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Stabilizing Design of Third-order Continuous-time Switched Linear Systems

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Abstract— **In this work, we address both the state feedback stabilization problem and the dynamic output feedback stabilization problem for third-order continuous-time switched linear systems. Based on the controllability normal form decomposition approach, we prove that any controllable system is state feedback stabilizable, and the rate of convergence could be arbitrarily pre-assigned. Furthermore, for observable switched systems, we propose a reduced-order observer that could asymptotically estimate the unmeasured states. The dynamic output feedback stabilization problem is solved by designing a common switching law that stabilizes both the state and the observer. The design process is completely constructive.**

I. Introduction

Switched systems are a class of dynamical systems with wide representability and powerful control ability. When both the control law and the switching law are design variables, the interaction between them poses interesting and challenging issues that are theoretically appealing [6], [9], [11].

For a switched linear system with freely designed control input and switching signal, the stabilization problem is to design, when possible, proper switching/control laws that make the system exponentially stable. The stabilization problem has long been a core & classical problem that attracts much attention in the literature, and huge progress has been made by means of various approaches, for example, the Lyapunov method [8], [10], [7], [12], [19], [20], the optimization approach [1], [22], the structural decomposition approach [15], [17], [24], the phase portrait method [18], [21], and the automata-driven switching scheme [5], to list only a few.

As the investigation extended, it was found that the stabilization problem is very involved. Basically, the problem is nonconvex in Lyapunov functions [4], time-varying or nonlinear in control [14, §5.4.2], which makes the standard tools, for instance, the Lyapunov method, the average approach, and the system decomposition scheme, are not readily applied to the design of stabilizing switching/control laws. As such, the investigation becomes harder even for lower-dimensional switched linear systems. In the recent work [13], the problem was solved in a constructive manner for third-order continuous-time switched linear systems with two subsystems.

In this work, we focus on the stabilization problem for third-order continuous-time switched linear systems with arbitrarily many subsystems, and present a constructive design scheme for solving the problem under the mild conditions of controllability/observability. The contributions of the work include: i) we prove that, for any controllable third-order continuous-time switched linear system, three linear state feedbacks are sufficient for achieving stability, and constructive design procedures are developed to stabilize the switched system with any pre-assigned rate of convergence; ii) for an observable third-order continuous-time switched linear system, we design a reduced-order observer that, together with the measured output, could asymptotically estimate the system state in any given time interval with any pre-assigned rate of accuracy; and iii) we propose an observer-driven switching law for the dynamic output feedback system that could stabilize both the the original system and the observer, and the switching law is always well-posed.

II. Preliminaries

Let n, m, p be positive integers with $m \geq 2$. The switched linear system in this work is described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad (1)$$

$$y(t) = C_{\sigma(t)}x(t), \quad (2)$$

where $x(t) \in \mathbf{R}^n$ is the system state, $\sigma(t) \in M \triangleq \{1, 2, \cdots, m\}$ is the switching signal to be designed, $u(t) \in \mathbf{R}^p$ is the control input, $y(t)$ is the measured output, and $A_1, \cdots, A_m$, $B_1, \cdots, B_m$, and $C_1, \cdots, C_m$ are real constant matrices with compatible dimensions. For convenience, we denote the system by $\Sigma(A_i, B_i, C_i)_M$.

A. Definitions

Definition 1: Switched system (1) is said to be (exponentially) stabilizable, if there are positive real numbers α and β, such that for any $x_0 \in \mathbf{R}^n$, there exist switching signal $\sigma : [0, +\infty) \to M$ and piecewise continuous control input $u : [0, +\infty) \to \mathbf{R}^p$ satisfying

$$\|x(t)\| \leq \beta e^{-\alpha t}\|x_0\|, \quad \forall\ t \geq 0.$$

Here, α is said to be the rate of convergence.

Definition 2: Let k be a positive integer. Switched system (1) is said to be k-linear-state-feedback (exponentially) stabilizable, if there are positive real numbers α, β, and gain matrices $F_{i,1}, \cdots, F_{i,k}$, $i \in M$, such that for any $x_0 \in \mathbf{R}^n$, there exist switching signal σ :

$[0, +\infty) \to M$ and piecewise continuous control input $u(t) \in \{F_{\sigma(t),1}x(t), \cdots, F_{\sigma(t),k}x(t)\}$ satisfying

$$\|x(t)\| \le \beta e^{-\alpha t}\|x_0\|, \quad \forall\ t \ge 0.$$

In particular, when $k = 1$, then the system is said to be piecewise-linear-state feedback stabilizable.

