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Robust tube MPC using gain-scheduled policies for a class of LPV systems

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Abstract— **This paper presents a method for robust model predictive control (MPC) of linear parameter varying (LPV) systems considering control policies that are affine functions of the parameter, which is possible when only the 'A' and not the 'B' matrix depends on the uncertain parameter (LPV-A systems). This is less conservative than formulations in which the policy is restricted to perturbations on a feedback law, as it includes such policies as a special case. State and input constraints are handled efficiently by bounding predicted states in a sequence of polyhedra (i.e. tube MPC), that are parameterised by variables in the online optimisation. The resulting controller can be implemented by online solution of a single quadratic programming problem and can exploit rate bounds on the LPV parameters, which requires a pre-processing step at each iteration. Recursive feasibility and exponential stability are proven and the approach is compared to existing methods in numerical examples drawn from other publications, showing reduced conservatism and improved regions of attraction.**

Index Terms— **Predictive control for linear systems, Linear parameter varying systems, Robust control.**

I. Introduction

A significant difficulty with robust MPC is that the control policy used for predictions must have a feedback component to achieve good performance in the presence of uncertainty. This stands in contrast to MPC for deterministic systems, where it is sufficient to optimise over open-loop control sequences. Computing optimal feedback policies is computationally a difficult problem, and to obtain tractable solutions in MPC, the policies considered are typically restricted to some fixed class, reducing performance somewhat. Much research effort in the control community has focused on reducing this conservatism. For additive disturbances, it is possible to optimise over policies that are affine functions of this disturbance [1]. However, when the uncertainty enters the state space equations multiplicatively as an unknown parameter in the state space matrices this is difficult, as the resulting optimisation problem becomes bilinear when predicting over two or more steps. For such systems, robust MPC formulations typically consider either: predicted control inputs from a linear state feedback law [2], perturbations on a fixed state feedback law [3], or interpolation between fixed state feedback laws [4]. A notable exception appears in [5], which considers more general control policies when the uncertainty is restricted to the 'A' matrix in a linear parameter varying (LPV) formulation (termed 'LPV-A' systems in that paper). However, the policy is calculated via multiparametric dynamic programming so is only suitable for systems with a low dimensional state space. In this paper we introduce an online MPC algorithm for such systems.

Nonlinear MPC methods are applicable to LPV models, but typically require solution of a (nonconvex) nonlinear programming problem online [6]. Typically the parameter in LPV models is assumed to be time-varying but measurable for the purposes of control, so in MPC of LPV models it is desirable to formulate controllers that exploit knowledge of this parameter in their prediction structure. To achieve tractable online optimisations, early approaches generally required conservative ellipsoidal bounding of the predicted state yielding LMI constraints in the online optimisation [2], [7]. A solution to these issues is to bound the predicted state in a sequence of polyhedra (often called a 'tube') and then use each polyhedron to tighten the constraints applied in the online optimisation, a method that was applied to systems with additive disturbances in [8], building on previous approaches that has considered constraint restrictions for systems with additive disturbances [9]. Conservatism can be reduced if the polyhedra are parameterised by variables in the online optimisation [10], a technique which also allows tube MPC to be applied to systems with time-varying parameters by solving a single convex QP problem [11]. A survey of such robust MPC techniques appears in [12]. Recently, tube MPC has also been extended to LPV systems considering rate bounds on the parameter [13] and more varied tube parameterisations have been introduced [14].

In the current work we introduce an MPC for LPV-A systems where the uncertain parameter is assumed measurable at each time. Novelly for a tube MPC, we optimise over policies in which the predicted control action is an affine function of the future uncertain parameters, and can achieve this by solving a single convex QP problem online. For LPV-A systems where the parameter is measurable, this is more general than optimising over perturbations on state feedback policies (e.g. [3], [11]). This is motivated by several systems of practical interest which are difficult to handle with existing robust MPC approaches and where gain-scheduled linear controllers have typically been applied until now, including rider assistance systems for motorcycles [15] and blade pitch control for offshore wind turbines [16], which require feedback controllers gain-scheduled on vehicle speed and wind speed respectively. To the best of the authors' knowledge, this is the first tube-based MPC for LPV systems that optimises a parameter-dependent control law online in a convex QP. This extends the robust MPC framework of [11] to LPV systems, with improvements including more general

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tube parameterisations, inclusion of rate bounds, relaxation of the contractivity requirement on tube cross-sections, and exponential (rather than asymptotic) stability results.

Notation

The subscript notation $v_{k|t}$ denotes the vector v at time k, as predicted at time t. Superscripts in round brackets are used to define affine dependencies of matrices on a time-dependent parameter $\theta_t \in \mathbb{R}^q$ in the form $M(\theta_t) = M^{(0)} + \sum_{i=1}^{q} \Delta_M^{(i)} \theta_{t,i}$, with $\theta_{t,i}$ denoting element i of the vector θ_t. The notation $Co(M^{[j]})$ denotes the convex hull of the vectors or matrices $M^{[j]}$ for $j = 1, 2, \ldots, r$, where the square brackets are intended to remind the reader that these are vertices of a polytope and are distinct from the superscripts in round brackets. We denote Minkowski summation for sets $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} + \mathcal{B}$. Finally, we use $\underline{1}$ to denote a column vector with 1 as each element.

