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Distributed optimal formation control of second-order multiagent systems with obstacle avoidance

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Abstract— **This paper formulates a class of generic optimal formation control problems for second-order multiagent systems, where agents are steered to achieve the optimal formation determined by a convex optimization problem with generic formation constraints and admissible range constraints. These constraints determine the geometric pattern and limit the range of the optimal formation, respectively. A generic optimal algorithm based on the primal-dual dynamics is proposed for various formation requirements. Based on Lyapunov stability and optimization theories, the states of the second-order multiagent system are shown to converge to the optimal solutions. Moreover, an obstacle avoidance mechanism based on the control barrier function is introduced to make our algorithm more practical. Finally, numerical simulations illustrate the effectiveness of the proposed algorithm.**

I. Introduction

Distributed formation control for various dynamical systems, e.g., second-order and Euler-Lagrange systems, has received increasing attention due to its wide applications in surveillance, disaster management, exploration, etc [1], [2].

The aim of formation control is to steer a group of agents to achieve the target formation determined by two aspects: 1) the desired geometric pattern and 2) the target geometric parameters including the centroid position, orientation, formation scale, etc. Different control approaches corresponding to the inter-agent constraints' invariance to translation, rotation, and scaling have been proposed to achieve the target formation. For example, the displacement-based method with Laplacian constraint achieves a target formation with translation invariance [3], [4]. The geometric parameter free to be determined in this method is the centroid position. Similarly, the distance-based method [5], [6] and the bearing-based method [7]–[9] can control the orientation and formation scale since the distance and bearing constraints have rotation and scaling invariance, respectively. Recently, stress matrix constraint has been proposed in [10] to characterize the affine span which is invariant to translation, rotation, scaling, and shear. Affine maneuvering for the second-order and unicycle agents is achieved by this method in [11]. The general approach adopted in these works is to control a few leaders to achieve target formation and the other agents to follow the leaders, which requires an extra control layer superimposed on the formation controller. Moreover, optimal performance is not considered in these approaches, e.g., they may fail to tackle the minimum distance problem to the initial formation [12] or the optimal coverage problem [13].

To achieve optimal performance, distributed optimization approaches can be used by encoding optimal geometric parameters into the cost functions and the desired geometric pattern as the formation constraints [14]–[19]. For example, a distributed coordination rule is proposed to steer agents to rendezvous at a point determined by strongly convex cost functions in [14]. In [16], the authors investigate the optimal rendezvous problem by considering not only Laplacian constraint but also local inequality constraints. However, the cost functions are required to contain certain strongly convex terms. Ref. [17] studies a distributed optimal translation formation problem for Euler-Lagrange systems by combining distributed optimization and adaptive control techniques. It is noted that the abovementioned works only deal with the standard Laplacian constraint. This means that they can only achieve optimal centroid position. To achieve the target formation with other geometric parameters, the stress matrix constraint is introduced in [18], so that the optimal orientation, scale, and degree of shear can also be achieved. A distributed formation control law to achieve collision avoidance is proposed in [19] by constructing novel safety barrier certificates for Euler-Lagrangian systems. Other related works on distributed optimization algorithms for second-order systems can be found in [20]–[22], where resource allocation problems are solved. However, the formulations are different from those of the optimal formation problems studied in this paper.

The algorithms proposed in all the abovementioned researches only target one specific formation pattern, such as rendezvous, translation, or affine formation. Moreover, strictly or strongly convex cost functions are required in these works and obstacle avoidance is less concerned except [19].

Our main purpose in this paper is to study a generic optimal formation control approach that can be applied to various formation requirements for second-order systems with obstacle avoidance in a unified way. The contributions of the paper include: **1)** We formulate a general formation problem to describe a wide range of formation tasks with varying feasible formation sets, including the translation formation set, translation and scaling formation set, affine formation set, and so on. By fully exploiting the relationships between the formation constraints and formation sets, we can deal with different formation problems in a unified

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manner. The proposed generic formation includes those in [14]–[19] as special cases. **2)** A distributed optimal formation controller is proposed for the second-order system. The proposed controller is a modified primal-dual algorithm, which consists of a primal variable update for seeking the optimal formation, and a dual variable update for the satisfactions of formation constraints and admissible range constraints. A novel Lyapunov function is adopted to prove the convergence of the overall closed-loop system. Then, by incorporating the control barrier function (CBF) constraints into the quadratic program (QP) to best approximate the optimal formation controller, obstacle avoidance can be achieved. Compared with [14], [16], [18], which can only deal with the first-order dynamics, the proposed algorithm can overcome the difficulty caused by the system inertia so that the stability and optimality of the equilibrium point can be ensured for second-order dynamics simultaneously. Moreover, the proposed algorithm relaxes the strict or strong convexity assumption on the cost functions required in [14]–[19] to be just convex.

