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Linear-quadratic mean-field-type difference games with coupled affine inequality constraints

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Abstract— **In this paper, we study a class of linear-quadratic mean-field-type difference games with coupled affine inequality constraints. We show that the mean-filed-type equilibrium can be characterized by the existence of a multiplier process which satisfies some implicit complementarity conditions. Further, we show that the equilibrium strategies can be computed by reformulating these conditions as a single large-scale linear complementarity problem. We illustrate our results with an energy storage problem arising in the management of microgrids.**

I. Introduction

Stochastic dynamic games offer a mathematical framework for modeling dynamic decision-making scenarios that involve multiple players interacting in uncertain environments. Mean-field-type dynamic games (MFTDGs) are a specific class of stochastic games that allow for the inclusion of not just the state and control terms, but also their distributions in the objective functionals and state dynamics [1], [2]. As a result, MFTDGs, when the mean and variance terms are considered, are related to the mean-variance paradigm developed by H. Markowitz [3]. MFTDGs differ from mean-field games [4] by accounting for inherent heterogeneities and finite decision-makers, unlike the approximation provided by mean-field games for problems with many symmetric players. MFTDGs have been increasingly utilized to model real-world engineering problems arising in water networks [5], smart-grids [6] and pedestrian flow [7]; see [2] and [8] for a detailed coverage.

MFTDGs have been solved using various approaches in the literature, including the stochastic maximum principle [9], dynamic programming [10], and direct method [11], [12]. In [13], the authors studied MFTDGs involving equality constraints on the control variables. Multi-agent decision problems in engineering and economics require incorporating inequality constraints on state and control variables like capacity, saturation, and budget constraints. The dynamic nature of these constraints poses technical challenges in characterizing admissible controls and establishing solvability conditions for equilibria, distinguishing them from unconstrained counterparts. Despite the significance of these challenges, the literature on MFTDGs involving inequality constraints is scarce, to the best of our knowledge.

This paper proposes a solution for linear-quadratic MFTDGs with inequality constraints. We focus on a discrete-time setting with finite horizon, scalar state dynamics, and quadratic objectives. Our approach incorporates coupled affine inequality constraints on the mean values of state and control variables. Using the direct method, also known as completion of squares, we establish a connection between the existence of a solution for these MFTDGs and the existence of a multiplier process satisfying implicit complementarity conditions. By leveraging an approach similar to [14], we transform these existence conditions into the solvability of a single large-scale linear complementarity problem, thereby providing a computational method for solving these games. The proposed approach can be easily extended to matrix-valued settings.

This paper is organized as follows. In Section II, we introduce a class of MFTDGs with coupled affine inequality constraints. In Section III, we provide conditions for the existence of an equilibrium in these games. In Section IV, we present an approach for reformulating these conditions as a linear complementarity problem. In Section V, we illustrate our method with numerical simulations and finally Section VI concludes.

Notation: We denote the transpose of any vector a or matrix A by a' and A' respectively. The identity matrix and the matrix of zeros are represented by $\mathbf{I}$ and $\mathbf{0}$, respectively, with dimensions determined from the context. $\mathbb{E}[x]$ denotes the expected value of x. Let $A \in \mathbb{R}^{n\times n}$ be partitioned as $n = n_1 + \cdots + n_K$. We represent $[A]_{ij}$ as the $n_i \times n_j$ sub-matrix associated with indices n_i (row) and n_j (column). We denote the column vector $[v_1', \cdots, v_n']'$ by $\mathrm{col}(v_k)_{k=1}^n$ and the row vector $[v_1 \ \cdots \ v_n]$ by $\mathrm{row}(v_k)_{k=1}^n$. The block diagonal matrix obtained by taking the matrices $M_1, \cdots, M_K$ as diagonal elements in this sequence, is represented by $\oplus_{k=1}^K M_k$. We represent the Kronecker product operation by $\otimes$. We call two vectors $x, y \in \mathbb{R}^n$ complementary if $x \geq 0,\ y \geq 0$ and $x'y = 0$, and we compactly denote these conditions by $0 \leq x \perp y \geq 0$.

II. Preliminaries

In this section we introduce a class of N-player scalar finite-horizon mean-field-type difference game with inequality constraints (MFTDGC). We denote the set of players by $\mathsf{N} = \{1, 2, \cdots, N\}$, the set of time instants or decision stages by $\mathsf{K} = \{0, 1, ..., K\}$. We define the following two sets as $\mathsf{K}_l := \mathsf{K} \setminus \{K\}$ and $\mathsf{K}_r := \mathsf{K} \setminus \{0\}$. At each time instant $k \in \mathsf{K}_l$, each player $i \in \mathsf{N}$ chooses an action $u_k^i \in \mathbb{R}$ and influences the evolution of state variable $x_k \in \mathbb{R}$ according to the following discrete-time linear dynamics

$$x_{k+1} = a_k x_k + \bar{a}_k \mathbb{E}[x_k] + \sum_{i\in\mathsf{N}} (b_k^i u_k^i + \bar{b}_k^i \mathbb{E}[u_k^i]) + c_k + \sigma_k w_k, \tag{1a}$$

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where the initial state x_0 is a scalar random variable with known finite mean and known finite variance, and a_k, $\bar{a}_k$, b_k^i, $\bar{b}_k^i$, c_k, $\sigma_k \in \mathbb{R}$, $i \in \mathsf{N}$. $w_k \in \mathbb{R}$ denotes a stochastic disturbance with zero-mean and finite variance. We assume that the decisions of each player $i \in \mathsf{N}$ additionally satisfy the following mixed affine coupled inequality constraints

$$\bar{m}_k^i \mathbb{E}[x_k] + \sum_{j\in\mathsf{N}} \bar{n}_k^{ij} \mathbb{E}[u_k^j] + p_k^i \geq 0, \tag{1b}$$

where $\bar{m}_k^i$, $\bar{n}_k^{ij}$, $p_k^i \in \mathbb{R}^{s_i}$, $k \in \mathsf{K}_l$. We denote the set of players excluding the player i by $-i := N \setminus \{i\}$. At any instant $k \in \mathsf{K}_l$ the collection of actions of all players excluding player i is denoted by $u_k^{-i} := \mathrm{col}(u_k^1, \cdots, u_k^{i-1}, u_k^{i+1}, \cdots, u_k^N)$. The profile of actions, also referred to as a strategy, of player $i \in \mathsf{N}$ is denoted by $\mathsf{u}^i := \mathrm{col}(u_k^i)_{k=0}^{K-1}$, and the strategy of all players except player i is denoted by $\mathsf{u}^{-i} := \mathrm{col}(u_k^{-i})_{k=0}^{K-1}$. Each player $i \in \mathsf{N}$ while choosing their actions seeks to minimize the following interdependent stage additive cost functional

$$J^i(\mathsf{u}^i, \mathsf{u}^{-i}) = \tfrac{1}{2} q_K^i (x_K)^2 + \tfrac{1}{2} \bar{q}_K^i \mathbb{E}[x_K]^2 + \tfrac{1}{2} \sum_{k\in\mathsf{K}_l} \big(q_k^i (x_k)^2 + \bar{q}_k^i \mathbb{E}[x_k]^2 \big) + \tfrac{1}{2} \sum_{k\in\mathsf{K}_l} \sum_{j\in\mathsf{N}} \big(r_k^{ij} (u_k^j)^2 + \bar{r}_k^{ij} \mathbb{E}[u_k^j]^2 \big), \tag{1c}$$

where $q_k^i, \bar{q}_k^i \in \mathbb{R}$, $k \in \mathsf{K}$ and $r_k^{ij}, \bar{r}_k^{ij} \in \mathbb{R}$, $k \in \mathsf{K}_l$.