II. Problem Formulation

We consider an LPV-A system with an input $u_t \in \mathbb{R}^m$, state $x_t \in \mathbb{R}^n$ and parameter $\theta_t \in \mathbb{R}^q$, which is available to the controller at any time t. The dynamics are given by

$$x_{t+1} = A(\theta_t) x_t + B u_t \tag{1}$$

where the matrix $A(\theta_t) \in \mathbb{R}^{n \times n}$ depends affinely on θ_t as $A(\theta_t) = A^{(0)} + \sum_{i=1}^{q} \Delta_A^{(i)} \theta_{t,i}$ where for all times t, θ_t lies within a polytope $\Theta = Co(\theta^{[j]})$. Matrix $B \in \mathbb{R}^{n \times m}$ has no such dependence. Optionally, we can impose rate bounds $\delta\theta_t = \theta_t - \theta_{t-1} \in \mathcal{D}$ for a polytope $\mathcal{D}$. The performance objective is to minimise a worst-case quadratic function,

$$J_0 = \max_{\theta_0, \theta_1, \ldots} \sum_{t=0}^{\infty} \left(x_t^T Q x_t + u_t^T R u_t \right) \tag{2}$$

in which matrix $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite and $R \in \mathbb{R}^{m \times m}$ is positive definite. This minimisation must be carried out while robustly (i.e. for all possible $\theta_0, \theta_1, \ldots$, etc.) satisfying the mixed state-input constraints

$$F(\theta_t) x_t + G u_t \leq \underline{1} \tag{3}$$

where matrix $F(\theta_t) \in \mathbb{R}^{n_c \times n}$ may depend affinely on θ_t as $F(\theta_t) = F^{(0)} + \sum_{i=1}^{q} \Delta_F^{(i)} \theta_{t,i}$, but the matrix $G \in \mathbb{R}^{n_c \times m}$ has no such dependence. Mixed state-input constraints are considered for additional generality, noting that they are required to constrain outputs of systems with direct feed-through. Separable state and input constraints $F_x x \leq \underline{1}$ and $G_u u \leq \underline{1}$ may also be expressed in this form by choosing e.g. $F^T = [F_x \; 0]^T$ and $G^T = [0 \; G_u]^T$. We make two assumptions on the system dynamics (1). Firstly, that the system is quadratically stabilisable by a gain-scheduled feedback $K(\theta_t)$ in the form: $K(\theta_t) = K^{(0)} + \sum_{i=1}^{q} \Delta_K^{(i)} \theta_{t,i}$. This will allow the performance objective (2) to be upper bounded by a quadratic function so that the MPC optimisation is a QP:

Assumption 1. *There exists a gain-scheduled state feedback $K(\theta_t) = K^{(0)} + \sum_{i=1}^{q} \Delta_K^{(i)} \theta_{t,i}$ such that the system $x_{t+1} = A(\theta_t) x_t + BK(\theta_t) x_t$ admits a quadratic Lyapunov function.*

We also make an observability assumption, which is required for closed-loop stability:

Assumption 2. *There exists some $\theta \in \Theta$ such that the pair $(Q^{1/2}, A(\theta))$ is observable.*

III. Parameter-dependent control policies

The proposed MPC uses a control policy affine in $\theta_{k|t}$

$$u_{k|t}(\theta_{k|t}) = K(\theta_{k|t}) x_{k|t} + c_{k|t}^{(0)} + \sum_{i=1}^{q} c_{k|t}^{(i)} \theta_{k,i} \tag{4}$$

where the values $c_{k|t}^{(0)}$ and $c_{k|t}^{(i)}$ are to be optimised by the MPC. To simplify notation in what follows, we will define $c_{k|t}(\theta_{k|t}) = c_{k|t}^{(0)} + \sum_{i=1}^{q} c_{k|t}^{(i)} \theta_{k,i}$ such that we may write $u_{k|t}(\theta_{k|t}) = K(\theta_{k|t}) x_{k|t} + c_{k|t}(\theta_{k|t})$. With this input parameterisation, we may write the prediction dynamics as

$$x_{k+1|t} = \bar{A}(\theta_{k|t}) x_{k|t} + B c_{k|t}(\theta_{k|t}) \tag{5}$$

and the constraint (3) applied to the predictions as

$$\bar{F}(\theta_{k|t}) x_{k|t} + G c_{k|t}(\theta_{k|t}) \leq \underline{1} \tag{6}$$

where we define $\bar{A}(\theta_{k|t}) = A(\theta_{k|t}) + BK(\theta_{k|t})$ and $\bar{F}(\theta_{k|t}) = F(\theta_{k|t}) + GK(\theta_{k|t})$.

Compared to robust MPC using $u_{k|t} = K x_{k|t} + c_{k|t}$ with $c_{k|t}$ independent of the parameters such as [3], [9], [11], the affine dependence of $c_{k|t}(\theta_{k|t})$ on $\theta_{k|t}$ provides the controller with some additional freedom to allow for different possible future values of $\theta_{k|t}$. Values of θ_k for $k > t$ are not known at time t, but a suitable interpretation is that the controller is planning different control inputs for different values of θ_k that may occur. This is an 'affine-in-the-parameter' formulation that is conceptually similar to 'affine-in-the-disturbance' formulations such as [1] in that predictions of states and inputs depend on values that are currently unknown, but will be known once time k is reached.