This paper is organized as follows. Section II formulates the optimal formation control problem. Section III proposes a continuous-time algorithm to solve the optimal formation problem with obstacle avoidance. Section IV analyzes the convergence of the proposed algorithm. Section V provides simulation results. Finally, Section VI concludes this paper.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Notations and Preliminaries

$\mathbb{R}$ and $\mathbb{R}_+$ represent the real and the nonnegative real numbers sets, respectively. 1_n and 0_n denote the n-dimensional vector with all 1 and 0 entries, respectively. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. $\mathrm{col}(x_1,\dots,x_n) = [x_1^T,\dots,x_n^T]^T$. $\mathrm{blkdiag}(A_1,\dots,A_n)$ is a block diagonal matrix with matrices $A_1,\dots,A_n$. $\otimes$ and $\circ$ denote the Kronecker product and the Hadamard product, respectively. $\|x\|$ represents the standard Euclidean norm of x. Given a function $h:\mathbb{R}^n\to\mathbb{R}$, $\mathcal{Q}_h := \{x\in\mathbb{R}^n : h(x)\geq 0\}$ and $\mathcal{Q}_{h,\sigma} := \{x\in\mathbb{R}^n : h(x)\geq \sigma\}$. $L_f h$ denotes the lie derivative of h with respect to f, i.e., $L_f h = \frac{\partial h}{\partial x} f(x)$.

The gradient of a function f with respect to $x\in\mathbb{R}^n$ is denoted by $\nabla f(x)$. A differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is convex over a convex set $K\subseteq\mathbb{R}^n$ if $f(x_1)-f(x_2)\geq \nabla f(x_2)^T(x_1-x_2), \forall x_1,x_2\in K$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is Lipschitz over a set S if $\|f(x_1)-f(x_2)\|\leq\tau\|x_1-x_2\|$, $\forall x_1,x_2\in S$, where $\tau>0$ is the Lipschitz constant.

The projection of x on a closed convex set K is defined by $P_K(x) := \arg\min_{y\in K}\|x-y\|$, having a basic property

$$(x-P_K(x))^T(y-P_K(x))\leq 0,\ \ \forall x\in\mathbb{R}^n, \forall y\in K. \quad (1)$$

B. Graph Theory and Generic Formation

Consider a group of N agents in $\mathbb{R}^n$. To describe the interaction among agents, an undirected graph is defined as $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\dots,N\}$ denotes the node set, and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ denotes the edge set. The edge $(i,j)\in\mathcal{E}$ indicates that agent i and j can communicate with each other. $\mathcal{N}_i$ denotes the set of neighbors of agent i.

TABLE I: Examples of generic formation patterns

H	d	Invariance	The described image $\mathcal{A}(p^\star)$
Standard Laplacian [4]	arbitray in $\mathbb{R}^{Nn}$	translation	$\{x\in\mathbb{R}^{Nn} \| x=p^\star+1_N\otimes b, b\in\mathbb{R}^n\}$
Bearing Laplacian [8]	0_{Nn}	translation, scaling	$\{x\in\mathbb{R}^{Nn} \| x=ap^\star+1_N\otimes b, a\in\mathbb{R}_+, b\in\mathbb{R}^n\}$
Stress matrix [11]	0_{Nn}	translation, rotation, scaling, shear	$\{x\in\mathbb{R}^{Nn} \| x=(I_N\otimes A)p^\star+1_N\otimes b, A\in\mathbb{R}^{n\times n}, b\in\mathbb{R}^n\}$

Assumption 1: $\mathcal{G}$ is undirected and connected.

Definition 1 (Generic Formation Pattern): A feasible image determined by the following equation

$$H(G(p^\star))x = d(p^\star) \quad (2)$$

with the nominal formation $p^\star=[p_1^{\star T},\dots,p_N^{\star T}]^T$ is called a generic formation pattern, if $d(p^\star)\in\mathbb{R}^{Nn}$ is a constant vector, $H(G(p^\star))\in\mathbb{R}^{Nn\times Nn}$ is a positive-semidefinite matrix with the ijth block submatrix as

$$[H(G(p^\star))]_{ij} = \begin{cases} \sum_{j\in\mathcal{N}_i} G_{ij}(p^\star), & i=j, \\ -G_{ij}(p^\star), & i\neq j, (i,j)\in\mathcal{E}, \\ 0_{n\times n}, & i\neq j, (i,j)\notin\mathcal{E}, \end{cases} \quad (3)$$

where $G_{ij}(p^\star)\in\mathbb{R}^{n\times n}$ is a matrix characterizing the interconnection between agent i and agent j in the nominal formation.

We emphasize that Definition 1 gives different desired geometric patterns by specifying H and d, some of which are summarized in Table. I. It should be noted that additional conditions on topology connectivity in Assumption 1 might be needed to determine some of the formation patterns in Table. I. For example, infinitesimal bearing rigidity and universal rigidity are required for bearing Laplacian and stress matrix, respectively [8], [11].