Remark 1. From (1a), the expected state dynamics is given by

$$\mathbb{E}[x_{k+1}] = (a_k + \bar{a}_k)\mathbb{E}[x_k] + \sum_{i\in\mathsf{N}} (b_k^i + \bar{b}_k^i)\mathbb{E}[u_k^i] + c_k. \tag{2}$$

As $\mathbb{E}[\mathbb{E}[x_k](x_k - \mathbb{E}[x_k])] = 0$ and $\mathbb{E}[\mathbb{E}[u_k^j](u_k^j - \mathbb{E}[u_k^j])] = 0$, the expected objective can also be represented with mean and variance terms of state and control variables as follows

$$\begin{aligned}\mathbb{E}[J^i(\mathsf{u}^i, \mathsf{u}^{-i})] &= \tfrac{1}{2} q_K^i \mathbb{E}[\big(x_K - \mathbb{E}[x_K]\big)^2] + \tfrac{1}{2}(q_K^i + \bar{q}_K^i)\mathbb{E}[x_K]^2 \\ &+ \tfrac{1}{2} \sum_{k\in\mathsf{K}_l} \big(q_k^i \mathbb{E}[(x_k - \mathbb{E}[x_k])^2] + (q_k^i + \bar{q}_k^i)\mathbb{E}[x_k]^2 \big) \\ &+ \tfrac{1}{2} \sum_{k\in\mathsf{K}_l} \sum_{j\in\mathsf{N}} \big(r_k^{ij} \mathbb{E}[(u_k^j - \mathbb{E}[u_k^j])^2] + (r_k^{ij} + \bar{r}_k^{ij})\mathbb{E}[u_k^j]^2 \big).\end{aligned} \tag{3}$$

The constraints given by (1b) are coupled i.e., at every stage $k \in \mathsf{K}_l$, the control actions u_k^{-i} of players in $-i$ impose a restriction on player i's control action u_k^i. Collecting the constraints of all the players, and by eliminating the expectation of state variable using (2), we get

$$\begin{aligned}&\tilde{\mathsf{M}}_k \big((a_{k-1} + \bar{a}_{k-1}) \cdots (a_0 + \bar{a}_0)\mathbb{E}[x_0] + (a_{k-1} + \bar{a}_{k-1}) \cdots \\ &\times (a_1 + \bar{a}_1)\bar{\mathsf{B}}_0 \mathbb{E}[\mathsf{u}_0] + \cdots + (a_{k-1} + \bar{a}_{k-1})\bar{\mathsf{B}}_{k-2}\mathbb{E}[\mathsf{u}_{k-2}] \\ &+ \bar{\mathsf{B}}_{k-1}\mathbb{E}[\mathsf{u}_{k-1}] \big) + \bar{\mathsf{N}}_k \mathbb{E}[\mathsf{u}_k] + \boldsymbol{p}_k \geq 0,\end{aligned} \tag{4}$$

where $\tilde{\mathsf{M}}_k := \mathrm{col}(\bar{m}_k^i)_{i=1}^N$, $\bar{\mathsf{N}}_k := \mathrm{col}(\mathrm{row}(\bar{n}_k^{ij})_{j=1}^N)_{i=1}^N$, $\bar{\mathsf{B}}_k := \mathrm{row}(b_k^i + \bar{b}_k^i)_{i=1}^N$, $\mathbb{E}[\mathsf{u}_k] = \mathrm{col}(\mathbb{E}[u_k^i])_{i=1}^N$ and $\boldsymbol{p}_k = \mathrm{col}(p_k^i)_{i=1}^N$. The joint feasible strategy space of the players is given by

$$R(\mathbb{E}[x_0]) := \{(\mathsf{u}^i, \mathsf{u}^{-i}) \in \mathbb{R}^{KN} : (4) \text{ holds } \forall k \in \mathsf{K}_l\}. \tag{5}$$

Using (5), the admissible strategy space of player $i \in \mathsf{N}$ for a given $\mathbb{E}[x_0] \in \mathbb{R}$ and u^{-i} is given by

$$\mathsf{U}^i(\mathsf{u}^{-i}) := \{\mathsf{u}^i \in \mathbb{R}^K : (\mathsf{u}^i, \mathsf{u}^{-i}) \in R(\mathbb{E}[x_0])\}. \tag{6}$$

Next, we have the following assumption.

Assumption 1. (i) For a given $\mathbb{E}[x_0] \in \mathbb{R}$, the joint admissible strategy set $R(\mathbb{E}[x_0]) \subseteq \mathbb{R}^{KN}$ is non-empty.

(ii) All the elements of the vector $\bar{n}_k^{ii} \in \mathbb{R}^{s_i}$ are non-zero for all $k \in \mathsf{K}_l$ and $i \in \mathsf{N}$.

(iii) For each player $i \in \mathsf{N}$, q_k^i, $q_k^i + \bar{q}_k^i \geq 0$, $k \in \mathsf{K}$ and r_k^{ii}, $r_k^{ii} + \bar{r}_k^{ii} > 0$, $k \in \mathsf{K}_l$.

Item (i) is required to guarantee the existence of a solution of (1a) satisfying (1b), for a given $\mathbb{E}[x_0] \in \mathbb{R}$. Item (ii) is required to satisfy the constraint qualification conditions. Item (iii) is a technical assumption which can be relaxed; see Remark 3.

The non-cooperative outcome, that is, mean-field-type Nash equilibrium associated with MFTDGC, described by (1), is defined as follows.

Definition 1. For a given $\mathbb{E}[x_0] \in \mathbb{R}$, an admissible strategy profile $(\mathsf{u}^{i\star}, \mathsf{u}^{-i\star}) \in R(\mathbb{E}[x_0])$ is a mean-field-type generalized Nash equilibrium (MFTGNE) for MFTDGC, if for each player $i \in \mathsf{N}$ the following condition holds

$$\mathbb{E}[J_i(\mathsf{u}^{i\star}, \mathsf{u}^{-i\star})] \leq \mathbb{E}[J_i(\mathsf{u}^i, \mathsf{u}^{-i\star})], \ \forall \mathsf{u}^i \in \mathsf{U}^i(\mathsf{u}^{-i\star}). \tag{7}$$

In this letter, we seek to obtain conditions for the existence of MFTGNE for MFTDGC.

III. Main Result

In this section, we present a characterization of MFTGNE for MFTDG. To this end, we employ the direct method, which involves a five-step procedure for finding the solution.

Theorem 1. *Let Assumption 1 holds. Assume there exist a multiplier process $\{\mu_k^{i\star} \in \mathbb{R}^{s_i},\ i \in \mathsf{N},\ k \in \mathsf{K}_l\}$ satisfying the following complementarity conditions*

$$0 \leq \big(\bar{m}_k^i + \sum_{j\in\mathsf{N}} \bar{n}_k^{ij}\delta_k^j\big)\mathbb{E}[x_k^\star] + \sum_{j\in\mathsf{N}} \bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i \perp \mu_k^{i\star} \geq 0, \tag{8}$$

where $\{x_k^\star,\ k \in \mathsf{K}_l\}$ evolves as follows

$$x_{k+1}^\star - \mathbb{E}[x_{k+1}^\star] = \mathsf{A}_k(x_k^\star - \mathbb{E}[x_k^\star]) + \sigma_k w_k,\ x_0^\star = x_0, \tag{9a}$$

$$\mathbb{E}[x_{k+1}^\star] = \bar{\mathsf{A}}_k \mathbb{E}[x_k^\star] + \big(\sum_{j\in\mathsf{N}} (b_k^j + \bar{b}_k^j)\bar{\delta}_k^j + c_k\big), \tag{9b}$$

with $\mathsf{A}_k := a_k + \sum_{j\in\mathsf{N}} b_k^j \eta_k^j$, $\bar{\mathsf{A}}_k := \big(a_k + \bar{a}_k + \sum_{j\in\mathsf{N}} (b_k^j + \bar{b}_k^j)\delta_k^j\big)$ and for each $i \in \mathsf{N}$, $\{\eta_k^i, \delta_k^i, \bar{\delta}_k^i,\ k \in \mathsf{K}_l\}$ satisfy the following algebraic equations

$$r_k^{ii}\eta_k^i + b_k^i \alpha_{k+1}^i \sum_{j\in\mathsf{N}} b_k^j \eta_k^j + b_k^i \alpha_{k+1}^i a_k = 0, \tag{10a}$$

$$(r_k^{ii} + \bar{r}_k^{ii})\delta_k^i + (b_k^i + \bar{b}_k^i)\bar{\alpha}_{k+1}^i \sum_{j\in\mathsf{N}} (b_k^j + \bar{b}_k^j)\delta_k^j + (b_k^i + \bar{b}_k^i)\bar{\alpha}_{k+1}^i (a_k + \bar{a}_k) = 0, \tag{10b}$$

$$(r_k^{ii} + \bar{r}_k^{ii})\bar{\delta}_k^i + (b_k^i + \bar{b}_k^i)\bar{\alpha}_{k+1}^i \sum_{j\in\mathsf{N}} (b_k^j + \bar{b}_k^j)\bar{\delta}_k^j + (b_k^i + \bar{b}_k^i)\big(\bar{\alpha}_{k+1}^i c_k + \beta_{k+1}^i\big) - \bar{n}_k^{ii\prime}\mu_k^{i\star} = 0, \tag{10c}$$