By the homogeneity of the switched system and the Heine-Borel Theorem, stabilizability of a switched linear system implies (and hence is equivalent to) k-linear-feedback stabilizability when k is sufficiently large.

Definition 3: A state observer for switched system (1) is a dynamical system

$$\dot{z}(t) = f(t, z(t), y(t), \sigma(t), u(t)), \quad z(0) = z_0, \tag{3}$$

where f is a proper vector function. For a switching signal σ, the observer is said to be σ-asymptotic if $\lim_{t\to+\infty} \|x(t) - z(t)\| = 0$ for any x_0 and z_0.

Definition 4: Switched system (1) is said to be dynamic output stabilizable, if there exist a state observer and an observer-driven switching signal that steer the switched system and the observer exponentially convergent.

Definition 5: Switched system (1) is said to be (completely) controllable, if for each $x_0 \in \mathbf{R}^n$, there exist time $T > 0$, switching signal σ, and control input u such that $x(T) = 0$.

Definition 6: For switched system (1), state x_0 is said to be unobservable if it is indistinguishable from the origin, that is, for any switching signal σ we have

$$C_{\sigma(t)}x(t) = 0, \quad \forall\ t \ge 0.$$

The switched system is said to be completely (switched) observable if the unobservable set is $\{0\}$.

Switched system (1) is said to be of single-input if $\sum_{i=1}^m \operatorname{rank} B_i = 1$. Otherwise is said to be of multi-input. Similarly, switched system (1-2) is said to be of single-output if $\sum_{i=1}^m \operatorname{rank} C_i = 1$.

A switching path is a switching signal defined over a finite time interval. Suppose that θ is a switching path defined over $[0, \varsigma)$, then the length of θ is $|\theta| = \varsigma$. Given two switching paths θ_1 and θ_2, the concatenation of θ_1 and θ_2, denoted by $\theta_1 \sqcup \theta_2$, is defined to be

$$\theta_1 \sqcup \theta_2 = \begin{cases} \theta_1(t) & \text{if } t \in [0, |\theta_1|) \\ \theta_2(t) & \text{if } t \in [|\theta_1|, |\theta_1| + |\theta_2|). \end{cases}$$

Concatenation of more than two switching paths could be defined in the same manner. In particular, denote $\mathcal{P}_\theta = \theta \sqcup \theta \sqcup \ldots$ the infinite concatenation of switching path θ, which is a periodic switching signal defined over $[0, +\infty)$.

Let θ be a switching path, and suppose that the switching times are $0 < t_1 < \ldots < t_l < |\theta|$. The state transition matrix of the switched system along θ is

$$\Phi(|\theta|) = e^{A_{\theta(t_l)}(|\theta|-t_l)} \cdots e^{A_{\theta(t_1)}(t_2-t_1)} e^{A_{\theta(0)}t_1}.$$

B. Controllability Normal Forms

We focus on third-order switched systems, and made the following assumptions:

Assumption 2.1: Switched system (1-2) is completely controllable and completely observable.

For controllable multi-input switched linear systems, it has been proved that controllability implies piecewise-linear stabilizability, and constructive stabilizing design was presented in [14, §5.4.2].

When the switched linear system is of single input, we could assume without loss generality that $B_1 \neq 0$ and $B_i = 0$, $i = 2, \ldots, m$. Under the controllability assumption, it follows from the controllability criterion [14, Remark 4.20] that, there are indices i_1 and i_2, both in M, such that

$$\operatorname{rank}[B_1, A_{i_1}B_1, A_{i_2}B_1, A_{i_2}A_{i_1}B_1] = 3.$$

By possible re-numerating the subsystems, we could classify the single-input switched system into the following cases:

a) $\operatorname{rank}[B_1, A_1B_1, A_1^2B_1] = 3$;
b) $\operatorname{rank}[B_1, A_1B_1, A_2B_1] = 3$;
c) $\operatorname{rank}[B_1, A_2B_1, A_1A_2B_1] = 3$;
d) $\operatorname{rank}[B_1, A_2B_1, A_2^2B_1] = 3$;
e) $\operatorname{rank}[B_1, A_2B_1, A_3A_2B_1] = 3$; and
f) $\operatorname{rank}[B_1, A_2B_1, A_3B_1] = 3$.