C. Problem Formulation

In this paper, we consider a distributed optimal formation problem for a second-order multiagent system as follows

$$\begin{aligned} \min \quad & f(x)=\textstyle\sum_{i=1}^N f_i(x_i) \\ \text{s.t.} \quad & H(G(p^\star))x=d(p^\star), \\ & B_i x_i\leq c_i, \qquad \forall i\in\mathcal{V}, \end{aligned} \quad (4)$$

where $x=\mathrm{col}(x_1,x_2,\dots,x_N)$ is the stack vector of all agents' positions, $f_i(x_i)$ is the local cost function of agent i. The equality imposes the generic formation constraint. For convenience, we write $H(G(p^\star))$ as H and $d(p^\star)$ as d in the sequel. The local inequality constraints restrict the workspace of agents with $B_i\in\mathbb{R}^{m_i\times n}$ and $c_i\in\mathbb{R}^{m_i}$. Each agent is driven by the second-order dynamic

$$\dot{x}_i=v_i,\quad \dot{v}_i=u_i, \quad (5)$$

where x_i, v_i, u_i are the position, velocity, and control input of agent i, respectively.

Our main goal in this paper is to design the control law u_i for each agent to achieve the optimal positions while forming the desired geometric pattern by solving (4).

Assumption 2: The local cost function $f_i(x_i)$ of each agent is differentiable and convex on $\mathbb{R}^n$. $\nabla f_i(x_i)$ is τ_i-Lipschitz on $\mathbb{R}^n$, where $\tau_i > 0$.

Assumption 2 implies that problem (4) is convex, and ensures that the optimal solution exists according to [23].

III. DISTRIBUTED CONTINUOUS-TIME ALGORITHM DESIGN

In this section, we propose a distributed optimal formation control algorithm to solve problem (4). The obstacle avoidance mechanism based on the CBF method is also given.

A. Algorithm Design

The Lagrangian function of problem (4) is given by

$$\begin{aligned} L(x,\lambda,e) =& f(x) + \sum_{i=1}^{N} \lambda_i^T (\sum_{j\in\mathcal{N}_i} G_{ij}(x_i - x_j) - d_i) \\ &+ \sum_{i=1}^{N} e_i^T (B_i x_i - c_i), \end{aligned} \tag{6}$$

where $\lambda_i \in \mathbb{R}^n$ and $e_i \in \mathbb{R}_+^{m_i}$ are the Lagrangian multipliers. Denote $\lambda = \mathrm{col}(\lambda_1,\dots,\lambda_N) \in \mathbb{R}^{Nn}$ and $e = \mathrm{col}(e_1,\dots,e_N) \in \mathbb{R}_+^{\sum_{i=1}^N m_i}$. Then, the optimal conditions of problem (4) are given as follows (see [23] for a proof):

Lemma 1: A point $x^\star \in \mathbb{R}^{Nn}$ is an optimal solution to problem (4) if and only if there exist $\lambda^\star \in \mathbb{R}^{Nn}$ and $e^\star \in \mathbb{R}_+^{\sum_{i=1}^N m_i}$ such that

$$\nabla f(x^\star) + H\lambda^\star + B^T e^\star = 0,\ Hx^\star = d,$$
$$B_i x_i^\star - c_i \le 0,\quad e_i^\star \circ (B_i x_i^\star - c_i) = 0,\quad \forall i \in \mathcal{V}.$$

Based on the optimal conditions in Lemma 1, we design the control input u_i of each agent as follows:

$$\begin{aligned} u_i =& - k_1 v_i - q_i - \sum_{j\in\mathcal{N}_i} G_{ij}(\lambda_i - \lambda_j) - B_i^T e_i \\ &- \sum_{j\in\mathcal{N}_i} G_{ij}(x_i - x_j + v_i - v_j) + d_i, \end{aligned} \tag{7a}$$

$$\dot{\lambda}_i = k_2 (\sum_{j\in\mathcal{N}_i} G_{ij}(x_i - x_j) + \sum_{j\in\mathcal{N}_i} G_{ij}(v_i - v_j) - d_i), \tag{7b}$$

$$\dot{\mu}_i = \frac{1}{2}(-\mu_i + e_i), \tag{7c}$$

$$e_i = P_{\mathbb{R}_+^{m_i}}[\mu_i + B_i(x_i + v_i) - c_i], \tag{7d}$$

where $q_i = \nabla f_i(x_i + v_i)$, k_1 and k_2 are the positive scalars.

Our algorithm is motivated by the primal-dual dynamics in [24]. Since the algorithm in [24] is proposed for the first-order dynamics and can only deal with the strictly convex cost function, it can not be directly applied to our optimal formation problem. In this case, we modify the algorithm in [24] to include a primal variable update for seeking the optimal formation, and a dual variable update for the satisfactions of formation constraints and admissible range constraints. In both variable updates, we introduce the velocity feedback terms to adapt to the second-order dynamics. Here, (7a) is the gradient descent of the Lagrangian function with respect to the primal variable x_i to guarantee the convergence of x_i to the optimal solution. The extra term $\sum_{j\in\mathcal{N}_i} G_{ij}(x_i - x_j + v_i - v_j) - d_i$ is equal to $\dot{\lambda}_i/k_2$, which can be viewed as a proportional compensation, together with the integral compensation $\sum_{j\in\mathcal{N}_i} G_{ij}(\lambda_i - \lambda_j)$ to tackle the formation constraint and only convex function. (7b) is the gradient ascent of (6) with respect to the dual variable λ_i, which is used to satisfy the formation constraint. (7c) and (7d) are the modified gradient ascent-based update rules to ensure that the dual variable e_i corresponding to the inequality is non-negative, where $\mu_i \in \mathbb{R}^{m_i}$ is an auxiliary variable for tracking e_i. Moreover, to deal with the second-order dynamics, the terms corresponding to v_i are added to our algorithm as state feedback for stability and $\nabla f_i(x_i)$ is modified as $\nabla f_i(x_i + v_i)$ in (7a) compared to the first-order dynamics-based algorithms.