where α_k^i, $\bar{\alpha}_k^i$ and β_k^i for $k \in \mathsf{K}_l$ are obtained by solving the following backward difference equations

$$\alpha_k^i = \alpha_{k+1}^i (\mathsf{A}_k)^2 + \sum_{j\in\mathsf{N}} r_k^{ij} (\eta_k^j)^2 + q_k^i, \tag{11a}$$

$$\bar{\alpha}_k^i = \bar{\alpha}_{k+1}^i(\bar{\mathsf{A}}_k)^2 + \sum_{j\in\mathsf{N}}(r_k^{ij}+\bar{r}_k^{ij})(\delta_k^j)^2 + (q_k^i+\bar{q}_k^i), \tag{11b}$$

$$\beta_k^i = \bar{\mathsf{A}}_k\beta_{k+1}^i + \sum_{j\in -i}\big((r_k^{ij}+\bar{r}_k^{ij})\delta_k^j + \bar{\mathsf{A}}_k\bar{\alpha}_{k+1}^i(b_k^j+\bar{b}_k^j)\big)\bar{\delta}_k^j - \big(\bar{m}_k^i + \sum_{j\in\mathsf{N}}\bar{n}_k^{ij}\delta_k^j\big)'\mu_k^{i\star} + \bar{\mathsf{A}}_k\bar{\alpha}_{k+1}^i c_k, \tag{11c}$$

with boundary conditions $\alpha_K^i = q_K^i$, $\bar{\alpha}_K^i = q_K^i + \bar{q}_K^i$ *and* $\beta_K^i = 0$. *Then, the MFTGNE strategy of each player* $i\in\mathsf{N}$ *is given by*

$$u_k^{i\star} - \mathbb{E}[u_k^{i\star}] = \eta_k^i(x_k^\star - \mathbb{E}[x_k^\star]), \tag{12a}$$

$$\mathbb{E}[u_k^{i\star}] = \delta_k^i\mathbb{E}[x_k^\star] + \bar{\delta}_k^i. \tag{12b}$$

Furthermore, the expected equilibrium cost of player $i\in\mathsf{N}$ *is given by* $\mathbb{E}[J^i(\mathsf{u}^{i\star},\mathsf{u}^{-i\star})] = \frac{1}{2}\alpha_0^i\mathbb{E}[(x_0-\mathbb{E}[x_0])^2] + \frac{1}{2}\bar{\alpha}_0^i\mathbb{E}[x_0]^2 + \beta_0^i\mathbb{E}[x_0] + \gamma_0^i$, *where* γ_k^i *for* $k\in\mathsf{K}_l$ *is obtained from the following backward difference equation with boundary condition* $\gamma_K^i = 0$.

$$\gamma_k^i = \gamma_{k+1}^i - \tfrac{1}{2}\beta_{k+1}^i(b_k^i+\bar{b}_k^i)\bar{\delta}_k^i + \tfrac{1}{2}\sum_{j\in -i}(r_k^{ij}+\bar{r}_k^{ij})(\bar{\delta}_k^j)^2 + \tfrac{1}{2}\big(\sum_{j\in\mathsf{N}}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big)\big(\bar{\alpha}_{k+1}^i\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big) + 2\beta_{k+1}^i\big) - \tfrac{1}{2}\mu_k^{i\star\prime}\big(2\sum_{j\in\mathsf{N}}\bar{n}_k^{ij}\bar{\delta}_k^j - \bar{n}_k^{ii}\bar{\delta}_k^i + 2p_k^i\big) + \tfrac{1}{2}\alpha_{k+1}^i(\sigma_k)^2\mathbb{E}[(w_k)^2]. \tag{13}$$

Proof. First, it is straightforward to verify that when all the players use strategies, given by (12), then $\{x_k^\star,\ k\in\mathsf{K}\}$, given by (9), is the generated state trajectory. Due to (8), the inequality constraints (1b) hold for all players with the strategy profile $(\mathsf{u}^{i\star},\mathsf{u}^{-i\star})$. This implies, $(\mathsf{u}^{i\star},\mathsf{u}^{-i\star})\in R(\mathbb{E}[x_0])$ or $\mathsf{u}^{i\star}\in\mathsf{U}^i(\mathsf{u}^{-i\star})$, and in particular, $\mathsf{U}^i(\mathsf{u}^{-i\star})\neq\emptyset$. Consider any admissible strategy profile $\mathsf{u}^i\in\mathsf{U}^i(\mathsf{u}^{-i\star})$, and let $\{x_k,\ k\in\mathsf{K}\}$ be the corresponding state trajectory. Using (12) for all players in $-i$, in (1a), (2), and (1b) we obtain the following relations

$$\mathbb{E}[x_{k+1}] = \big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)\mathbb{E}[x_k] + (b_k^i+\bar{b}_k^i)\mathbb{E}[u_k^i] + \big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big), \tag{14a}$$

$$\begin{aligned}\mathbb{E}[x_{k+1}]^2 &= \big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)^2\mathbb{E}[x_k]^2 + (b_k^i+\bar{b}_k^i)^2\mathbb{E}[u_k^i]^2 + \big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big)^2\\ &\quad + 2\big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)(b_k^i+\bar{b}_k^i)\mathbb{E}[x_k]\mathbb{E}[u_k^i]\\ &\quad + 2\big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big)\\ &\quad \times\mathbb{E}[x_k] + 2(b_k^i+\bar{b}_k^i)\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big)\mathbb{E}[u_k^i],\end{aligned} \tag{14b}$$

$$\begin{aligned}\mathbb{E}[\big(x_{k+1}-\mathbb{E}[x_{k+1}]\big)^2] &= \big(a_k+\sum_{j\in -i}b_k^j\eta_k^j\big)^2\mathbb{E}[(x_k-\mathbb{E}[x_k])^2]\\ &\quad + (b_k^i)^2\mathbb{E}[(u_k^i-\mathbb{E}[u_k^i])^2] + (\sigma_k)^2\mathbb{E}[(w_k)^2]\\ &\quad + 2\big(a_k+\sum_{j\in -i}b_k^j\eta_k^j\big)b_k^i\mathbb{E}[(x_k-\mathbb{E}[x_k])(u_k^i-\mathbb{E}[u_k^i])],\end{aligned} \tag{14c}$$

$$\big(\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j\big)\mathbb{E}[x_k] + \bar{n}_k^{ii}\mathbb{E}[u_k^i] + \sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i \geq 0. \tag{14d}$$

Next we use the direct method to complete the proof.

Step 1 – (Defining a guess functional): We first define a quadratic guess functional of the following form

$$f^i(k,x_k) = \tfrac{1}{2}\alpha_k^i(x_k-\mathbb{E}[x_k])^2 + \tfrac{1}{2}\bar{\alpha}_k^i\mathbb{E}[x_k]^2 + \beta_k^i\mathbb{E}[x_k] + \gamma_k^i.$$

Step 2 – (Telescopic sum of the guess functional): Upon taking the telescopic sum of $f^i(k,x_k)$ over $k\in\mathsf{K}$ we obtain

$$\begin{aligned}f^i(0,x_0) &= f^i(K,x_K) - \sum_{k\in\mathsf{K}_l}\big(f^i(k+1,x_{k+1}) - f^i(k,x_k)\big)\\ &= \tfrac{1}{2}\alpha_K^i(x_K-\mathbb{E}[x_K])^2 + \tfrac{1}{2}\bar{\alpha}_K^i\mathbb{E}[x_K]^2 + \beta_K^i\mathbb{E}[x_K] + \gamma_K^i\\ &\quad - \tfrac{1}{2}\sum_{k\in\mathsf{K}_l}\big(\alpha_{k+1}^i(x_{k+1}-\mathbb{E}[x_{k+1}])^2 - \alpha_k^i(x_k-\mathbb{E}[x_k])^2\big)\\ &\quad - \tfrac{1}{2}\sum_{k\in\mathsf{K}_l}\big(\bar{\alpha}_{k+1}^i\mathbb{E}[x_{k+1}]^2 - \bar{\alpha}_k^i\mathbb{E}[x_k]^2\big)\\ &\quad - \tfrac{1}{2}\sum_{k\in\mathsf{K}_l}\big(\beta_{k+1}^i\mathbb{E}[x_{k+1}] - \beta_k^i\mathbb{E}[x_k] + \gamma_{k+1}^i - \gamma_k^i\big).\end{aligned} \tag{15}$$