IV. Bound on minimax cost function

We now develop an autonomous form of the LPV prediction dynamics (5) for quadratic bounding of the minimax objective (2). To this end, we introduce the concatenation of the variables $c_{k|t}^{(i)}$ into a vector $d_{k|t} \in \mathbb{R}^{mq}$ which specifies the control policy at a single prediction time k, and a further concatenation of the $d_{k|t}$ for $k = t$ to $k = t + N - 1$ into a single vector $\underline{d}_t \in \mathbb{R}^{Nmq}$ that specifies the entire control policy over a prediction horizon of length N at a time t:

$$d_{k|t} = \begin{bmatrix} c_{k|t}^{(0)} \\ c_{k|t}^{(1)} \\ \cdots \\ c_{k|t}^{(q)} \end{bmatrix}, \quad \underline{d}_t = \begin{bmatrix} d_{t|t} \\ d_{t+1|t} \\ \vdots \\ d_{t+N-1|t} \end{bmatrix}$$

We also define the unique matrices T and $S^{(i)}$ (for $i = 0, 1, \ldots$) to shift up and select elements of $\underline{d}_t$ such that

$$T \underline{d}_t = \begin{bmatrix} d_{t+1|t} \\ \vdots \\ d_{t+N-1|t} \\ 0 \end{bmatrix}, \quad S^{(i)} \underline{d}_t = c_{t|t}^{(i)}$$

i.e. $T \in \mathbb{R}^{Nmq\times Nmq}$ is a block matrix with $mq \times mq$ identity matrices on its superdiagonal and $S^{(0)} = [I\ 0\ 0\ \ldots]$, $S^{(1)} = [0\ I\ 0\ \ldots]$ etc. Finally, we introduce $\Delta_S^{(i)} = S^{(i)} - S^{(0)}$ and a matrix-valued function $S(\theta) \in \mathbb{R}^{m\times Nmq}$ defined by $S(\theta_t) = S^{(0)} + \sum_{i=1}^{q} \Delta_S^{(i)}\theta_{t,i}$. With these definitions, the prediction dynamics (5) under the control policy (4) can be restated in the equivalent form

$$\begin{bmatrix} x_{k+1|t} \\ \underline{d}_{k+1|t} \end{bmatrix} = \begin{bmatrix} \bar{A}(\theta_{k|t}) & BS(\theta_{k|t}) \\ 0 & T \end{bmatrix} \begin{bmatrix} x_{k|t} \\ \underline{d}_{k|t} \end{bmatrix} \tag{7}$$

which is an autonomous LPV system , in which we have inductively defined $\underline{d}_{k+1|t} = T\underline{d}_{k|t}$ with $\underline{d}_{t|t} = \underline{d}_t$.

Proposition 1 (Minimax performance bound)**.** *Let $W \succeq 0$ be a matrix satisfying the linear matrix inequalities*

$$\begin{aligned} W - \begin{bmatrix} \bar{A}(\theta^{[j]}) & BS(\theta^{[j]}) \\ 0 & T \end{bmatrix}^T W \begin{bmatrix} \bar{A}(\theta^{[j]}) & BS(\theta^{[j]}) \\ 0 & T \end{bmatrix} \\ \succeq \begin{bmatrix} Q + (K(\theta^{[j]}))^T RK(\theta^{[j]}) & 0 \\ 0 & (S(\theta^{[j]}))^T RS(\theta^{[j]}) \end{bmatrix} \end{aligned} \tag{8}$$

for $j = 1, \ldots, r$, then the minimax performance objective (2) is upper bounded as

$$\max_{\theta_0,\theta_1,\ldots} \sum_{k=t}^{\infty} \left(x_{k|t}^T Q x_{k|t} + u_{k|t}^T R u_{k|t} \right) \leq \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix} \tag{9}$$

under the scheduled control policy (4).

Proof. If (8) holds for all j, then the analogous expression with $A(\theta^{[j]}), S(\theta^{[j]}), K(\theta^{[j]})$ replaced by $A(\theta_{k|t}), S(\theta_{k|t}), K(\theta_{k|t})$ holds at all prediction times k as $A(\theta) \in Co(A(\theta^{[j]}))$, $S(\theta) \in Co(S(\theta^{[j]}))$ and $K(\theta) \in Co(K(\theta^{[j]}))$. The results follow by multiplying (8) on the left and right with $[x_t^T \underline{d}_t^T]$ and its transpose respectively, then summing over $k = t, \ldots, \infty$. The matrix T is nilpotent, such that $\underline{d}_t$ converges to 0 in finite time, and therefore by Assumption 1, the autonomous system (7) is asymptotically stable so that $\lim_{t\to\infty}[x_t^T \underline{d}_t^T] = 0$. □

The expression (9) is a convex quadratic function of x_t and $\underline{d}_t$ and can be used as the objective function of a QP to optimise over control policies of the form (4).