System (5) with algorithm (7) is rewritten in a concatenated form as follows:

$$\dot{x} = v, \tag{8a}$$
$$\dot{v} = - k_1 v - q - H\lambda - B^T e - [H(x+v) - d], \tag{8b}$$
$$\dot{\lambda} = k_2 [H(x+v) - d], \tag{8c}$$
$$\dot{\mu} = \frac{1}{2}(-\mu + e), \tag{8d}$$
$$e = P_{\mathbb{R}_+^M}[\mu + B(x+v) - c], \tag{8e}$$

where $M = \sum_{i=1}^N m_i$, $q = \mathrm{col}(q_1,\dots,q_N) \in \mathbb{R}^{Nn}$, $v = \mathrm{col}(v_1,\dots,v_N) \in \mathbb{R}^{Nn}$, $d = \mathrm{col}(d_1,\dots,d_N) \in \mathbb{R}^{Nn}$, $\mu = \mathrm{col}(\mu_1,\dots,\mu_N) \in \mathbb{R}^M$, $c = \mathrm{col}(c_1,\dots,c_N) \in \mathbb{R}^M$, $B = \mathrm{blkdiag}(B_1,\dots,B_N) \in \mathbb{R}^{M\times Nn}$.

B. Optimality Analysis

The following lemma gives the relationship between the equilibrium of closed-loop system (8) and the optimal solution to problem (4).

Lemma 2: Suppose Assumptions 1-2 hold. $\overline{x}$ is an optimal solution to problem (4) if and only if there exist $\overline{v} \in \mathbb{R}^{Nn}$, $\overline{\lambda} \in \mathbb{R}^{Nn}$ and $\overline{\mu} \in \mathbb{R}^M$ such that $(\overline{x},\overline{v},\overline{\lambda},\overline{\mu})$ is an equilibrium of closed-loop system (8).

Proof: *Sufficiency*: Let $(\overline{x},\overline{v},\overline{\lambda},\overline{\mu})$ be an equilibrium point of system (8). It satisfies

$$0 = \overline{v}, \tag{9a}$$
$$0 = -\overline{q} - H\overline{\lambda} - B^T\overline{e} - k_1\overline{v} - [H(\overline{x}+\overline{v}) - d], \tag{9b}$$
$$0 = k_2 [H(\overline{x}+\overline{v}) - d], \tag{9c}$$
$$0 = \frac{1}{2}(-\overline{\mu} + \overline{e}), \tag{9d}$$

where $\overline{q} = \nabla f(\overline{x}+\overline{v}) = \nabla f(\overline{x})$.

From (9a), (9b) and (9c), we have $-\nabla f(\overline{x}) - H\overline{\lambda} - B^T\overline{e} = 0$. From (9a) and (9c), we have $H\overline{x} = d$. Furthermore, (9d) implies that $\overline{e} = \overline{\mu}$. Since $\overline{e}_i = P_{\mathbb{R}_+^{m_i}}[\overline{\mu}_i + B_i(\overline{x}_i + \overline{v}_i) - c_i]$, $\overline{e}_i = \overline{\mu}_i$ and $\overline{v}_i = 0$, we have the following two cases: 1) $\overline{\mu}_i + B_i\overline{x}_i - c_i \ge 0 \Rightarrow \overline{e}_i \ge 0$ and $B_i\overline{x}_i - c_i = 0$, 2) $\overline{\mu}_i + B_i\overline{x}_i - c_i < 0 \Rightarrow \overline{e}_i = 0$ and $B_i\overline{x}_i - c_i < 0$. It can be concluded that $\overline{e}_i \ge 0$, $B_i\overline{x}_i - c_i \le 0$ and $\overline{e}_i \circ (B_i\overline{x}_i - c_i) = 0$. By Lemma 1, $\overline{x}$ is an optimal solution to problem (4).