Step 3 – (Difference between the cost and the guess functional): Next, using the expressions (3), (14a)-(14c) and (15) we compute $\mathbb{E}[J^i(\mathsf{u}^i,\mathsf{u}^{-i}) - f^i(0,x_0)]$ as follows

$$\begin{aligned}&\mathbb{E}[J^i(\mathsf{u}^i,\mathsf{u}^{-i\star}) - f^i(0,x_0)] = \tfrac{1}{2}(q_K^i-\alpha_K^i)\mathbb{E}[\big(x_K-\mathbb{E}[x_K]\big)^2]\\ &+ \tfrac{1}{2}(q_K^i+\bar{q}_K^i-\bar{\alpha}_K^i)\mathbb{E}[x_K]^2 - \beta_K^i\mathbb{E}[x_K] - \gamma_K^i\\ &+ \tfrac{1}{2}\sum_{k\in\mathsf{K}_l}\big((C_k^i+\tfrac{(B_k^i)^2}{A_k^i})\mathbb{E}[(x_k-\mathbb{E}[x_k])^2] + (L_k^i+\tfrac{(F_k^i)^2}{D_k^i})\mathbb{E}[x_k]^2\\ &+ \sum_{k\in\mathsf{K}_l}\big(M_k^i + \mu_k^{i\star\prime}(\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j) + \tfrac{F_k^iG_k^i}{D_k^i}\big)E[x_k]\\ &+ \sum_{k\in\mathsf{K}_l}\big(N_k^i + \mu_k^{i\star\prime}\big(\sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i\big) + \tfrac{1}{2}\tfrac{(G_k^i)^2}{D_k^i}\big)\\ &+ \sum_{k\in\mathsf{K}_l}\big(\tfrac{1}{2}A_k^i\mathbb{E}[(u_k^i-\mathbb{E}[u_k^i])^2] + B_k^i\mathbb{E}[(x_k-\mathbb{E}[x_k])(u_k^i-\mathbb{E}[u_k^i])]\big)\\ &+ \sum_{k\in\mathsf{K}_l}\big(\tfrac{1}{2}D_k^i\mathbb{E}[u_k^i]^2 + \big(F_k^i\mathbb{E}[x_k] + G_k^i + \mu_k^{i\star\prime}\bar{n}_k^{ii}\big)\mathbb{E}[u_k^i]\big),\end{aligned} \tag{16}$$

where $A_k^i := r_k^{ii} + \alpha_{k+1}^i(b_k^i)^2$, $B_k^i := \alpha_{k+1}^i\big(a_k + \sum_{j\in -i}b_k^j\eta_k^j\big)b_k^i$, $C_k^i := \big(q_k^i + \sum_{j\in -i}r_k^{ij}(\eta_k^j)^2\big) + \alpha_{k+1}^i\big(a_k + \sum_{j\in -i}b_k^j\eta_k^j\big)^2 - \alpha_k^i - \frac{(B_k^i)^2}{A_k^i}$, $D_k^i := r_k^{ii} + \bar{r}_k^{ii} + \bar{\alpha}_{k+1}^i(b_k^i+\bar{b}_k^i)^2$,
$F_k^i := \bar{\alpha}_{k+1}^i\big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)(b_k^i+\bar{b}_k^i)$,
$G_k^i := (b_k^i+\bar{b}_k^i)\Big(\bar{\alpha}_{k+1}^i\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big) + \beta_{k+1}^i\Big) - \mu_k^{i\star\prime}\bar{n}_k^{ii}$, $L_k^i := (q_k^i+\bar{q}_k^i) + \sum_{j\in -i}(r_k^{ij}+\bar{r}_k^{ij})(\delta_k^j)^2 - \bar{\alpha}_k^i + \bar{\alpha}_{k+1}^i\big(a_k+\bar{a}_k+\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)^2 - \frac{(F_k^i)^2}{D_k^i}$,
$M_k^i := \sum_{j\in -i}(r_k^{ij}+\bar{r}_k^{ij})\delta_k^j\bar{\delta}_k^j - \mu_k^{i\star\prime}\big(\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j\big) + \big(a_k + \bar{a}_k + \sum_{j\in -i}(b_k^j+\bar{b}_k^j)\delta_k^j\big)\big(\bar{\alpha}_{k+1}^i\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big) + \beta_{k+1}^i\big) - \beta_k^i - \frac{F_k^iG_k^i}{D_k^i}$ and
$N_k^i := \frac{1}{2}\sum_{j\in -i}(r_k^{ij}+\bar{r}_k^{ij})(\bar{\delta}_k^j)^2 - \mu_k^{i\star\prime}\big(\sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i\big) + \frac{1}{2}\alpha_{k+1}^i(\sigma_k)^2\mathbb{E}[(w_k)^2] + \frac{1}{2}\bar{\alpha}_{k+1}^i\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big)^2 + \beta_{k+1}^i\big(\sum_{j\in -i}(b_k^j+\bar{b}_k^j)\bar{\delta}_k^j + c_k\big) + \gamma_{k+1}^i - \gamma_k^i - \frac{1}{2}\frac{(G_k^i)^2}{D_k^i}$.

Step 4 – (Incorporation of inequality constraints and completion of squares): We add and subtract the term $\sum_{k\in \mathsf{K}_l} \mu_k^{i\star\prime}\big((\bar{m}_k^i + \sum_{j\in -i} \bar{n}_k^{ij}\delta_k^j)\mathbb{E}[x_k] + \bar{n}_k^{ii}\mathbb{E}[u_k^i] + \sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i\big)$ to the right-hand-side of the expression $\mathbb{E}[J^i(\mathsf{u}^i,\mathsf{u}^{-i}) - f^i(0,x_0)]$ in (16). Then, we perform the completion of squares of the terms involving $(u_k^i - \mathbb{E}[u_k^i])$ and $\mathbb{E}[u_k^i]$ as follows

$$\begin{aligned}
&\tfrac{1}{2}A_k^i\mathbb{E}[(u_k^i-\mathbb{E}[u_k^i])^2] + B_k^i\mathbb{E}[(x_k-\mathbb{E}[x_k])(u_k^i-\mathbb{E}[u_k^i])]\\
&\quad= \tfrac{1}{2}A_k^i\mathbb{E}\Big[\Big((u_k^i-\mathbb{E}[u_k^i]) + \tfrac{B_k^i}{A_k^i}(x_k-\mathbb{E}[x_k])\Big)^2\Big]\\
&\qquad -\tfrac{1}{2}\tfrac{(B_k^i)^2}{A_k^i}\mathbb{E}[(x_k-\mathbb{E}[x_k])^2], \qquad (17a)\\
&\tfrac{1}{2}D_k^i\mathbb{E}[u_k^i]^2 + \big(F_k^i\mathbb{E}[x_k]+G_k^i\big)\mathbb{E}[u_k^i]\\
&\quad= \tfrac{1}{2}D_k^i\big(\mathbb{E}[u_k^i] + \tfrac{1}{D_k^i}\big(F_k\mathbb{E}[x_k]+G_k^i\big)\big)^2 - \tfrac{1}{2}\tfrac{(F_k^i)^2}{D_k^i}\mathbb{E}[x_k]^2\\
&\qquad -\tfrac{F_k^iG_k^i}{D_k^i}\mathbb{E}[x_k] - \tfrac{1}{2}\tfrac{(G_k^i)^2}{D_k^i}. \qquad (17b)
\end{aligned}$$

After performing the above calculations we obtain

$$\begin{aligned}
&\mathbb{E}[J^i(\mathsf{u}^i,\mathsf{u}^{-i\star})] = \mathbb{E}[f^i(0,x_0)] + \sum_{k\in\mathsf{K}_l}\mu_k^{i\star\prime}\big((\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j)\mathbb{E}[x_k]\\
&+\bar{n}_k^{ii}\mathbb{E}[u_k^i] + \sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i\big) + \tfrac{1}{2}(q_K^i-\alpha_K^i)\mathbb{E}[\big(x_K-\mathbb{E}[x_K]\big)^2]\\
&+\tfrac{1}{2}(q_K^i+\bar{q}_K^i-\bar{\alpha}_K^i)\mathbb{E}[x_K]^2 - \beta_K^i\mathbb{E}[x_K] - \gamma_K^i\\
&+\tfrac{1}{2}\sum_{k\in\mathsf{K}_l}\big(C_k^i\mathbb{E}[(x_k-\mathbb{E}[x_k])^2] + L_k^i\mathbb{E}[x_k]^2 + 2M_k^iE[x_k] + 2N_k^i\big)\\
&+\tfrac{1}{2}\sum_{k\in\mathsf{K}_l}A_k^i\mathbb{E}\big[\big((u_k^i-\mathbb{E}[u_k^i]) + \tfrac{B_k^i}{A_k^i}(x_k-\mathbb{E}[x_k])\big)^2\big]\\
&+\tfrac{1}{2}\sum_{k\in\mathsf{K}_l}D_k^i\big(\mathbb{E}[u_k^i] + \tfrac{1}{D_k^i}\big(F_k^i\mathbb{E}[x_k]+G_k^i\big)\big)^2. \qquad (18)
\end{aligned}$$

Step 5 – (Verification of MFTGNE (7)): Next, using the definitions of A_k^i, B_k^i, D_k^i, F_k^i and G_k^i in (10) it is easy to verify that $\eta_k^i = -B_k^i/A_k^i$, $\delta_k^i = -F_k^i/D_k^i$ and $\bar{\delta}_k^i = -G_k^i/D_k^i$. Then, using (11) and (13), it is verified that the terms C_k^i, L_k^i, M_k^i and N_k^i are zero for $k\in\mathsf{K}_l$ (see the online supplementary version [15] for these straightforward but lengthy calculations).

