})|, k), \lambda\left(\frac{|\bar{x}(T_{i-1})|}{2}\right)\right\}. \quad (33)$$

As $\hat{\beta}(r/2, s) = 0$ when $s \geq \hat{T}/2$ if $\hat{\beta}(r, s) = 0$ when $t \geq \hat{T}$, together with $\hat{\beta}(r, s) = 0$ for $s \geq T(r)$ and $\bar{x}(T_{i-1})$ introduced in (32), we have $\hat{\beta}(|\bar{x}(T_{i-1})|, k) = 0$ for $k \geq T(r)/2^{i-1}$. Then $T_{r/2^j}$ defined in (32) obeys $T_{r/2^j} = T(r)/2^j$, and thus

$$\lim_{i\to\infty} T_i = \lim_{i\to\infty} \sum_{j=0}^{i-1} T_{r/2^j} = \lim_{i\to\infty} \sum_{j=0}^{i-1} T(r)/2^j = 2T(r),$$

implying the finite time convergence of (27). Further, as (25) implies that $|\bar{x}(k)| \leq \hat{\beta}(|\xi|, 0)$ for all $k \in \mathbb{Z}_+$, all $d \in \mathcal{M}$ and all $\xi \in \mathbb{R}^n$, (27) is also Lyapunov stable. Then (27) is GFTS, i.e. (2) is weakly robust finite time stable by Definition 3. ■

Our converse FTISS Lyapunov functions mainly rely on the weakly robust stability as defined in Definition 3, which is implied by GFTS of (4) as in Definition 2. Recall the settling time function $\bar{K}_d(\xi)$ defined in (7), denote $\bar{K}_d(\xi)$ as the settling time function for (4) with $\xi \in \mathbb{R}^n$ the initial state and a given $d \in \mathcal{M}$. By Definition 2, there holds

$$\bar{K}_*(\xi) := \sup_{d\in\mathcal{M}} \bar{K}_d(\xi) < \bar{K} < \infty, \ \forall \xi \in \mathbb{R}^n, \quad (34)$$

and it follows from (4) that $\bar{K}_d(0) = 0$ for all $d \in \mathcal{M}$.

Obviously, $\bar{K}_*(0) = 0$. For all $\xi \in \mathbb{R}^n$, by defining $\mathcal{K}_\infty$ function κ_1 with $\kappa_1(r) \leq \inf_{|\xi|=r} \bar{K}_*(\xi)$ and $\mathcal{K}_\infty$ function κ_2 with $\kappa_2(r) \geq \sup_{|\xi|=r} \bar{K}_*(\xi)$, there holds

$$\kappa_1(|\xi|) \leq \bar{K}_*(\xi) \leq \kappa_2(|\xi|), \ \forall \xi \in \mathbb{R}^n. \quad (35)$$

When $\bar{x}(k) \neq 0$, for a given $d \in \mathcal{M}$ we have $\bar{K}_d(\bar{x}(k+1)) = \bar{K}_d(\bar{x}(k)) - 1$, leading to

$$\bar{K}_*(\bar{x}(k)) = \sup_{\substack{\forall d(s)\\ s\in\mathbb{Z}_{\geq k}}} \bar{K}_d(\bar{x}(k)) = \sup_{\substack{\forall d(s)\\ s\in\mathbb{Z}_{\geq k}}} \bar{K}_d(\bar{x}(k+1)) + 1 \geq$$

$$1 + \sup_{\forall d(s), s\in\mathbb{Z}_{\geq k+1}} \bar{K}_d(\bar{x}(k+1)) = 1 + \bar{K}_*(\bar{x}(k+1)) \quad (36)$$

We present our first FTISS converse Lyapunov function with respect to implication-form FTISS Lyapunov function II.