Necessity: For an optimal solution $\overline{x}$, one has $H\overline{x} = d$ and $B\overline{x} - c \le 0$. Furthermore, according to Lemma 1, there exist $\overline{\lambda} \in \mathbb{R}^{Nn}$ and $\overline{e} \in \mathbb{R}_+^M$ such that $\nabla f(\overline{x}) + H\overline{\lambda} + B^T\overline{e} = 0$, $\overline{e} \circ (B\overline{x} - c) = 0$. Let $\overline{\mu}_i = \overline{e}_i, \forall i \in \mathcal{V}$, then there are two cases to be considered: 1) If $B_i\overline{x}_i - c_i = 0$, we obtain

$\overline{\mu}_i = \overline{e}_i \geq 0$; 2) If $B_i\overline{x}_i - c_i < 0$, then $\overline{\mu}_i = \overline{e}_i = 0$. From the above, it can be concluded that $\overline{e}_i = P_{\mathbb{R}_+^{m_i}}[\overline{\mu}_i + B_i\overline{x}_i - c_i]$. Since $\overline{v} = 0$, we have $-\nabla f(\overline{x}+\overline{v}) - H\overline{\lambda} - B^T\overline{e} - k_1\overline{v} - [H(\overline{x}+\overline{v}) - d] = 0$, $H(\overline{x}+\overline{v}) - d = 0$, $-\overline{\mu} + \overline{e} = 0$, where $\overline{e} = P_{\mathbb{R}_+^M}[\overline{\mu} + B(\overline{x}+\overline{v}) - c]$. Hence, $(\overline{x}, \overline{v}, \overline{\lambda}, \overline{\mu})$ is an equilibrium point of system (8). ■

C. Obstacle Avoidance Mechanism

In practical situations, obstacle avoidance is necessary for physical safety. To this end, the control barrier function (CBF) method is adopted for obstacle avoidance during the achievement of the optimal formation.

The multiagent system is said to be safe if $Z_i(x_i) = \|x_i - x_{obs}\|^2 - \rho^2 \geq 0,\ \forall i \in \mathcal{V}$ with ρ being the safe distance between the agent i and the obstacle x_{obs}. Then we define the safe region of admissible states $\mathcal{Q}_{Z_i} = \{x_i \in \mathbb{R}^n | Z_i(x_i) \geq 0\}$ for the agent i. Note that $Z_i(x_i)$ for the second-order system is of relative degree 2, the traditional CBF in [25] is not applicable since the control input will vanish in this case. Thus, we design a high-order CBF (HOCBF) as in [26]:

$$h_i(x_i) = \begin{cases} (Z_i(x_i)/\sigma_i - 1)^3 + 1, & Z_i(x_i) \leq \sigma_i, \\ 1, & Z_i(x_i) > \sigma_i, \end{cases} \tag{10}$$

where σ_i is a positive constant.

Proposition 1: Consider the second-order dynamics (5) and the safety region $\mathcal{Q}_{Z_i}$, the function h_i is a HOCBF where h_i denotes $h_i(x_i)$.

The proof is similar to [26, Proposition 4] and hence omitted here.

For obstacle avoidance, it suffices to find an admissible distributed control law $\tau_i \in U_i(x_i)$ where

$$\begin{aligned} U_i(x_i) = \{\tau_i \in \mathbb{R}^n | \mathcal{L}_{f_i}\dot{h}_i + \mathcal{L}_{g_i}\dot{h}_i\tau_i + \frac{\partial\alpha(h_i)}{\partial h_i}\dot{h}_i \\ + \beta(\dot{h}_i + \alpha(h_i)) \geq 0\}, \end{aligned} \tag{11}$$

with $f_i = [v_i, 0]^T$, $g_i = [0, 1]^T$ and extended class-$\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$. However, the complex nonlinearity with x_i in (11) brings the difficulty to design an analytical controller. The basic idea is to regard u_i in (7) as a nominal controller and modify it as little as possible to ensure the original optimal formation control objective. Thus, we rewrite $\tau_i \in U_i(x_i)$ equivalently as a linear constraint $A_i\tau_i \leq \nu_i$, where $A_i = -\mathcal{L}_{g_i}\dot{h}_i$ and $\nu_i = \mathcal{L}_{f_i}\dot{h}_i + \frac{\partial\alpha(h_i)}{\partial h_i}\dot{h}_i + \beta(\dot{h}_i + \alpha(h_i))$. Then, a QP problem is constructed as [27], i.e.,

$$\begin{aligned} \tau_i^\star = \text{argmin} \quad & \|\tau_i - u_i\|^2 \\ \text{s.t.} \quad & A_i\tau_i \leq \nu_i \end{aligned} ,\forall i \in \mathcal{V}. \tag{12}$$

IV. STABILITY ANALYSIS

In this part, we analyze the stability of the closed-loop system (8) based on Lyapunov stability theory. Then, we will prove that there is no collision with the obstacle under the controller solved by the QP problem in (12).

Theorem 1: If Assumptions 1-2 are satisfied, the closed-loop system (8) is stable. Moreover, the trajectories of $x(t)$ can converge to an optimal solution to the problem (4), provided that the following condition holds:

$$k_1 > 1, \quad 0 < k_2 < 1/\lambda_N(H), \tag{13}$$

where $\lambda_N(H)$ is the largest eigenvalue of H.