For Cases a)-d), it has been proven in [14], [13] that the system is exponentially stabilizable, and constructive design procedures were developed.

For Case e), it can be easily verified that there exists a real number η such that linear system $(\eta A_2 + A_3, B_1)$ is controllable. It follows from Theorem 5.24 in [14] that the switched system is piecewise-linear feedback stabilizable.

For Case f), we could transform the system into the controllability normal form (Cf. [14, Sec. 4.5.2]). By applying Theorem 5.24 in [14], we need only to consider the normal form given by

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & b_1 & b_2 \\ 1 & b_3 & b_4 \\ 0 & 0 & b_5 \end{bmatrix}, \ B_2 = 0, \\ A_3 &= \begin{bmatrix} 0 & c_1 & c_2 \\ 0 & c_3 & 0 \\ 1 & c_4 & c_5 \end{bmatrix}, \ B_3 = 0, \end{aligned} \tag{4}$$

where a, $b_1, \ldots, b_5$, and $c_1, \ldots, c_5$ are real constants.

III. State Feedback Stabilizing Design

Based on the discussion in the previous section, we assume that the switched system admits normal form (4).

Lemma 1: For any $t > 0$, $\gamma_1 > 0$ and $\gamma_2, \gamma_3 \in \mathbf{R}$, there is a row vector $F \in \mathbf{R}^3$, such that

$$\exp[(A_1 + B_1F)t] = e^{at} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Proof: Let

$$f_1 = \frac{\ln\gamma_1}{t} + a,\ f_2 = \frac{\gamma_2 \ln\gamma_1}{t(\gamma_1 - 1)},\ f_3 = \frac{\gamma_3 \ln\gamma_1}{t(\gamma_1 - 1)}.$$

Define a function

$$g(t, \gamma_1, \gamma_2, \gamma_3) = [f_1, f_2, f_3].$$

Let $F = g(t, \gamma_1, \gamma_2, \gamma_3)$. It can be computed that

$$e^{(A_1+B_1F)t} = e^{at}\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Proposition 1: Switched system (4) is 3-linear-feedback (exponentially) stabilizable.

Proof: Fix positive real times $h_{1,1}, h_{1,2}, h_{1,3}$. We are to design positive real numbers h_2, h_3 and feedback gain matrices F_1, F_2, F_3, such that

$$e^{(A_1+B_1F_3)h_{1,3}} e^{A_2h_2} e^{(A_1+B_1F_2)h_{1,2}} e^{A_3h_3} e^{(A_1+B_1F_1)h_{1,1}}$$

is norm contractive, thus a periodic switching signal could steer the system exponentially convergent. For this, let

$$F_i = g(h_{1,i}, \rho_{i,1}, \rho_{i,2}, \rho_{i,3}),$$

where $\rho_{i,j},\ i, j = 1, 2, 3$, are real numbers to be designed.

For almost any $h_2 > 0$, the $(1, 1)$-th and $(2, 1)$-th entries of matrix $e^{A_2h_2}$ are nonzero. Fix such an h_2, and denote

$$e^{A_2h_2} = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ 0 & 0 & v_7 \end{bmatrix}.$$

It can be seen that $v_7 = e^{b_5h_2} > 0$.

For almost any $h_3 > 0$, the $(3, 1)$-th entry of matrix $e^{A_3h_3}$ is nonzero. Fix such an h_3, and denote

$$e^{A_3h_3} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & \omega_4 & 0 \\ \omega_5 & \omega_6 & \omega_7 \end{bmatrix}.$$

It can be seen that $\omega_4 = e^{c_3h_3} > 0$. Note that the determinant of matrix $\exp\left(\begin{bmatrix} 0 & c_2 \\ 1 & c_5 \end{bmatrix} h_3\right)$ is $\omega_3\omega_5 - \omega_1\omega_7 \neq 0$.

Denote $T = h_{1,1} + h_{1,2} + h_{1,3} + h_2 + h_3$. Fix $\delta < 1$, and let $\delta_0 = e^{-a(h_{1,1}+h_{1,2}+h_{1,3})}\delta$.