Next, as $\alpha_K^i = q_K^i$, $\bar{\alpha}_K^i = q_K^i + \bar{q}_K^i$, $\beta_K^i = 0$, $\gamma_K^i = 0$, the expression (18) is simplified as follows

$$\begin{aligned}
&\mathbb{E}[J^i(\mathsf{u}^i,\mathsf{u}^{-i\star})] = \mathbb{E}[f^i(0,x_0)] + \sum_{k\in\mathsf{K}_l}\mu_k^{i\star\prime}\big((\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j)\\
&\quad\times\mathbb{E}[x_k] + \bar{n}_k^{ii}\mathbb{E}[u_k^i] + \sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i\big)\\
&\quad+\tfrac{1}{2}\sum_{k\in\mathsf{K}_l}A_k^i\mathbb{E}\big[\big((u_k^i-\mathbb{E}[u_k^i]) - \eta_k^i(\bar{x}_k-\mathbb{E}[x_k])\big)^2\big]\\
&\quad+\tfrac{1}{2}\sum_{k\in\mathsf{K}_l}D_k^i\big(\mathbb{E}[u_k^i] - \delta_k^i\mathbb{E}[x_k] - \bar{\delta}_k^i\big)^2. \qquad (19)
\end{aligned}$$

If u^i is set as $\mathsf{u}^{i\star}$ (given by (12)) in (19), then we know $\{x_k^\star,\ k\in\mathsf{K}\}$ is the corresponding state trajectory which satisfies the complementarity condition (8) and evolves according to (9). Then, the second term on the right-hand-side of the expression in (19) vanishes. Besides this, the third and the fourth terms also vanish as $\mathsf{u}^{i\star}$ satisfies (12). So, we have

$$\mathbb{E}[J^i(\mathsf{u}^{i\star},\mathsf{u}^{-i\star})] = \mathbb{E}[f^i(0,x_0)]. \qquad (20)$$

From Assumption 1.(i), every admissible $\mathsf{u}^i\in\mathsf{U}^i(\mathsf{u}^{-i\star})$ satisfies $(\bar{m}_k^i + \sum_{j\in -i}\bar{n}_k^{ij}\delta_k^j)\mathbb{E}[x_k] + \bar{n}_k^{ii}\mathbb{E}[u_k^i] + \sum_{j\in -i}\bar{n}_k^{ij}\bar{\delta}_k^j + p_k^i \geq 0$. Further, the multipliers in (8) satisfy $\mu_k^{i\star}\geq 0$ for all $k\in\mathsf{K}_l$. This implies, the second term on right-hand-side of the expression in (19) is non-negative. Besides this, as $A_k^i, D_k^i > 0$ for all $k\in\mathsf{K}_l$ the third and the fourth terms are also non-negative. Consequently, comparing (19) and (20), we obtain

$$\mathbb{E}[J_i(\mathsf{u}^{i\star},\mathsf{u}^{-i\star})] \leq \mathbb{E}[J_i(\mathsf{u}^i,\mathsf{u}^{-i\star})],\ \forall \mathsf{u}^i\in\mathsf{U}^i(\mathsf{u}^{-i\star}).$$

As the choice of player i is arbitrary, the above condition holds for each player $i\in\mathsf{N}$. So, from Definition 1, the strategy profile $\{u_k^{i\star},\ i\in\mathsf{N},\ k\in\mathsf{K}_l\}$ given by (12) is indeed a MFTGNE. ■

Remark 2. Using the algebraic equations (10), at each stage $k\in\mathsf{K}_l$, we note that the variables $\{\eta_k^i,\delta_k^i,\bar{\delta}_k^i, i\in\mathsf{N}\}$ can be solved interms of the variables $\{\alpha_{k+1}^i,\bar{\alpha}_{k+1}^i,\beta_{k+1}^i, i\in\mathsf{N}\}$. Then, using these solutions in the backward difference equations (11) the variables $\{\alpha_k^i,\bar{\alpha}_k^i,\beta_k^i, i\in\mathsf{N}\}$ are evaluated. So, starting with the boundary conditions $\alpha_K^i = q_K^i$, $\bar{\alpha}_K^i = q_K^i + \bar{q}_K^i$ and $\beta_K^i = 0$, and using the above mentioned recursive procedure the variables $\{\alpha_k^i,\bar{\alpha}_k^i,\beta_k^i,\eta_k^i,\delta_k^i,\bar{\delta}_k^i,\ k\in\mathsf{K}_l,\ i\in\mathsf{N}\}$ are determined. In particular, from (10c) and (11c), the variables $\{\bar{\delta}_k^i,\beta_k^i,\ k\in\mathsf{K}_l,\ i\in\mathsf{N}\}$ contain linear terms involving the multipliers $\{\mu_k^{i\star},\ k\in\mathsf{K},\ i\in\mathsf{N}\}$. Further, substituting $\{\bar{\delta}_k^i,\ k\in\mathsf{K}_l,\ i\in\mathsf{N}\}$ in (9), the MFTGNE state trajectory $\{x_k^\star,\ k\in\mathsf{K}\}$ is expressed linearly in terms of the multipliers $\{\mu_k^{i\star},\ k\in\mathsf{K},\ i\in\mathsf{N}\}$. Upon eliminating these state variables in (8) we obtain (implicit) complementarity conditions involving only the multipliers and $\mathbb{E}[x_0]$ (which is known). In Section IV, under a few assumptions on the problem data, we illustrate the above mentioned procedure towards determining the multipliers.

Remark 3. Assumption 1.(iii) can be relaxed with a less stringent condition by requiring that the solutions $\{\alpha_k^i,\bar{\alpha}_k^i,\ i\in\mathsf{N},\ k\in\mathsf{K}\}$ of the backward difference equations (11a)-(11b) are such that $A_k^i = r_k^{ii} + \alpha_{k+1}^i(b_k^i)^2$ and $D_k^i = r_k^{ii} + \bar{r}_k^{ii} + \bar{\alpha}_{k+1}^i(b_k^i+\bar{b}_k^i)^2$ are positive for all $k\in\mathsf{K}_l$ and $i\in\mathsf{N}$. We notice that $\{\alpha_k^i,\bar{\alpha}_k^i,\ i\in\mathsf{N},\ k\in\mathsf{K}\}$ depend only on the problem data associated with state dynamics (1a) and objectives (1c). So, using the recursive procedure mentioned in Remark 2, the required positivity condition can be verified numerically using the problem data; see also Remark 5.

Remark 4. From Remark 3 and (18), we have that player i's expected cost function is a strictly convex function in her decision variables u^i.