Theorem 4. *If the discrete time system (2) is FTISS, then it admits an implication-form FTISS Lyapunov function II.*

Proof. From Theorem 3, (4) is weakly robust finite time stable in this case, i.e., (4), the special case of (2), is GFTS per Definition 2 with the settling time function obeying (7). Define $V : \mathbb{R}^n \to \mathbb{R}_+$ by $V(x) = b\bar{K}_*(x)$ with $b > 0$ and $\bar{K}_*$ defined in (34). Then it follows from (35) that V is proper. Further, from (36), when $\bar{x}(k) \neq 0$ there holds

$$V(\bar{x}(k+1)) = b\bar{K}_*(\bar{x}(k+1)) \leq \max\{b(\bar{K}_*(\bar{x}(k)) - 1), 0\}$$
$$= \max\{V(\bar{x}(k)) - b, 0\} \quad (37)$$

Then it follows from (2) and (4) that

$$V(x(k+1)) \leq \max\{V(x(k)) - b, 0\} \quad (38)$$

whenever $|u(k)| \leq \varphi(x(k))$, with φ, $x(k)$ and $u(k)$ defined in (4) and (2), respectively. From Lemma 3, there exists a $\mathcal{K}_\infty$ function $\hat{\rho}$ such that $\varphi(\xi) \geq \hat{\rho}(|\xi|)$ for all $\xi \in \mathbb{R}^n$. Thus, (38) holds whenever $|u(k)| \leq \hat{\rho} \circ \alpha_2^{-1}(V(x(k)))$. Let $\phi_{\text{imp2}} = \alpha_2 \circ \hat{\rho}^{-1}$. Then V is an implication-form FTISS Lyapunov function II for (2). ■

Our second FTISS converse Lyapunov function is with respect to the implication-form FTISS Lyapunov function I.

Theorem 5. *If the discrete time system (2) is FTISS, then it admits an implication-form FTISS Lyapunov function I.*

Proof. Consider (4) which is GFTS by Definition 2. Define $V : \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(\bar{x}(k)) = \sup_{k\in\mathbb{Z}_+} \frac{1 + p_1 k}{1 + p_2 k} \bar{K}_*(\bar{x}(k))^a \quad (39)$$

with $\bar{K}_*$ defined in (34), $1 > p_1 > 0.5 > p_2 > 0$ and $a > 2$. Then it follows from (35) that V is proper.

When $\bar{x}(k) = 0$, we have $\bar{x}(k+1) = 0$ by (4), implying $V(\bar{x}(k+1)) = 0$. When $\bar{x}(k) \neq 0$, $\bar{K}_*(\bar{x}(k)) \in \mathbb{Z}_{\geq 1}$, and we consider two case: 1) $\bar{x}(k+1) = 0$; and 2) $\bar{x}(k+1) \neq 0$. In the former case there holds $V(\bar{x}(k+1)) = 0$. In the latter case, suppose $V(\bar{x}(k+1))$ is achieved at $\hat{k}$, we have

$$V(\bar{x}(k+1)) = \frac{1 + p_1\hat{k}}{1 + p_2\hat{k}} \bar{K}_*(\bar{x}(\hat{k}+1))^a =$$

$$\Big(1 - \frac{p_1 - p_2}{(1 + p_1\hat{k} + p_1)(1 + p_2\hat{k})}\Big) \frac{1 + p_1\hat{k} + p_1}{1 + p_2\hat{k} + p_2} \bar{K}_*(\bar{x}(\hat{k}+1))^a$$

$$\leq \Big(1 - \frac{p_1 - p_2}{(1 + p_1\hat{k} + p_1)(1 + p_2\hat{k})}\Big) V(\bar{x}(k)) \quad (40)$$

$$\leq \Big(1 - \frac{p_2(p_1 - p_2)}{p_1(1 + p_2\hat{k})^2}\Big) V(\bar{x}(k)) \quad (41)$$

$$\leq \Big(1 - \frac{p_2(p_1 - p_2)}{p_1\big(1 + p_2\bar{K}_*(\bar{x}(k))\big)^2}\Big) V(\bar{x}(k)) \quad (42)$$

where (40) uses the fact that $V(\bar{x}(k))$ is achieved at $\hat{k} + 1$, (41) uses $\frac{p_1}{p_2} > 1 + p_1$, and (42) uses the fact that $\hat{k} < \bar{K}_*(\bar{x}(k))$, otherwise $\bar{x}(\hat{k}+1) = 0$, leading to $V(\bar{x}(k+1)) = 0$, which contradicts our assumption. As in this case $\bar{x}(k) \neq 0$, we have $\bar{K}_*(\bar{x}(k)) \geq 1$. From (39), $V(\bar{x}(k)) \geq \bar{K}_*(\bar{x}(k))^a$ and

$$\begin{aligned}(1 + p_2\bar{K}_*(\bar{x}(k)))^a &\leq \big(\bar{K}_*(\bar{x}(k)) + p_2\bar{K}_*(\bar{x}(k))\big)^a \\ &\leq (1 + p_2)^a V(\bar{x}(k)) \quad (43)\end{aligned}$$