V. Constraint handling using tubes

To apply constraints, we bound the predicted state $x_{k|t}$ of the system in polyhedral tube cross-sections $\mathcal{T}_{k|t}$ defined by

$$\mathcal{T}_{k|t} = \mathcal{T}(\alpha_{k|t}) = \{x \in \mathbb{R}^n |\ Vx \leq g(\alpha_{k|t})\} \tag{10}$$

where $\alpha_{k|t} \in \mathbb{R}^s$ is a vector of variables in the online optimisation that determines the shape and size of the tube cross-section at prediction time k. The function $g(\alpha)$, which parameterises the class of possible tube cross-sections, is assumed to be affine in α. We also assume that there exists some value α^* such that the set $\mathcal{T}(\alpha^*)$ is an admissible set for $u_{k|t} = K(\theta_{k|t})x_{k|t}$ (i.e. positively invariant and within the constraint bounds). This implies that V cannot be chosen arbitrarily but must be constructed through an algorithm to find an admissible polyhedron. Such a polyhedron exists due to Assumption 1. Considering possible parameterisations, if we choose $g(\alpha) = Vz + \gamma g_0$ where $\alpha^T = [z^T, \gamma]$ for some vector z and scalar γ, this gives the 'homothetic' parameterisation described e.g. in [17] which is produced by translating and scaling a polyhedron $\{x \in \mathbb{R}^n |\ Vx \leq h\}$ around the state space. For a state space of dimension n, this implies $n+1$ decision variables $\alpha_{k|t}$ in the online optimisation to parameterise the tube cross section $\mathcal{T}_{k|t}$ for each prediction time k. A second possibility is to choose the dimension of α to match the number of rows of V and define $g(\alpha) = \alpha$, which gives the parameterisation found in [11]. In that case, each tube cross-section $\mathcal{T}(\alpha_{k|t})$ is defined by $Vx \leq \alpha_{k|t}$. This is more flexible, but could require more decision variables in the online optimisation.

To ensure that the sequence of tube cross-sections $\mathcal{T}(\alpha_{k|t})$ contains all possible future states under the dynamics (1), we apply the following conditions for all $k = t, \ldots, t+N-1$ and all possible $\theta_{k|t}$ over the prediction horizon:

$$\mathcal{T}_{k|t} \subseteq \{x \in \mathbb{R}^n |\ \bar{A}(\theta_{k|t})x + Bc_{k|t}(\theta_{k|t}) \in \mathcal{T}_{k+1|t}\} \tag{11}$$

$$\mathcal{T}_{k|t} \subseteq \{x \in \mathbb{R}^n |\ \bar{F}(\theta_{k|t})x_t + Gc_{k|t}(\theta_{k|t}) \leq \underline{1}\} \tag{12}$$

In words, (11) ensures that $\mathcal{T}_{k|t}$ is a subset of the preimage of $\mathcal{T}_{k+1|t}$ under the system dynamics and (12) ensures that constraints (3) will be satisfied. For the final predicted set $\mathcal{T}_{t+N|t}$, we apply the corresponding terminal conditions

$$\mathcal{T}_{t+N|t} \subseteq \{x \in \mathbb{R}^n |\ \bar{A}(\theta_{k|t})x \in \mathcal{T}_{t+N|t}\} \tag{13}$$

$$\mathcal{T}_{t+N|t} \subseteq \{x \in \mathbb{R}^n |\ \bar{F}(\theta_{k|t})x_t \leq \underline{1}\} \tag{14}$$

which ensure that this final tube cross section is an admissible set under the feedback $u_{k|t} = K(\theta_{k|t})x_{k|t}$.

These set theoretic conditions can be converted to constraints on the parameters $\alpha_{k|t}$ using the following lemma, which gives a sufficient condition:

Lemma 1 (Subsets of parameterised sets)**.** *Let $\mathcal{P}_1(\beta_1) = \{x : V_1 x \leq g_1(\beta_1)\}$ and $\mathcal{P}_2(\beta_2) = \{x : V_2 x \leq g_2(\beta_2)\}$, where g_1 and g_2 are affine functions of vectors β_1 and β_2, and let $H^*(g_1(\beta_1^*), V_1, V_2)$ be a matrix with rows h_i^T as*

$$h_i^T = \underset{h\geq 0,\ h^T V_1 = v_{2,i}^T}{\arg\min}\ h^T g_1(\beta_1^*) \tag{15}$$

for any $\beta_1^* \in \mathbb{R}^s$, where $v_{2,i}^T$ denotes the i'th row of V_2. Then $\mathcal{P}_1(\beta_1) \subseteq \mathcal{P}_2(\beta_2)$ if the inequality

$$H^*(g_1(\beta_1^*), V_1, V_2) g_1(\beta_1) \leq g_2(\beta_2). \tag{16}$$

holds for the given values of β_1 and β_2.