IV. SOLVABILITY

In this section, we present an approach for reformulating the equations (8)-(11) as single large-scale linear complementarity problem. This procedure is based on [14], and involves elimination of state variables in the equations (8)-(9); see also Remarks 2 and 3. Due to space constraints, we provide only a brief overview of this approach (see the online supplementary version [15] for details). We define all the notations used in this section as follows: $\mathbf{R}_k := \oplus_{i=1}^N r_k^{ii}$, $\bar{\mathbf{R}}_k := \oplus_{i=1}^N(r_k^{ii}+\bar{r}_k^{ii})$, $\mathbf{B}_k := \oplus_{i=1}^N b_k^i$,

$\bar{\mathbf{B}}_k := \oplus_{i=1}^{N}(b_k^i+\bar{b}_k^i)$, $\mathsf{B}_k := \mathrm{row}(b_k^i)_{i=1}^{N}$, $\mathsf{N}_k := \oplus_{i=1}^{N}\bar{n}_k^{ii}$, $\bar{\mathsf{M}}_k := \mathrm{col}(\bar{m}_k^i + \sum_{j\in\mathsf{N}}\bar{n}_k^{ij}\delta_k^j)_{i=1}^{N}$, $\boldsymbol{\alpha}_k := \mathrm{col}(\alpha_k^i)_{i=1}^{N}$, $\bar{\boldsymbol{\alpha}}_k := \mathrm{col}(\bar{\alpha}_k^i)_{i=1}^{N}$, $\boldsymbol{\eta}_k := \mathrm{col}(\eta_k^i)_{i=1}^{N}$, $\boldsymbol{\beta}_k := \mathrm{col}(\beta_k^i)_{i=1}^{N}$, $\boldsymbol{\delta}_k := \mathrm{col}(\delta_k^i)_{i=1}^{N}$, $\bar{\boldsymbol{\delta}}_k := \mathrm{col}(\bar{\delta}_k^i)_{i=1}^{N}$, $\boldsymbol{\mu}_k^\star := \mathrm{col}(\mu_k^{i\star})_{i=1}^{N}$, $\Lambda_k := \mathbf{R}_k + \mathbf{B}_k'\boldsymbol{\alpha}_{k+1}\mathsf{B}_k$, $\bar{\Lambda}_k := \bar{\mathbf{R}}_k + \bar{\mathbf{B}}_k'\bar{\boldsymbol{\alpha}}_{k+1}\bar{\mathsf{B}}_k$, $[\mathsf{P}_k^1]_{ij} = (r_k^{ij}+\bar{r}_k^{ij})\delta_k^j + \bar{\mathsf{A}}_k\bar{\alpha}_{k+1}^i(b_k^j+\bar{b}_k^j)$ for $i\neq j$, $[\mathsf{P}_k^1]_{ii} = 0$, $\mathsf{P}_k^2 = \oplus_{i=1}^{N}(\bar{m}_k^i + \sum_{j\in\mathsf{N}}\bar{n}_k^{ij}\delta_k^j)'$, $\mathsf{P}_k^3 = \mathrm{col}(\bar{\mathsf{A}}_k\bar{\alpha}_{k+1}^i)_{i=1}^{N}$. Next, for all $k\in\mathsf{K}_l$, we aggregate the variables as $x_\mathsf{K}^\star = \mathrm{col}(x_k^\star)_{k=0}^{K-1}$, $\mathbb{E}[x_\mathsf{K}^\star] = \mathrm{col}(\mathbb{E}[x_k^\star])_{k=0}^{K-1}$, $\mathsf{u}_\mathsf{K}^\star = \mathrm{col}(\mathsf{u}_k^\star)_{k=0}^{K-1}$, $\mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = \mathrm{col}(\mathbb{E}[\mathsf{u}_k^\star])_{k=0}^{K-1}$, $\boldsymbol{\mu}_\mathsf{K}^\star = \mathrm{col}(\boldsymbol{\mu}_k^\star)_{k=0}^{K-1}$, $\boldsymbol{c}_\mathsf{K} = \mathrm{col}(c_k)_{k=0}^{K-1}$, $\bar{\boldsymbol{\delta}}_\mathsf{K} = \mathrm{col}(\bar{\boldsymbol{\delta}}_k)_{k=0}^{K-1}$, $\boldsymbol{p}_\mathsf{K} = \mathrm{col}(\boldsymbol{p}_k)_{k=0}^{K-1}$, $\boldsymbol{w}_\mathsf{K} = \mathrm{col}(w_k)_{k=0}^{K-1}$, $\boldsymbol{\eta}_\mathsf{K} = \oplus_{k=0}^{K-1}\boldsymbol{\eta}_k$, $\boldsymbol{\delta}_\mathsf{K} = \oplus_{k=0}^{K-1}\boldsymbol{\delta}_k$, $\bar{\mathsf{M}}_\mathsf{K} = \oplus_{k=0}^{K-1}\bar{\mathsf{M}}_k$, $\bar{\mathsf{N}}_\mathsf{K} = \oplus_{k=0}^{K-1}\bar{\mathsf{N}}_k$. Let $\psi(k,\tau)$ and $\phi(k,\tau)$ be the state transition matrices associated with the system dynamics (9a) and (9b), respectively i.e., $\psi(k,\tau) = \mathsf{A}_{k-1}\mathsf{A}_{k-2}\cdots\mathsf{A}_\tau$ for any $k>\tau$ and $\psi(k,\tau) = \mathbf{I}$ for $k=\tau$, $\phi(k,\tau) = \bar{\mathsf{A}}_{k-1}\bar{\mathsf{A}}_{k-2}\cdots\bar{\mathsf{A}}_\tau$ for any $k>\tau$ and $\phi(k,\tau) = \mathbf{I}$ for $k=\tau$. Using these, we define $[\mathsf{P}_\mathsf{K}^1]_{k\tau} = \bar{\mathbf{B}}_{k+1}'\big(\mathbf{I}_N\otimes\phi(\tau-1,k)\big)\mathsf{P}_{\tau-1}^1$, $[\mathsf{P}_\mathsf{K}^2]_{k\tau} = -\bar{\mathbf{B}}_{k+1}'\big(\mathbf{I}_N\otimes\phi(\tau-1,k)\big)\mathsf{P}_{\tau-1}^2$, $[\mathsf{P}_\mathsf{K}^3]_{k\tau} = -\bar{\mathbf{B}}_{k+1}'\big(\mathbf{I}_N\otimes\phi(\tau-1,k)\big)\mathsf{P}_{\tau-1}^3$ for $\tau>k$, $[\mathsf{P}_\mathsf{K}^1]_{kk} = \bar{\Lambda}_k$, $[\mathsf{P}_\mathsf{K}^2]_{kk} = \mathsf{N}_k'$, $[\mathsf{P}_\mathsf{K}^3]_{kk} = -\bar{\mathbf{B}}_k'\bar{\boldsymbol{\alpha}}_{k+1}$ and $[\mathsf{P}_\mathsf{K}^1]_{k\tau} = \mathbf{0}$, $[\mathsf{P}_\mathsf{K}^2]_{k\tau} = \mathbf{0}$, $[\mathsf{P}_\mathsf{K}^3]_{k\tau} = \mathbf{0}$ for $\tau<k$, $[\boldsymbol{\Psi}_0]_k = \psi(k-1,0)$, $[\boldsymbol{\Psi}_1]_{k\tau} = \psi(k-1,\tau)\sigma_{\tau-1}$, for $k>\tau$, $[\boldsymbol{\Psi}_1]_{k\tau} = \mathbf{0}$ for $k\leq\tau$, $[\boldsymbol{\Phi}_0]_k = \phi(k-1,0)$, $[\boldsymbol{\Phi}_1]_{k\tau} = \phi(k-1,\tau)\bar{\mathsf{B}}_{\tau-1}$, $[\boldsymbol{\Phi}_2]_{k\tau} = \phi(k-1,\tau)$ for $k>\tau$ and $[\boldsymbol{\Phi}_1]_{k\tau} = \mathbf{0}$, $[\boldsymbol{\Phi}_2]_{k\tau} = \mathbf{0}$ for $k\leq\tau$ with $k,\tau\in\mathsf{K}_r$.