With (43), (42) becomes

$$\begin{aligned} V(\bar{x}(k+1)) &= V(\bar{x}(k)) - \\ &\frac{p_2(p_1-p_2)}{p_1} V(\bar{x}(k))\big(1+p_2\bar{K}_*(\bar{x}(k))\big)^{-2} \\ &\leq V(\bar{x}(k)) - \frac{p_2(p_1-p_2)}{p_1(1+p_2)^2} V(\bar{x}(k)) V(\bar{x}(k))^{-\frac{2}{a}} \\ &= V(\bar{x}(k)) - cV(\bar{x}(k))^{\bar{a}} \end{aligned} \quad (44)$$

where in (44) $c = \frac{p_2(p_1-p_2)}{p_1(1+p_2)^2} > 0$ and $0 < \bar{a} = \frac{a-2}{a} < 1$.

Following the steps in Theorem 4, V is an implication-form FTISS Lyapunov function I for (2) by Definition 5. ■

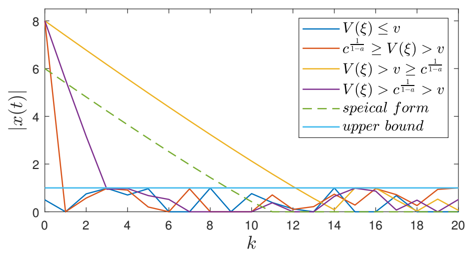

Fig. 1: Trajectories of states of the form (2) using different parameters and the state of the special form (4).

Consider the following scalar discrete time system

$$\begin{aligned} x(k+1) = \max\big\{&x(k) - c\,\text{sign}(x(k))\min\{|x(k)|/c, \\ &|x(k)|^a\}, \sin(v)\big\},\ k \in \mathbb{Z}_+, \end{aligned} \quad (45)$$

where $c > 0, a \in (0,1)$ and $u(k) = \sin(v)$ with v randomly generated between 0 and k. Let $V = |\cdot|$ be the Euclidean norm. Then there holds

$$\begin{aligned} &V(x(k+1)) = |x(k+1)| = \\ &|\max\big\{x(k) - c\,\text{sign}(x(k))\min\{|x(k)|/c, |x(k)|^a\}, \sin(v)\big\}| \leq \\ &\max\Big\{\underbrace{|x(k) - c\,\text{sign}(x(k))\min\{|x(k)|/c, |x(k)|^a\}|}_{(A)}, |\sin(v)|\Big\}. \end{aligned}$$

It follows from (18) in [14] that either $(A) = 0$ when $|x(k)| \leq c^{\frac{1}{1-a}}$, or $(A) = |x(k)(1 - c|x(k)|^{a-1})|$ when $|x(k)| > c^{\frac{1}{1-a}}$. In the latter case $(A) \leq |x(k)||1 - c|x(k)|^{a-1}| = |x(k)|(1 - c|x(k)|^{a-1})$ where the equality results from $1 - c|x(k)|^{a-1} > 0$ when $|x(k)| > c^{\frac{1}{1-a}}$. Thus, in both cases we have

$$V(x(k+1)) \leq \max\{V(x(k)) - cV(x(k))^a, |\sin(v)|\}. \quad (46)$$

Thus from Definition 5 that V is a max-form FTISS Lyapunov function I for the considered system in question.

Figure 1 depicts trajectories of states of the discrete time system (2) using different parameters ($\xi = 0.5, 8, 8$ and 8, $c = 2, 4, 0.5$ and 2, $a = 0.1, 0.5, 0.1$ and 0.1). These cover all cases of (23) in Theorem 1, as well as the trajectory of the state of the special form (4). According to Figure 1, cases $V(\xi) > v \geq c^{\frac{1}{1-a}}$ and $v > c^{\frac{1}{1-a}} > v$ need 13 and 3 rounds to drop below the upper bound, respectively, while the theoretical time needed using (23) are 26 and 5, respectively. One can capture (4) by choosing $d(k)$ in (4) as $\sin(v)$ defined in (45), and $\varphi(\bar{x}(k))$ as $\varphi(a) = b|a|$ for all $a \in \mathbb{R}$ with $|\cdot|$ the Euclidean norm and b a small positive real number such that (26) is satisfied. It can be seen in Figure 1 that the trajectory of (4) converges to zero within 12 rounds.