Proof: Define $W(x+v, \lambda, \mu) = f(x+v) + \frac{1}{2}\|e\|^2 + \frac{1}{2}(x+v+\lambda)^T H(x+v+\lambda)$ with $e = P_{\mathbb{R}_+^M}[\mu + B(x+v) - c]$ and

$$\begin{aligned} V_1 =& W(x+v, \lambda, \mu) - W(\overline{x}+\overline{v}, \overline{\lambda}, \overline{\mu}) \\ & - \nabla_{(x+v)}^T W(\overline{x}+\overline{v}, \overline{\lambda}, \overline{\mu})(x - \overline{x} + v - \overline{v}) \\ & - \nabla_\lambda^T W(\overline{x}+\overline{v}, \overline{\lambda}, \overline{\mu})(\lambda - \overline{\lambda}) - \nabla_\mu^T W(\overline{x}+\overline{v}, \overline{\lambda}, \overline{\mu})(\mu - \overline{\mu}). \end{aligned}$$

It is straightforward to observe that the function $W(x+v, \lambda, \mu)$ is convex with respect to $x+v$, λ, μ, and there holds $V_1 \geq 0$ in light of the property of convexity.

Considering the following functions:

$$V_2 = \frac{k_1 - 1}{2}\|x - \overline{x}\|^2 + \frac{k_1 - 1}{2}\|v - \overline{v}\|^2 + \frac{1}{2}\|x - \overline{x} + v - \overline{v}\|^2,$$

$$V_3 = \frac{1}{2}(\lambda - \overline{\lambda})^T M_1 (\lambda - \overline{\lambda}), \quad V_4 = \frac{1}{2}\|\mu - \overline{\mu}\|^2,$$

where $M_1 = \frac{1}{k_2}I - H$, which is positive definite under the second condition in (13). Then, define the Lyapunov function candidate as $V = V_1 + V_2 + V_3 + V_4$, which is positive definite under the condition (13).

The derivative of V_1 along the trajectories of (8) is

$$\begin{aligned} \dot{V}_1 =& (\nabla_{(x+v)}W - \nabla_{(x+v)}\overline{W})^T(\dot{x} + \dot{v}) \\ & + (x - \overline{x} + v - \overline{v} + \lambda - \overline{\lambda})^T H\dot{\lambda} + (e - \overline{\mu})^T\dot{\mu}, \end{aligned} \tag{14}$$

where, for simplicity, $\nabla_{(x+v)}\overline{W}$ denotes $\nabla_{(x+v)}W(\overline{x} + \overline{v}, \overline{\lambda}, \overline{\mu})$. Let ϑ_1, ϑ_2 and ϑ_3 denote the first, second and last terms of $\dot{V}_1$ in (14), respectively.

By expanding the terms in ϑ_1, we have

$$\begin{aligned} \vartheta_1 = & -\|\underbrace{q + B^T e + H\lambda + H(x+v) - d + (k_1 - 1)v}_{\xi}\|^2 \\ & - (k_1 - 1)v^T(v + \dot{v}). \end{aligned}$$

Next, take the derivative of V_2 and V_3 along the trajectories of (8), and denote $S_1 := \vartheta_1 + \dot{V}_2 + \dot{V}_3$, we have

$$\begin{aligned} S_1 =& -\|\xi\|^2 - 2(k_1 - 1)\|v\|^2 - (x - \overline{x} + v - \overline{v})^T(q - \overline{q}) \\ & - (x - \overline{x} + v - \overline{v})^T B^T(e - \overline{\mu}) \\ & - (x - \overline{x} + v - \overline{v})^T H(x - \overline{x} + v - \overline{v} + \lambda - \overline{\lambda}) \\ & + (\lambda - \overline{\lambda})^T M_1\dot{\lambda} \\ \leq & -\|\xi\|^2 - 2(k_1 - 1)\|v\|^2 - (x - \overline{x} + v - \overline{v})^T B^T(e - \overline{\mu}) \\ & - (x - \overline{x} + v - \overline{v})^T H(x - \overline{x} + v - \overline{v} + \lambda - \overline{\lambda}) \\ & + (\lambda - \overline{\lambda})^T M_1\dot{\lambda}, \end{aligned} \tag{15}$$

where the last inequality is obtained by using $-(x - \overline{x} + v - \overline{v})^T(q - \overline{q}) \leq 0$ in light of the convexity of the function $f(\cdot)$. Let ϑ_4 and ϑ_5 denote the last two terms in the last inequality of (15), respectively.

To proceed, it can be calculated that $\vartheta_2 + \vartheta_4 + \vartheta_5 = -(x + v - x^o)^T(H - k_2H^2)(x + v - x^o)$ with $Hx^o = d$. Combining $\vartheta_2 + \vartheta_4 + \vartheta_5$ and (15), we have

$$\begin{aligned} S_1 + \vartheta_2 \leq & -\|\xi\|^2 - 2(k_1 - 1)\|v\|^2 \\ & - (x + v - x^o)^T M_2(x + v - x^o) \\ & - (x - \overline{x} + v - \overline{v})^T B^T(e - \overline{\mu}). \end{aligned} \tag{16}$$

where $M_2 := H - k_2H^2$ is positive semidefinite due to (13).