Define

$$\begin{aligned}
\rho_{1,1} &= \frac{\delta_0}{v_7\omega_5},\ \rho_{1,2} = -\frac{\omega_6}{\omega_5},\ \rho_{1,3} = -\frac{\omega_7}{\omega_5}, \\
\rho_{2,1} &= \frac{\omega_5\delta_0}{v_4(\omega_3\omega_5 - \omega_1\omega_7)}, \\
\rho_{2,2} &= -\frac{v_5}{v_4} - \frac{\delta_0(\omega_2\omega_5 - \omega_1\omega_6)}{v_4\omega_4(\omega_3\omega_5 - \omega_1\omega_7)}, \\
\rho_{2,3} &= -\frac{v_6}{v_4} - \frac{\omega_1\delta_0}{v_4(\omega_3\omega_5 - \omega_1\omega_7)}, \\
\rho_{3,1} &= \frac{|v_4|\delta_0}{|v_1v_2v_4\omega_4 - v_1v_5\omega_4| + |v_1\delta_0|}, \\
\rho_{3,2} &= 0, \\
\rho_{3,3} &= \frac{|v_4|(v_1v_6 - v_3v_4)\delta_0}{v_4v_7(|v_1v_2v_4\omega_4 - v_1v_5\omega_4| + |v_1\delta_0|)}.
\end{aligned}$$

It can be verified that

$$e^{(A_1+B_1F_3)h_{1,3}} e^{A_2h_2} e^{(A_1+B_1F_2)h_{1,2}} e^{A_3h_3} e^{(A_1+B_1F_1)h_{1,1}} = \begin{bmatrix} 0 & \delta_1 & \delta_2 \\ 0 & 0 & \delta \\ \delta & 0 & 0 \end{bmatrix}$$

with $|\delta_1| + |\delta_2| = \delta$.

As a result, the matrix

$$e^{(A_1+B_1F_3)h_{1,3}} e^{A_2h_2} e^{(A_1+B_1F_2)h_{1,2}} e^{A_3h_3} e^{(A_1+B_1F_1)h_{1,1}}$$

admits infinity norm $\delta < 1$. This means that system (4) is 3-linear-feedback (exponentially) stabilizable with convergence rate $\frac{-\ln\delta}{T}$.

Let $t_1 = h_{1,1}$, $t_2 = t_1 + h_3$, $t_3 = t_2 + h_{1,2}$, $t_4 = t_3 + h_2$, and $T = t_4 + h_{1,3}$. Define switching path

$$\theta(t) = \begin{cases} 1 & \text{when } t \in [0, t_1) \cup [t_2, t_3) \cup [t_4, T) \\ 3 & \text{when } t \in [t_1, t_2) \\ 2 & \text{when } t \in [t_3, t_4). \end{cases} \tag{6}$$

Accordingly, define control law

$$u(t) = \begin{cases} F_1x(t) & \text{when} \mod(t, T) \in [0, t_1) \\ F_2x(t) & \text{when} \mod(t, T) \in [t_2, t_3) \\ F_3x(t) & \text{when} \mod(t, T) \in [t_4, T), \end{cases} \tag{7}$$

where $\mod(t, T)$ is the reminder of t divided by T.

By integrating Proposition 1 and [13, Thm. 1], we have the following result.

Theorem 1: Any controllable third-order continuous-time switched linear system is 3-linear-feedback (exponentially) stabilizable, and the rate of convergence could be arbitrarily pre-assigned.

Proof: Without loss of generality, we assume that $\sum(A_i, B_i)_{M_0}$ is controllable, where the cardinality of M_0, denoted by m_0, is two or three.

First, when there exist non-negative real numbers $\eta_1, \cdots, \eta_{m_0}$ such that linear system $(\sum_{i=1}^{m_0} \eta_iA_i, [B_1, \cdots, B_{m_0}])$ is controllable, by applying the average approach (Cf. [14, P.185]), we could design m_0 feedback gain matrices and a (high-frequency) periodic switching law such that the switched system is exponentially stable with any pre-assigned rate of convergence.

Then, for form (3) in [13] when $m_0 = 2$, it has been established that two feedback gain matrices could be designed such that the switched system is exponentially stable with any pre-assigned rate of convergence.

Finally, for normal form (4) with $m_0 = 3$, the state transition matrix along switching path $\mathcal{P}_\theta$ and control law u in (7) is exactly

$$e^{(A_1+B_1F_3)h_{1,3}} e^{A_2h_2} e^{(A_1+B_1F_2)h_{1,2}} e^{A_3h_3} e^{(A_1+B_1F_1)h_{1,1}},$$

which is norm contractive. This means that the switched system is exponentially stable with any pre-assigned rate of convergence.