V. Conclusion

This paper has addressed the notion of FTISS of discrete time autonomous systems. Specifically, Lyapunov functions are provided to show the FTISS of discrete time systems, and the relations between those Lyapunov functions are also characterized. This paper further proves that a feedback form of the discrete time system is finite time stable $if$the original system is FTISS, and the settling time of this feedback system is used to provide two converse Lyapunov theorems.

Our future work would be developing Lyapunov-based small gain theorems for networked discrete time systems and their applications in distributed algorithms with finite time convergence, e.g., [20].

References

[1] Y. Mo and S. Dasgupta, "Small gain conditions for finite time input-to-state stability of interconnected discrete time systems," in *2022 IEEE 61st Conference on Decision and Control (CDC)*, 2022, pp. 7714–7719.
[2] E. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.
[3] L. Praly and Z. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics," *Systems & Control Letters*, vol. 21, no. 1, pp. 19–33, 1993.
[4] L. Praly and Y. Wang, "Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability," *Mathematics of Control, Signals and Systems*, vol. 9, pp. 1–33, 1996.
[5] R. Li, Q. Zhang, and T. Chu, "Input-output-to-state stability of systems related through simulation relations," *SIAM Journal on Control and Optimization*, vol. 59, no. 1, pp. 614–634, 2021.
[6] Z. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.
[7] R. Geiselhart and N. Noroozi, "Equivalent types of ISS Lyapunov functions for discontinuous discrete-time systems," *Automatica*, vol. 84, pp. 227–231, 2017.
[8] N. Noroozi, R. Geiselhart, L. Grüne, B. Rüffer, and F. Wirth, "Non-conservative discrete-time ISS small-gain conditions for closed sets," *IEEE Transactions on Automatic Control*, vol. 63, no. 5, pp. 1231–1242, 2017.
[9] A. Mironchenko, *Input-to-State Stability*, Springer Cham, 2023.
[10] Y. Hong, Y. Xu, and J. Huang, "Finite-time control for robot manipulators," *Systems & Control Letters*, vol. 46, no. 4, pp. 243–253, 2002.
[11] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.
[12] W. Haddad and A. L'Afflitto, "Finite-time partial stability and stabilization, and optimal feedback control," *Journal of the Franklin Institute*, vol. 352, no. 6, pp. 2329–2357, 2015.
[13] T. Rajpurohit and W. Haddad, "Stochastic finite-time partial stability, partial-state stabilization, and finite-time optimal feedback control," *Mathematics of Control, Signals, and Systems*, vol. 29, no. 2, pp. 10, 2017.
[14] W. Haddad and J. Lee, "Finite-time stability of discrete autonomous systems," *Automatica*, vol. 122, pp. 109282, 2020.
[15] Y. Hong, Z. Jiang, and G. Feng, "Finite-time input-to-state stability and applications to finite-time control design," *SIAM Journal on Control and Optimization*, vol. 48, no. 7, pp. 4395–4418, 2010.
[16] X. He, X. Li, and S. Song, "Finite-time input-to-state stability of nonlinear impulsive systems," *Automatica*, vol. 135, pp. 109994, 2022.
[17] S. Liang and J. Liang, "Finite-time input-to-state stability of nonlinear systems: the discrete-time case," *International Journal of Systems Science*, pp. 1–11, 2022.
[18] Y. Mo, S. Dasgupta, and J. Beal, "Stability and resilience of distributed information spreading in aggregate computing," *IEEE Transactions on Automatic Control*, 2022.
[19] E. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–360, 1995.
[20] Y. Mo, S. Dasgupta, and J. Beal, "Robustness of the adaptive Bellman–Ford algorithm: Global stability and ultimate bounds," *IEEE Transactions on Automatic Control*, vol. 64, no. 10, pp. 4121–4136, 2019.