Proof. Note that if $\mathcal{P}_1(\beta_1)$ is an empty set, the subset condition holds trivially. If $\mathcal{P}_1(\beta_1) \neq \emptyset$, then considering the rows $v_{2,i}^T$ of V_2, the subset relation holds if and only if

$$\max_{x\in\mathcal{P}_1(\beta_1)} v_{2,i}^T x \leq g_{2,i}(\beta_2) \tag{17}$$

for all i where $g_{2,i}(\beta_2)$ denotes the i'th element of $g_2(\beta_2)$. We define a Lagrange dual function (see e.g. [18]) associated with the maximisation as $L(x, h) = v_{2,i}^T x + h^T(g_1(\beta_1) -$

$V_1x)$ in which $h \geq 0$ denotes the Lagrange multipliers of the constraints $V_1x \leq g_1(\beta_1)$ defining $\mathcal{P}_1(\beta_1)$. As the term $h^T(g_1(\beta_1) - V_1x)$ is nonnegative we have the bound $v_{2,i}^T x \leq L(x,h)$ for any $h \geq 0$ and any $x \in \mathcal{P}_1(\beta_1)$. If we choose h according to (15) such that $h^T V_1 = v_{2,i}^T$ then $L(x,h) = h^T g_1(\beta_1)$ as the terms involving x cancel. Hence for any $x \in \mathcal{P}_1(\beta_1)$, we have the bound $v_{2,i}^T x \leq h^T g_1(\beta_1)$, but also (16) implies $h^T g_1(\beta_1) \leq g_{2,i}(\beta_2)$ so that (17) holds as required. As this holds for all rows i, the result follows. □

As the rows of $H^*(g_1(\beta_1^*), V_1, V_2)$ are Lagrange multipliers of (17), at most $n = \dim(x)$ constraints will be active at the maximum and the complementary slackness condition $h^T(g_1(\beta_1) - V_1x) = 0$ implies that h contains at most n nonzeros, so the resulting matrix $H^*(g_1(\beta_1^*), V_1, V_2)$ is sparse. Using Lemma 1, we define parameter-dependent matrices $H_p(\theta_{k|t})$ and $H_f(\theta_{k|t})$ to apply (11) and (12) as $H_p(\theta) = H_p^{(0)} + \sum_{i=1}^q \Delta_{H_p}^{(i)} \theta_i$, $H_f(\theta) = H_f^{(0)} + \sum_{i=1}^q \Delta_{H_f}^{(i)} \theta_i$ where $\Delta_{H_p}^{(i)}$ and $\Delta_{H_f}^{(i)}$ for $i = 1, \dots, q$ are

$$\Delta_{H_p}^{(i)} = H^*(g(\alpha^*),\ V,\ V\Delta_{\bar{A}}^{(i)}) \tag{18}$$

$$\Delta_{H_f}^{(i)} = H^*(g(\alpha^*),\ V,\ \Delta_{\bar{F}}^{(i)}) \tag{19}$$

where we have defined $\Delta_{\bar{A}}^{(i)} = \Delta_A^{(i)} + B\Delta_K^{(i)}$ and $\Delta_{\bar{F}}^{(i)} = \Delta_F^{(i)} + G\Delta_K^{(i)}$. Matrices $H_p^{(0)}$ and $H_f^{(0)}$ are similarly

$$H_p^{(0)} = H^*(g(\alpha^*),\ V,\ V\bar{A}^{(0)}) \tag{20}$$

$$H_f^{(0)} = H^*(g(\alpha^*),\ V,\ \bar{F}^{(0)}) \tag{21}$$

where we have defined $\bar{A}^{(0)} = A^{(0)} + BK^{(0)}$ and $\bar{F}^{(0)} = F^{(0)} + GK^{(0)}$. Using $H_p(\theta_{k|t})$ and $H_f(\theta_{k|t})$, we may state sufficient conditions for (11) and (12):

Proposition 2 (Tube constraints)**.** *If, for a given $\theta_{k|t}$,*

$$H_p(\theta_{k|t})g(\alpha_{k|t}) + VBc_{k|t}(\theta_{k|t}) \leq g(\alpha_{k+1|t}) \tag{22a}$$

$$H_f(\theta_{k|t})g(\alpha_{k|t}) + Gc_{k|t}(\theta_{k|t}) \leq \underline{1} \tag{22b}$$

then (11) and (12) hold, i.e. the tube cross sections will bound the predicted state and satisfy state and input constraints.

Proof. The subset relations (11) and (12) may be written as

$$\{x \in \mathbb{R}^n |\ Vx \leq g(\alpha_{k|t})\} \subseteq \{x \in \mathbb{R}^n |\ V(\bar{A}(\theta_{k|t})x + Bc_{k|t}(\theta_{k|t})) \leq g(\alpha_{k+1|t})\}$$

$$\{x \in \mathbb{R}^n |\ Vx \leq g(\alpha_{k|t})\} \subseteq \{x \in \mathbb{R}^n |\ \bar{F}(\theta_{k|t})x + Gc_{k|t}(\theta_{k|t}) \leq \underline{1}\}$$

from which the results follow by applying Lemma 1. □

Proposition 3 (Terminal constraints)**.** *If, for a given $\theta_{k|t}$,*

$$H_p(\theta_{k|t})g(\alpha_{N|t}) \leq g(\alpha_{N|t}) \tag{23a}$$

$$H_f(\theta_{k|t})g(\alpha_{N|t}) \leq \underline{1} \tag{23b}$$

then conditions (13) and (14) hold, i.e. the terminal tube cross-section $\mathcal{T}_{t+N}$ is an admissible set.