Let ϑ_6 denote the last term of (16). Then, take the derivative of V_4, and combine it with ϑ_3 and ϑ_6, we have $\dot{V}_4+\vartheta_3+\vartheta_6 \leq -\frac{1}{2}\|e-\mu\|^2$, where the inequality is obtained by using $(e-\overline{\mu})^T[e-\mu-B(x+v)+c] \leq 0$ in light of the projection property (1), and $(e-\overline{\mu})^T[B(\overline{x}+\overline{v})-c] \leq 0$ since $e \geq 0$, $B(\overline{x}+\overline{v})-c \leq 0$, $\overline{\mu}^T[B(\overline{x}+\overline{v})-c]=0$.

To sum up, the derivative of V is calculated as

$$\dot{V} \leq -\|\xi\|^2 - 2(k_1-1)\|v\|^2 - (x+v-x^o)^T M_2 (x+v-x^o) - \frac{1}{2}\|e-\mu\|^2, \quad (17)$$

which is negative under condition (13).

Since V is positive definite, the trajectories $(x(t), v(t), \lambda(t), \mu(t))$ of (8) are bounded for all $t \geq 0$. Next, according to the LaSalle invariance principle [28, Theorem 4.4], (x, v, λ, μ) converge to the largest invariant set in the closure of $\Phi := \{(x, v, \lambda, \mu) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^M : \dot{V}=0\}$. It can be obtained from (17) that $\dot{V}=0$ results in $v=0$, $\xi=(k_1-1)v+q+B^Te+H\lambda+H(x+v)-d=0$, $H(x+v-x^o)=H(x+v)-d=0$, $e-\mu=0$, which means that the point (x, v, λ, μ) satisfying $\dot{V}=0$ is an equilibrium point of system (8). Furthermore, according to Lemma 2, the trajectories $x(t)$ can converge to an optimal solution to problem (4). This completes the proof. ■

Next, we will prove that obstacle avoidance can be achieved successfully under the controller $\tau^\star$ in (12). We make an assumption as in [26]:

Assumption 3: Denote the performance-critical region where the nominal control signal u_i should be utilized as $\mathcal{Q}_{s_i}$. There exists $\sigma_i > 0$ such that $\mathcal{Q}_{s_i} \subseteq \mathcal{Q}_{Z_i,\sigma_i}$, $i \in \mathcal{V}$. And the equilibrium point of system (8) is in the set $\bigcap_{i\in\mathcal{V}} \mathcal{Q}_{s_i}$.

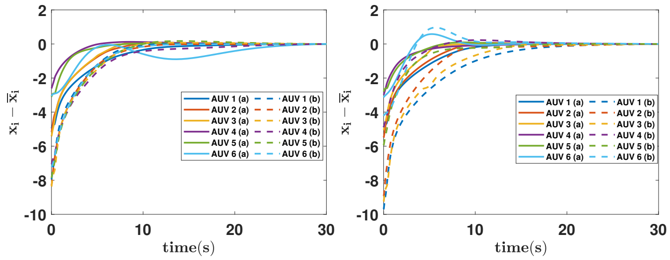
Theorem 2: Consider the second-order dynamics (5) with distributed control law (7) and safe region $\mathcal{Q}_{Z_i}$. The modified control input $\tau_i^\star$ obtained by solving QP (12) is Lipschitz continuous and renders set $\mathcal{Q}_{Z_i}$ forward invariant. Furthermore, if Assumption 3 holds, $\tau_i^\star(x_i) = u_i(x_i)$ for all $x_i \in \mathcal{Q}_{s_i}$, $i \in \mathcal{V}$, and the optimal formation can be achieved while the safety is always guaranteed.

Proof: From the above analysis, it can be obtained that nominal controller u_i in (7) is bounded and locally Lipschitz continuous. Define $\overline{h}_i = \dot{h}_i(x_i) + \alpha(h_i(x_i))$. From the definition of h_i, it can be obtained that $\mathcal{L}_{f_i}\overline{h}_i$ and $\mathcal{L}_{g_i}\overline{h}_i$ are both locally Lipschitz continuous. According to [26, Corollary 1], the control law $\tau_i^\star$ obtained by (12) is Lipschitz continuous and renders the set $\mathcal{Q}_{Z_i}$ forward invariant. Also based on [26, Corollary 1], $\tau_i^\star(x_i) = u_i(x_i)$ for all $x_i \in \mathcal{Q}_{s_i}$, $i \in \mathcal{V}$ under Assumption 3. This means that the nominal controller can be recovered in $\mathcal{Q}_{s_i}$. Thus, the optimal formation can be achieved while safety can be always guaranteed since the equilibrium point of system (8) is in the region $\bigcap_{i\in\mathcal{V}} \mathcal{Q}_{s_i}$. ■