Remark 1: Theorem 1 extends [13, Thm. 3.1], which addressed the special class of third-order switched systems each with two subsystems. The extension is far from trivial from both the design scheme and technical development perspectives. Indeed, for the general case, the classification of the normal forms is much more involved, and the design of gain matrices is by means of directly assigning the spectra of the state transition matrix, which allows us to thoroughly assign the convergence rate of the designed system.

Remark 2: Note that the design approach relies heavily on the classification of the controllability normal forms. Indeed, here we classify the controllable systems into 6 categories according to the rank conditions, each category admits one controllability normal form. For fourth or higher-order switched systems, the number of the controllability normal forms grows fast as the system dimension and the number of subsystem grow, which prohibits the current approach from extending to fourth or higher-order switched systems.

Remark 3: When the switched system is not completely controllable, we could decompose the system into the controllable submode and uncontrollable submode. We could prove that, the switched system is stabilizable if and only if the uncontrollable submode is stabilizable as a switched autonomous system. The design of the control law and switching law is similar to [13, §III.B], and is omitted here due to space limit.

IV. Dynamic Output Feedback Stabilization

A. Observer Design

When the system state is not totally available, state observers/estimators could be designed to asymptotically estimate the state. In Ref. [16], an observer design method was proposed for a class of hybrid systems with both switching and impulse. The observer itself is also a hybrid system, which could asymptotically converge to the actual state under persistent switching. Other observer design schemes could be found in [2], [3], [23].

For switched linear systems, it has been established that observability is dual with reachability, and observer design is dual with state feedback stabilizing design [14]. By the duality property, the design of an asymptotic observer could be addressed for third-order continuous-time switched linear systems. The resultant observer is also a third-order continuous-time switched linear system with Luenburger-like subsystems, and the observer exponentially approaches the state of the original switched system along a proper periodic switching signal. The design of such a full-order observer is totally parallel to the stabilizing design as investigated in the previous section, and we will not repeat the details. Instead, we focus on a specific yet practically important case that the output is independent of the switching law, that is, $y = Cx$ where $C \neq 0$.

Let us examine the third-order switched linear system with a single output, $y = Cx$, where C is a non-zero row vector. Under Assumption 2.1, we have indices i_1 and i_2, both in M, such that either $\operatorname{rank}\begin{bmatrix} C \\ CA_{i_1} \\ CA_{i_2} \end{bmatrix} = 3$ or $\operatorname{rank}\begin{bmatrix} C \\ CA_{i_1} \\ CA_{i_1}A_{i_2} \end{bmatrix} = 3$. By possible re-numerating the subsystems, and with a proper coordinate change $\bar{x} = Tx$, the single-output switched system could be transformed into

$$\dot{\bar{x}}(t) = \bar{A}_{\sigma(t)}\bar{x}(t) + \bar{B}_{\sigma(t)}u(t), \quad (8)$$
$$y(t) = \bar{C}\bar{x}(t) \quad (9)$$

where $\bar{C} = [1 \quad 0 \quad 0]$. Furthermore, we have either

$$\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A}_1 \\ \bar{C}\bar{A}_2 \end{bmatrix} = I_3$$

or

$$\begin{bmatrix} \bar{C} \\ \bar{C}\bar{A}_1 \\ \bar{C}\bar{A}_1\bar{A}_2 \end{bmatrix} = I_3. \quad (10)$$

Rewrite matrices $\bar{A}_i$, $\bar{B}_i$ and $\bar{C}$, $i \in M$ into block forms

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i,1} & \bar{A}_{i,2} \\ \bar{A}_{i,3} & \bar{A}_{i,4} \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} \bar{B}_{i,1} \\ \bar{B}_{i,2} \end{bmatrix}, \quad \bar{C} = [\bar{C}_1 \ \bar{C}_2],$$

where $\bar{A}_{i,1}$, $\bar{B}_{i,1}$ and $\bar{C}_1$, $i \in M$ are scalers, and other blocks are of compatible dimensions.

Construct the observer that is the second-order switched linear system given by

$$\dot{\bar{z}}(t) = (\bar{A}_{\sigma(t),4} - L_{\sigma(t)}\bar{A}_{\sigma(t),2})\bar{z}(t) - (L_{\sigma(t)}\bar{A}_{\sigma(t),1} + L_{\sigma(t)} \bar{A}_{\sigma(t),2}L_{\sigma(t)} - \bar{A}_{\sigma(t),3})y(t) + (L_{\sigma(t)}\bar{B}_{\sigma(t),1} + \bar{B}_{\sigma(t),2})u,$$

where $\bar{z}(0) = \bar{z}_0 = 0$, $L_1 \in \mathbf{R}^{1\times 2}$ is to be designed, and $L_i = 0$, $i \neq 1$.