Proof. Follows from Proposition 2 by substituting $k = N$, $k+1 = N$ and $c_{k|t}(\theta_{k|t}) = 0$ □

VI. MPC ALGORITHM AND CLOSED-LOOP PROPERTIES

Online, we perform a pre-processing step before the QP solution to consider rate bounds in a similar manner to [13]. The preprocessing determines $\theta_{k|t}^{[j]}$ for $j = 1, \dots, r$ such that $\theta_{k|t} \in \mathrm{Co}\left(\theta_{k|t}^{[j]}\right) = \Theta_{k|t}$ at each prediction time k. This is conservative but straightforward, and we calculate $\Theta_{k|t}$ as

$$\Theta_{k|t} = (\{\theta_t\} + (k-t)\mathcal{D}) \cap \Theta \tag{24}$$

in which '+' denotes the Minkowski sum of sets. This calculation can be simplified in some cases, for example if both Θ and $\mathcal{D}$ are hyper-rectangles as in [13].

Optimisation 1.

$$\begin{aligned}
&\underset{x,\alpha_{k|t},c_{k|t}^{(i)}}{\textit{minimise}} && \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix} \\
&\textit{subject to} && Vx_t \leq g(\alpha_t), \\
& && H_p(\theta_{k|t}^{[j]})g(\alpha_{k|t}) + VBc_{k|t}(\theta_{k|t}^{[j]}) \leq g(\alpha_{k+1|t}), \\
& && H_f(\theta_{k|t}^{[j]})g(\alpha_{k|t}) + Gc_{k|t}(\theta_{k|t}^{[j]}) \leq \underline{1}, \\
& && H_p(\theta_{N|t}^{[j]})g(\alpha_{N|t}) \leq g(\alpha_N), \\
& && H_f(\theta_{N|t}^{[j]})g(\alpha_{N|t}) \leq \underline{1}, \\
& && \textit{for all } j = 1, \dots, r \\
& && \textit{for all } k = t, \dots, t+N-1
\end{aligned}$$

The MPC algorithm can now be stated as:

Algorithm 1. *Repeat for all times $t = 0, 1, \dots$*

1) *Measure the current state x_t and θ_t,*
2) *Perform pre-processing to apply rate bounds and find $\theta_{k|t}^{[j]}$ for $k = t, t+1, \dots, t+N$ and $j = 1, 2, \dots, r$,*
3) *Solve Optimisation 1 and apply the control $u_t = K(\theta_t) + c_t(\theta_t)$ to the system.*

We prove that Algorithm 1 remains feasible and stabilises the system, with the state approaching the origin exponentially in closed-loop if Optimisation 1 can be solved at $t = 0$.

Theorem 1 (Recursive feasibility)**.** *If there is a solution to Optimisation 1 at time t and the state x_t evolves according to the dynamics (1) with the control policy (4), then there exists a solution to Optimisation 1 at time $t+1$.*

Proof. Considering the sets $\Theta_{k|t}$ resulting from pre-processing, we have $\Theta_{k|t+1} \subseteq \Theta_{k|t}$. Hence due to the affine dependence of the constraints in Optimisation 1 on $\theta_{k|t}^{[j]}$, if the variables $c_{k|t}^{(i)}$ and $\alpha_{k|t}$ satisfy constraints at time t, the same values will satisfy these constraints at time $t+1$. This is the case for all variables except for $\alpha_{t+N+1|t+1}$ and $c_{t+N+1|t+1}^{(i)}$ which did not appear in Optimisation 1 at time t. For these we choose $\alpha_{t+N+1|t+1} = \alpha_{t+N|t}$ and $c_{t+N+1|t+1}^{(i)} = 0$ as the terminal conditions (13) and (14) ensure that $\mathcal{T}(\alpha_{t+N|t})$ is an admissible set for $u_t = K(\theta_t)x$. We have therefore constructed a feasible point at time $t+1$. □

Theorem 2 (Exponential stability)**.** *When the scheduled control policy (4) is applied using the solution of Optimisation*

1, and the state x_t follows the LPV-A dynamics (1), then

$$\begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix}^T W \begin{bmatrix} x_t \\ \underline{d}_t \end{bmatrix} \leq \varepsilon^{\frac{t}{n}-1} \begin{bmatrix} x_0 \\ \underline{d}_0 \end{bmatrix}^T W \begin{bmatrix} x_0 \\ \underline{d}_0 \end{bmatrix}$$

for some $0 \leq \varepsilon < 1$ where $n = \dim(x)$, so both x_t and $\underline{d}_t$ converge exponentially to zero in closed-loop.