V. SIMULATIONS

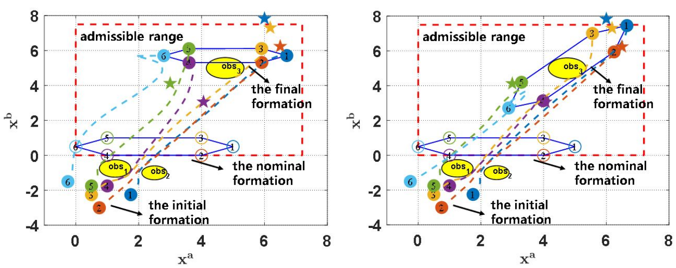
In our simulation, we consider the scenario that six autonomous underwater vehicles (AUVs) operate in a channel or sea-route with mission-driven formation configuration [29]. The mission-driven formation configuration is $p_1^\star = [5, 0.5]^T$, $p_2^\star = [4, 0]^T$, $p_3^\star = [4, 1]^T$, $p_4^\star = [1, 0]^T$, $p_5^\star = [1, 1]^T$, $p_6^\star = [0, 0.5]^T$. There are anchors in the ocean to help AUVs to obtain location information of AUVs themselves. Each AUV needs to be as close as possible to the anchor to mitigate the energy consumption on signal transmitting [30]. Thus, the local cost function of AUV, $i = 1, ..., 5$ is defined as $f_i(x_i) = \|x_i - AP_i\|^2$, $x_i = [x_i^a, x_i^b]^T \in \mathbb{R}^2$ and $AP_i \in \mathbb{R}^2$ denote the positions of AUV i and the corresponding anchor, respectively. $f_6(x_6)$ is set as constant which means AUV 6 loses the information of its anchor. The formation constraints are selected as bearing Laplacian constraints and stress matrix constraints, respectively. Let $B_i = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^T$ and $c_i = [7.2, 0, 7.5, 0]^T$, $i \in \mathcal{V}$, which constrain the admissible working ranges of each AUV. AUVs are required to avoid seafloor obstacles to guarantee safety. There are three obstacles at and positions $[1.25, -0.75]^T$, $[2.5, -1]^T$, $[4.75, 5]^T$ with $\rho_1 = 0.5$, $\rho_2 = 0.4$, $\rho_3 = 0.6$, respectively. The parameters σ_{1_i}, σ_{2_i}, σ_{3_i} in the HOCBF (10) are all chosen as 3, $i \in \mathcal{V}$. For convenience, we define $Zmin_i = \min\{Z_{1_i} - \sigma_{1_i}, Z_{2_i} - \sigma_{2_i}, Z_{3_i} - \sigma_{3_i}\}$.

The simulation results are shown in Figs. 1-3. Fig. 1 illustrates that the states of the AUVs both can converge to the optimal solution $\overline{x}_i$ under different types of formation constraints. Fig. 2 gives the nominal formation, initial formation, and final formation. The five-pointed stars in Fig. 2a and Fig. 2b represent the anchors and the color of each is similar to the corresponding AUV. It can be shown that the final formation under bearing Laplacian constraint can be transformed into the nominal formation by translation and scaling, while the case of stress matrix constraint can be transformed by translation, rotation, scaling. All 6 AUVs can



(a) (b)

Fig. 1: Trajectories of $x_i - \overline{x}_i$ under the bearing Laplacian and stress matrix constraints, respectively. The solid lines denote $x_i^a - \overline{x}_i^a$, and the dashed lines denote $x_i^b - \overline{x}_i^b$.



(a) (b)

Fig. 2: Formations under the bearing Laplacian and stress matrix constraints, respectively.

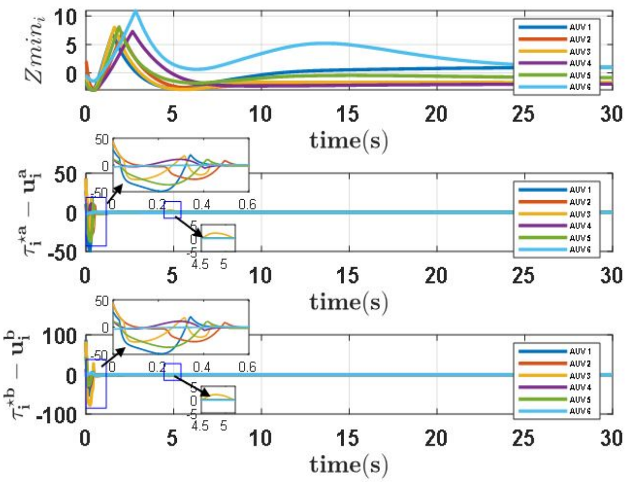


Fig. 3: The relationship between $Zmin_i(t)$ and $\tau_i^\star(t) - u(t)$ under bearing Laplacian constraints.

reach the optimal formation that is closest to the anchors with the desired formation shape in the admissible range in both cases. Collision avoidance is also guaranteed. Fig. 3 shows that the relationship between $Z_i(t) - \sigma_i$ and $\tau_i^\star(t) - u(t)$ for bearing Laplacian constraints case. It can be seen that the actual control input $\tau_i^\star$ coincides with the nominal control input u_i whenever $Z_i(t) \geq \sigma_i = 3$, which validates that the property of the performance-critical region. Similar results can be obtained for the stress matrix constraint case.

VI. Conclusion

In this paper, we formulate a generic formation control problem where agents are required to achieve an optimal formation depending on the optimization problem. Notably, our algorithm can tackle the generally convex functions and various formation constraints, which can be applied to more general formation problems. An optimal formation control law is proposed for the second-order multiagent system and the convergence is proved by Lyapunov theory. By virtue of the CBF-based method, obstacle avoidance is achieved.

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