Using the above notations for all $i\in\mathsf{N}$, (12) can be written compactly for all $k\in\mathsf{K}_l$ as

$$\mathsf{u}_\mathsf{K}^\star - \mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = \boldsymbol{\eta}_\mathsf{K}(x_\mathsf{K}^\star - \mathbb{E}[x_\mathsf{K}^\star]), \tag{21a}$$
$$\mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = \boldsymbol{\delta}_\mathsf{K}\mathbb{E}[x_\mathsf{K}^\star] + \bar{\boldsymbol{\delta}}_\mathsf{K}. \tag{21b}$$

Similarly, for all $i\in\mathsf{N}$, (10) are given by

$$\Lambda_k\boldsymbol{\eta}_k = -\mathbf{B}_k'\boldsymbol{\alpha}_{k+1}a_k, \tag{22a}$$
$$\bar{\Lambda}_k\boldsymbol{\delta}_k = -\bar{\mathbf{B}}_k'\bar{\boldsymbol{\alpha}}_{k+1}(a_k+\bar{a}_k), \tag{22b}$$
$$\bar{\Lambda}_k\bar{\boldsymbol{\delta}}_k = -\bar{\mathbf{B}}_k'\bar{\boldsymbol{\alpha}}_{k+1}c_k - \bar{\mathbf{B}}_k'\boldsymbol{\beta}_{k+1} + \mathsf{N}_k'\boldsymbol{\mu}_k^\star. \tag{22c}$$

Further, the vector form representation of (11c) is given by $\boldsymbol{\beta}_k = (\mathbf{I}_N\otimes\bar{\mathsf{A}}_k)\boldsymbol{\beta}_{k+1} + \mathsf{P}_k^1\bar{\boldsymbol{\delta}}_k + \mathsf{P}_k^2\boldsymbol{\mu}_k^\star + \mathsf{P}_k^3 c_k = \sum_{\tau=k}^{K-1}\big(\mathbf{I}_N\otimes\phi(\tau,k)\big)\big(\mathsf{P}_\tau^1\bar{\boldsymbol{\delta}}_\tau + \mathsf{P}_\tau^2\boldsymbol{\mu}_\tau^\star + \mathsf{P}_\tau^3 c_\tau\big)$, along with boundary condition $\boldsymbol{\beta}_K = \mathbf{0}$. Using this in (22c), and collecting all the terms for $k\in\mathsf{K}_l$ we obtain

$$\mathsf{P}_\mathsf{K}^1\bar{\boldsymbol{\delta}}_\mathsf{K} = \mathsf{P}_\mathsf{K}^2\boldsymbol{\mu}_\mathsf{K}^\star + \mathsf{P}_\mathsf{K}^3\boldsymbol{c}_\mathsf{K}. \tag{23}$$

We have the following assumption.

Assumption 2. The matrices $\{\Lambda_k,\ \bar{\Lambda}_k,\ k\in\mathsf{K}_l\}$ are invertible.

In (23), P_K^1 is a upper triangular matrix (as $[\mathsf{P}_\mathsf{K}^1]_{k\tau} = \mathbf{0}$ for $\tau<k$) with $\bar{\Lambda}_k$, $k\in\mathsf{K}_l$ as the block diagonal elements. From Assumption 2, the matrix P_K^1 is also invertible. Then, from (23) we have

$$\bar{\boldsymbol{\delta}}_\mathsf{K} = (\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^2\boldsymbol{\mu}_\mathsf{K}^\star + (\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^3\boldsymbol{c}_\mathsf{K}. \tag{24}$$

Similarly, the equilibrium trajectory (9) and the complementarity condition (8) for all $k\in\mathsf{K}_l$ are represented as

$$x_\mathsf{K}^\star - \mathbb{E}[x_\mathsf{K}^\star] = \boldsymbol{\Psi}_0(x_0 - \mathbb{E}[x_0]) + \boldsymbol{\Psi}_1\boldsymbol{w}_\mathsf{K}, \tag{25a}$$
$$\mathbb{E}[x_\mathsf{K}^\star] = \boldsymbol{\Phi}_0\mathbb{E}[x_0] + \boldsymbol{\Phi}_1\bar{\boldsymbol{\delta}}_\mathsf{K} + \boldsymbol{\Phi}_2\boldsymbol{c}_\mathsf{K}, \tag{25b}$$
$$0 \leq \bar{\mathsf{M}}_\mathsf{K}\mathbb{E}[x_\mathsf{K}^\star] + \bar{\mathsf{N}}_\mathsf{K}\bar{\boldsymbol{\delta}}_\mathsf{K} + \boldsymbol{p}_\mathsf{K} \perp \boldsymbol{\mu}_\mathsf{K}^\star \geq 0. \tag{25c}$$

Theorem 2. *Let Assumptions 1 and 2 hold. Then, the MFTGNE strategy profile for MFTDG is given by*

$$\mathsf{u}_\mathsf{K}^\star - \mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = \boldsymbol{\eta}_\mathsf{K}\boldsymbol{\Psi}_0(x_0 - \mathbb{E}[x_0]) + \boldsymbol{\eta}_\mathsf{K}\boldsymbol{\Psi}_1\boldsymbol{w}_\mathsf{K}, \tag{26a}$$
$$\mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = \mathsf{F}\boldsymbol{\mu}_\mathsf{K}^\star + \mathsf{P}, \tag{26b}$$

with $\boldsymbol{\mu}_\mathsf{K}^\star$ being the solution of the following single large-scale linear complementarity problem

$$\mathtt{LCP}:\quad 0 \leq \mathsf{M}\boldsymbol{\mu}_\mathsf{K}^\star + \mathsf{Q} \perp \boldsymbol{\mu}_\mathsf{K}^\star \geq 0, \tag{27}$$

where $\mathsf{M} = (\bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_1 + \bar{\mathsf{N}}_\mathsf{K})(\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^2$, $\mathsf{Q} = \bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_0\mathbb{E}[x_0] + \big((\bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_1 + \bar{\mathsf{N}}_\mathsf{K})(\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^3 + \bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_2\big)\boldsymbol{c}_\mathsf{K} + \boldsymbol{p}_\mathsf{K}$, $\mathsf{F} = (\boldsymbol{\delta}_\mathsf{K}\boldsymbol{\Phi}_1 + \mathbf{I})(\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^2$, $\mathsf{P} = \boldsymbol{\delta}_\mathsf{K}\boldsymbol{\Phi}_0\mathbb{E}[x_0] + \big((\boldsymbol{\delta}_\mathsf{K}\boldsymbol{\Phi}_1 + \mathbf{I})(\mathsf{P}_\mathsf{K}^1)^{-1}\mathsf{P}_\mathsf{K}^3 + \boldsymbol{\delta}_\mathsf{K}\boldsymbol{\Phi}_2\big)\boldsymbol{c}_\mathsf{K}$.

Proof. Substituting (25a) in (21a) results in (26a). Using (25b) in (21b) and (25c) we get $\mathbb{E}[\mathsf{u}_\mathsf{K}^\star] = (\boldsymbol{\delta}_\mathsf{K}\boldsymbol{\Phi}_1 + \mathbf{I})\bar{\boldsymbol{\delta}}_\mathsf{K} + \boldsymbol{\delta}_\mathsf{K}(\boldsymbol{\Phi}_0\mathbb{E}[x_0] + \boldsymbol{\Phi}_2\boldsymbol{c}_\mathsf{K})$ and $0 \leq (\bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_1 + \bar{\mathsf{N}}_\mathsf{K})\bar{\boldsymbol{\delta}}_\mathsf{K} + \bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_0\mathbb{E}[x_0] + \bar{\mathsf{M}}_\mathsf{K}\boldsymbol{\Phi}_2\boldsymbol{c}_\mathsf{K} + \boldsymbol{p}_\mathsf{K} \perp \boldsymbol{\mu}_\mathsf{K}^\star \geq 0$. Finally, using the expression for $\bar{\boldsymbol{\delta}}_\mathsf{K}$ from (24) in the above equations we obtain (26b) and (27), respectively. ■

Remark 5. We note that equations (22) are a matrix representation of the algebraic equations (10) at stage $k\in\mathsf{K}_l$, with $\Lambda_k = \mathbf{R}_k + \mathbf{B}_k'\boldsymbol{\alpha}_{k+1}\mathsf{B}_k$ and $\bar{\Lambda}_k = \bar{\mathbf{R}}_k + \bar{\mathbf{B}}_k'\bar{\boldsymbol{\alpha}}_{k+1}\bar{\mathsf{B}}_k$. Following the recursive procedure outlined in Remarks 2 and 3, the invertibilty of these matrices at every stage $k\in\mathsf{K}_l$, as required by Assumption 2, can be verified using the problem data without the need for solving the `LCP`.

Remark 6. We note that the `LCP` given by (27) is an implicit representation of (8)-(9). Further, if the `LCP` has multiple solutions, then, from Theorem 1, each one of these solutions constitutes a MFTGNE. The existence conditions and numerical methods for the `LCP` have been extensively studied in the optimization community; see [16] for details.