Proof. Multiplying (8) on the left and right by $[x_{k|t}^T \underline{d}_{k|t}^T]$ and its transpose respectively, noting that as $\theta_{k|t} \in Co(\theta_{k|t}^{[j]})$ the analogous inequality holds for any given $\theta_{k|t}$, then summing that inequality over $k, \dots, k+n-1$ gives:

$$\begin{bmatrix} x_{k|t} \\ \underline{d}_{k|t} \end{bmatrix}^T W \begin{bmatrix} x_{k|t} \\ \underline{d}_{k|t} \end{bmatrix} - \begin{bmatrix} x_{k+n|t} \\ \underline{d}_{k+n|t} \end{bmatrix}^T W \begin{bmatrix} x_{k+n|t} \\ \underline{d}_{k+n|t} \end{bmatrix}$$
$$\geq \max_{\theta_{k|t}, \theta_{k+1|t}, \dots} \sum_{j=k}^{k+n-1} \left(x_{j|t}^T Q x_{j|t} + u_{j|t}^T R u_{j|t} \right) \quad (25)$$

The observability condition of Assumption 2 implies that the right hand side of this expression is positive definite in $x_{k|t}$ and $\underline{d}_{k|t}$. Denoting $[x_{k|t}^T \underline{d}_{k|t}^T]$ by $s_{k|t}$, we can therefore find constants δ_1 and δ_2 such that $\delta_1 \|s\|^2 \leq s^T W s \leq \delta_2 \|s\|^2$ for all s and $\max_{\theta_{k|t}, \theta_{k+1|t}, \dots} \sum_{j=k}^{k+n-1} \left(x_{j|t}^T Q x_{j|t} + u_{j|t}^T R u_{j|t} \right) \geq \delta_1 \|s_{k|t}\|^2$. Hence, considering predicted values at time t,

$$s_{k+n|t}^T W s_{k+n|t} \leq (1 - \delta_1/\delta_2) s_{k|t}^T W s_{k|t} \quad (26)$$

for all t. Due to recursive feasibility, the actual (not predicted) values satisfy $s_{t+n}^T W s_{t+n} \leq s_{t+n|t}^T W s_{t+n|t}$ so by putting $k = t$ in (26), $s_{t+n}^T W s_{t+n} \leq (1 - \delta_1/\delta_2) s_t^T W s_t$ holds in closed-loop operation of the MPC. Assumption 2 implies that the quantity $x_t^T Q x_t + u_t^T R u_t$ can equal zero for at most n time steps in closed-loop operation for a nonzero x_t. From (25) the same therefore applies to $s_t^T W s_t$, and (26) implies that after n timesteps it must decrease by at least a factor of $\varepsilon = 1 - \delta_1/\delta_2$. The given result follows. □

VII. SIMULATION EXAMPLES

A. Example 1

The first comparison uses a system from [2], such that $A^{(1)}$, $A^{(2)}$, B and x_0 are defined by:

$$A^{(1)} = \begin{bmatrix} 0.2730 & 0.0660 & 0.3021 & -0.5012 \\ 0.2717 & 0.4416 & 0.5602 & -0.7123 \\ 0.3051 & -0.7865 & 0.7651 & -0.3121 \\ 0.7962 & -0.1452 & 0.5231 & -0.9345 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 0.2093 & -0.1981 & 0.2394 & 0.5671 \\ 0.2717 & 0.4598 & 0.5602 & 1.3782 \\ -0.4700 & 0.6700 & -0.8600 & -1.2400 \\ 0.3456 & -0.6312 & -1.4594 & 1.8936 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.2300 \\ 0.2601 \\ 0.1213 \\ 1.3452 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -0.3964 \\ 0.4377 \\ -1.0905 \\ 1.1137 \end{bmatrix}$$

and the uncertain matrix $A(\theta)$ is given by the expression $A(\theta) = \theta A^{(1)} + (1 - \theta) A^{(2)}$. The control objectives are specified by $Q = I$, $R = 1$, and the constraints $|u| \leq 2$ and $|x_3| \leq 1.4$. The time variation of $\theta \in [0, 1]$ is the sigmoid $\theta = 1 - \frac{1}{1 - e^{(t-10)}}$ centered at $t = 10$. For this system, a suitable state feedback $K(\theta) = \theta K^{(1)} + (1 - \theta) K^{(2)}$ may be found by solving the LMI optimisation (see [19]):

$$\text{maximise} \quad Trace(X)$$
$$\text{subject to:}$$
$$\begin{bmatrix} X & 0 & 0 & A^{(j)} X + B Y^{(j)} \\ 0 & Q^{-1} & 0 & X \\ 0 & 0 & R^{-1} & Y^{(j)} \\ * & * & * & X \end{bmatrix} \succeq 0$$

and defining $K^{(j)} = Y^{(j)} X^{-1}$. We compute the maximal admissible set, $Vx \leq g_0$, define $H_p^{(j)}$ and $H_f^{(j)}$ according to (18) and (19), and use a homothetic tube, $g(\alpha) = Vz + \gamma g_0$ where $\alpha^T = [z^T, \gamma]$. Table I and Figure 1 show the results of the LPV-A tube MPC of Algorithm 1 compared with the LMI-based technique for MPC of LPV systems from [2]. Yalmip [20] and the solvers OSQP [21] and MOSEK were used for numerical experiments in MATLAB, using the default solver settings in each case. For $N = 5$, Algorithm 1 achieves better performance in terms of computation time and closed-loop cost than the LMI-based algorithm from [2]. For Algorithm 1, the constraint $u_0 \geq -2$ is active at the initial time in Fig 1, while the LMI-based method shows conservatism as $u_0 \neq -2$.

B. Example 2

We additionally consider the example system from [11]:

$$A^{(0)} = \begin{bmatrix} 0.261 & -1.098 \\ 0.891 & 0.419 \end{bmatrix} \quad B = \begin{bmatrix} 0.319 \\ -1.308 \end{bmatrix}$$
$$\Delta_A^{(1)} = \begin{bmatrix} 0 & 0.125 \\ -0.125 & 0 \end{bmatrix} \quad \Delta_A^{(2)} = \begin{bmatrix} 0.025 & 0 \\ 0.125 & -0.025 \end{bmatrix}$$
$$\Delta_A^{(3)} = \begin{bmatrix} -0.025 & -0.125 \\ 0 & 0.025 \end{bmatrix}$$

with $\Theta \subseteq \mathbb{R}^3$ defined as the unit simplex. The control objectives are given by $Q = I$ and $R = 1$ and K was chosen as the corresponding LQR feedback $K = \begin{bmatrix} 0.484 & -0.440 \end{bmatrix}$. There is an input constraint $|u| \leq 12.5$ and a state constraint $|x_2| \leq 41.7$. The tube is parameterised as $Vx \leq \alpha$, with V defined by the maximal admissible set, as in [11], but additionally we apply rate bounds $|\delta\theta_t| \leq 0.2$.

We consider 4 different MPC controllers using natural modifications of Optimisation 1: Robust MPC, i.e. the controller does not know parameter θ_t at each time t; LPV MPC, which exploits knowledge of this parameter and optimises a control policy $u_{k|t} = Kx_{k|t} + c_{k|t}$; LPV with rate bounds, which further uses knowledge of the rate bounds to relax constraints; and finally, the LPV-A MPC of this paper which uses both rate bound knowledge and a scheduled policy to optimise a policy $u_{k|t} = Kx_{k|t} + c(\theta_{k|t})$. Additionally, we compare with the robust MPC algorithm of [11], choosing $N = 6$ for all methods. For each we perform a closed-loop simulation from the initial condition $x_0 = [-40 - 20]^T$ for the same randomly-chosen sequence θ_t satisfying the rate bounds, with the results shown in Table II. Inspecting Table II, the methods of this paper require less computation time

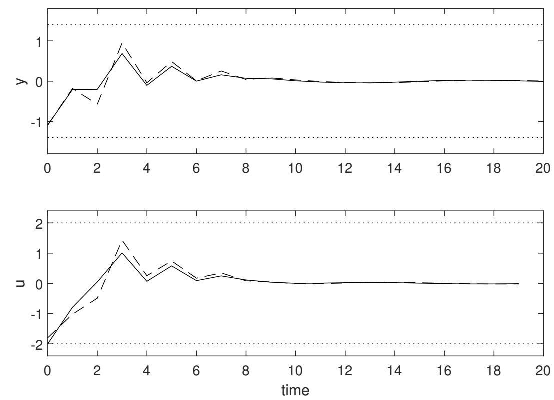

Fig. 1. Performance comparison of LPV MPC controllers (Example 1). Dashed: Quasi-min-max MPC, Solid: LPV Tube MPC (N=5) with scheduled policy. Constraint boundaries are also shown as dotted lines.

MPC Algorithm	N=5 (Alg. 1)	N=10 (Alg. 1)	Quasi-Min-Max [2]
Solver	OSQP	OSQP	MOSEK
Closed-loop cost	11.2	11.1	14.8
Computation time /ms	5.41	10.43	7.81

TABLE I

COMPARISON OF COST AND COMPUTATION TIME FOR EXAMPLE 1

MPC Algorithm	Robust (Alg. 1)	LPV (Alg. 1)	+ rate bound	+ scheduling	Robust [11]
Solver	OSQP	OSQP	OSQP	OSQP	OSQP
ROA volume	5863	7731	8979	9824	6198
Closed-loop cost	4530	4522	3045	3002	4546
Comp. time /ms	3.08	3.16	3.19	4.37	5.70

TABLE II

COMPARISON OF COST AND COMPUTATION TIME FOR EXAMPLE 2

than those of [11] yet can provide reductions in closed-loop cost and regions of attraction (ROA) when considering this LPV system. We also calculate the ROA for $\theta_0 = 0$, which are shown in Figure 2. Compared to robust MPC, the assumption that the LPV parameter is measurable with rate bounds provides a larger ROA. Using the gain-scheduled policy of this paper provides a further improvement, with no additional assumptions required on the system dynamics.

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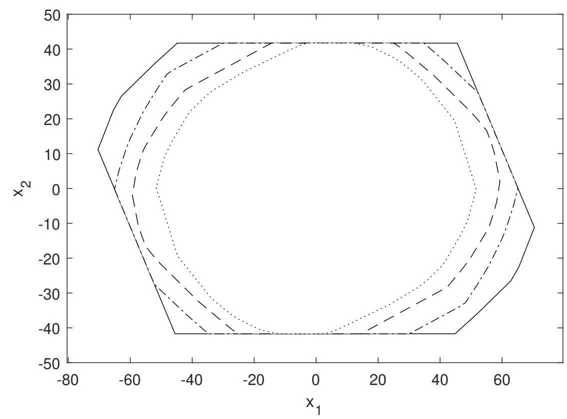

Fig. 2. Comparison of regions of attraction of Tube MPCs (Example 2). Dotted: Robust MPC of [11], Dashed: LPV MPC of Algorithm 1, Dot-dash: LPV with rate bounds, Solid: LPV with scheduled policy.

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