V. Numerical Illustration

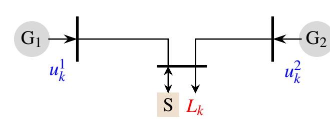
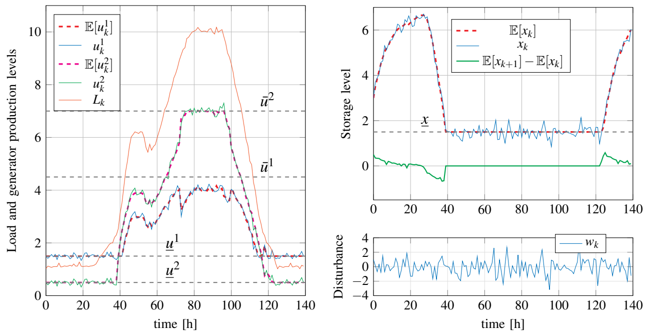


Fig. 1. A microgrid with two generators and one storage unit

In recent years, game theory has been extensively used to analyze energy storage issues that arise in microgrid management; see [17] and [18]. Motivated by these studies, we consider a simplified microgrid model, as shown in Fig. 1. The model comprises two generators, G_1 and G_2, with generation levels u_k^1 and u_k^2, respectively. These generators supply power to a time-varying load represented by L_k, through transmission lines. A storage unit, S, is installed near the load, which can either store surplus generator output (through charging) or supply the load (through discharging) when demand is not met. Let x_k denote the storage level at time instant k, and its evolution due to charging and discharging be given by $x_{k+1} = ax_k + \sum_{i=1}^{2}b^iu_k^i - L_k + \sigma w_k$, where $a\in(0,1)$ and $b^1,b^2\in(0,1)$ account for the natural storage



(a) (b)

Fig. 2. Panel (a) depicts time varying load and generator outputs, and panel (b) depicts the battery storage level and disturbance signal.

depreciation and transmission line losses respectively. We assume that the uncertainties in power generation and storage device operation are modeled by the disturbance process $w_k,\ k \in \mathsf{K}$. We consider the following constraints

$$\text{Storage: } \underline{x} \le \mathbb{E}[x_{k+1}] = a\ \mathbb{E}[x_k] + \sum_{i=1}^{2} b^i\ \mathbb{E}[u_k^i] - L_k, \quad (28a)$$

$$\text{Generation: } \underline{u}^i \le \mathbb{E}[u_k^i] \le \bar{u}^i,\ i = 1,2. \quad (28b)$$

The mixed (coupled) constraint (28a) indicates the reserve level of the storage unit, that is, the mean storage level cannot go below $\underline{x}$. Further, (28b) represent the operational constraints of the generators, that is, the mean/expected production level $\mathbb{E}[u_k^i]$ of each generator $i = 1,2$ cannot go above $\bar{u}^i$ and below $\underline{u}^i$. The generating units seek to minimize their production costs which are proportional to their generation levels. Further, they try to minimize variance in their generation levels. The generating units wish not to have high storage levels when they are able to meet the demand, and also wish to reduce the variance of the storage level. We assume there are no terminal costs. So, the cost functional of each generating unit $i = 1,2$ is given as $J^i = \frac{1}{2}\sum_{k\in\mathsf{K}_l}\big(q_k^i\mathbb{E}[(x_k - \mathbb{E}[x_k])^2] + (q_k^i + \bar{q}_k^i)\mathbb{E}[x_k]^2\big) + \frac{1}{2}\sum_{k\in\mathsf{K}_l}\big(r_k^{ii}\mathbb{E}[(u_k^i - \mathbb{E}[u_k^i])^2] + (r_k^{ii} + \bar{r}_k^{ii})\mathbb{E}[u_k^i]^2\big)$.

For numerical illustration, we consider the following parameter values: $r_k^{ii} = 0.5,\ \bar{r}_k^{ii} = 2.5,\ q_k^i = 3.5,\ \bar{q}_k^i = 0.5,\ i = 1,2,\ k \in \mathsf{K}_l,\ a = 0.9,\ b^1 = 0.90,\ b^2 = 0.94,\ \underline{x} = 1.5,\ \underline{u}^1 = 1.5,\ \underline{u}^2 = 0.5,\ \bar{u}^1 = 4.5,\ \bar{u}^2 = 7,\ K = 140,\ \sigma = 0.2,\ x_0 = 3$ (deterministic). We consider the disturbance signal $w_k,\ k \in \mathsf{K}_l$ to be a white Gaussian noise process. For the chosen parameter values, we note that the conditions required in Assumption 2 are satisfied. We used the freely available PATH solver (available at `https://pages.cs.wisc.edu/~ferris/path.html`) for solving the LCP (27). Fig. 2a illustrates the time varying load and generator production levels. We observe that the generators vary their production levels while satisfying the generation constraints (28b). Fig. 2b illustrates the battery storage levels and the disturbance signal. In particular, we observe that when $\mathbb{E}[x_{k+1}] - \mathbb{E}[x_k] = (a-1)\mathbb{E}[x_k] + b^1\mathbb{E}[u_k^1] + b^2\mathbb{E}[u_k^2] - L_k < 0$, the storage unit discharges towards meeting the demand and thereby reaches its reserve level $\underline{x}$, satisfying the mixed coupled constraint (28a).

VI. Conclusion

We have characterized the solution for a class of linear-quadratic MFTDGs with coupled affine inequality constraints. This involves a multiplier process satisfying implicit complementarity conditions. By reformulating these conditions as a single large-scale linear complementarity problem, we enable computation of these solutions. A numerical example has illustrated our proposed approach. In future work, we aim to generalize the constraint structure to include state and control terms, alongside their mean terms. Further, we also plan to explore the problem in the continuous-time formulation.

References

[1] H. Tembine, "Mean-field-type games," *AIMS Mathematics*, vol. 2, no. 4, pp. 706–735, 2017.

[2] J. Barreiro-Gomez and H. Tembine, *Mean-field-type Games for Engineers*. CRC Press, 2021.

[3] H. Markowitz, "Portfolio selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77–91, 1952.

[4] P. E. Caines, M. Huang, and R. P. Malhamé, *Mean Field Games*. In Handbook of Dynamic Game Theory. Springer International Publishing, Cham, 2018, pp. 345–372.

[5] J. Barreiro-Gomez and H. Tembine, "Mean-field-type model predictive control: An application to water distribution networks," *IEEE Access*, vol. 7, pp. 135 332–135 339, 2019.

[6] B. Djehiche, J. Barreiro-Gomez, and H. Tembine, "Electricity price dynamics in the smart grid: A mean-field-type game perspective," in *23rd International Symposium on Mathematical Theory of Networks and Systems, Hong Kong*, 2018.

[7] A. Aurell and B. Djehiche, "Mean-field type modeling of nonlocal crowd aversion in pedestrian crowd dynamics," *SIAM Journal on Control and Optimization*, vol. 56, no. 1, pp. 434–455, 2018.

[8] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, ser. SpringerBriefs in Mathematics. Springer New York, 2013.

[9] B. Djehiche, H. Tembine, and R. Tempone, "A stochastic maximum principle for risk-sensitive mean-field type control," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2640–2649, 2015.

[10] H. Pham, "Linear quadratic optimal control of conditional mckean-vlasov equation with random coefficients and applications," *Probability, Uncertainty and Quantitative Risk*, vol. 1, pp. 1–26, 2016.

[11] T. E. Duncan and H. Tembine, "Linear–quadratic mean-field-type games: A direct method," *Games*, vol. 9, no. 1, p. 7, 2018.

[12] J. Barreiro-Gomez, T. E. Duncan, B. Pasik-Duncan, and H. Tembine, "Semiexplicit solutions to some nonlinear nonquadratic mean-field-type games: a direct method," *IEEE Transactions on Automatic Control*, vol. 65, no. 6, pp. 2582–2597, 2019.

[13] J. Barreiro-Gomez, T. E. Duncan, and H. Tembine, "Linear-quadratic mean-field-type games with multiple input constraints," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 511–516, 2019.

[14] P. V. Reddy and G. Zaccour, "Open-loop Nash equilibria in a class of linear-quadratic difference games with constraints," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2559–2564, 2015.

[15] P. S. Mohapatra and P. V. Reddy, "Linear-quadratic mean-field-type difference games with coupled affine inequality constraints," 2023. Online supplementary version – https://arxiv.org/abs/2303.07824.

[16] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*. Society for Industrial and Applied Mathematics, 2009.

[17] J. Chen and Q. Zhu, "A game-theoretic framework for resilient and distributed generation control of renewable energies in microgrids," *IEEE Transactions on Smart Grid*, vol. 8, no. 1, pp. 285–295, 2016.

[18] J. Barreiro-Gomez, T. E. Duncan, and H. Tembine, "Linear-quadratic mean-field-type games-based stochastic model predictive control: A microgrid energy storage application," in *2019 American Control Conference (ACC)*, 2019, pp. 3224–3229.