Denote the observer error as

$$\bar{e}(t) = \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} - \bar{z}(t).$$

Lemma 2: Suppose that switched system (1-2) is completely observable. There exist a switching path θ with any pre-assigned length, such that $\bar{e}(t)$ exponentially approach zero with any pre-assigned rate of convergence.

Proof: It can be verified that

$$\dot{\bar{e}}(t) = (\bar{A}_{\sigma(t),4} - L_{\sigma(t)}\bar{A}_{\sigma(t),2})\bar{e}(t).$$

Thus the error dynamics is a switched autonomous system. By virtual of the complete observability of switched system (1-2), we could express the subsystem matrices as

$$\bar{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 1 \\ \eta_4 & \eta_5 & \eta_6 \end{bmatrix},$$

where $\gamma_1, \ldots, \gamma_5$ and $\eta_1, \ldots, \eta_6$ are real numbers.

Denote $L_1 = [l_1 \quad l_2]$. For any positive real number h_1, it is straightforward that

$$e^{(\bar{A}_{1,4} - L_1\bar{A}_{1,2})h_1} = \begin{bmatrix} e^{(\gamma_2 - l_1)h_1} & 0 \\ (\gamma_4 - l_2)\frac{e^{(\gamma_2 - l_1)h_1} - e^{\gamma_5 h_1}}{\gamma_2 - l_1 - \gamma_5} & e^{\gamma_5 h_1} \end{bmatrix}.$$

Further, for almost any positive real number h_2, the $(1, 2)$ entry of $e^{\bar{A}_{2,4}h_2}$ is non-zero. Fix such an h_2. Design $l_2 =$

$\gamma_4 - \frac{e^{\gamma_5 h_1}\gamma_8}{\gamma_7}$, where γ_7 and γ_8 denote the $(1,2)$ and $(2,2)$ entries of $e^{\bar{A}_{2,4}h_2}$, respectively. Routine calculation shows that the entries of matrix

$$e^{(\bar{A}_{1,4}-L_1\bar{A}_{1,2})h_1}e^{\bar{A}_{2,4}h_2}e^{(\bar{A}_{1,4}-L_1\bar{A}_{1,2})h_1} \tag{11}$$

is either zero or sufficiently approaching zero when $(l_1 - \gamma_2)h_1$ is sufficiently large. Design l_1 such that the norm of the matrix in (11) is smaller than or equal to $e^{-(2h_1+h_2)\nu}$, where ν is any pre-assigned rate of convergence. Define switching path

$$\theta = \begin{cases} 1 & \text{if } t \in [0, h_1) \\ 2 & \text{if } t \in [h_1, h_1 + h_2) \\ 1 & \text{if } t \in [h_1 + h_2, 2h_1 + h_2). \end{cases}$$

It is clear that the $\bar{e}(t)$ exponentially approach zero under the periodic switching law $\mathcal{P}_\theta$.

Remark 4: It follows from the proof that the reduced-order observer could exponentially track the state along a periodic switching law with any pre-assigned rate of tracking accuracy. The base switching path is with two switches, and its length could be chosen to be arbitrarily short. In contrast, to design a full-order observer we need up to five switches in a base switching path.

Remark 5: The design procedure could be extended to more general situation that the output relies on the switching law. In fact, detailed classification shows that, the only exception is when the system is equivalent to the case with $C_1 = [1 \quad 0 \quad 0]$, $C_i = [0 \quad 0 \quad 0]$, $i = 2, \ldots, m$, and both the $(1,2)$ and $(1,3)$ entries of A_1 are zeros. Whether the system in this case admits a reduced-order observer or not is an interesting issue to be addressed.

B. Observer-driven Switching Design

Fix $\upsilon_0 < 1$.

Suppose that we have design a full-order or reduced-order observer with base switching path θ_2. Denote

$$\upsilon_1 = \|\bar{\Phi}(|\theta_2|)\|, \;\; \upsilon_2 = \|\Phi(|\theta_2|)\|,$$

where Φ and $\bar{\Phi}$ are the state transition matrices of the original switched system and the error system, respectively.

Fix υ_3 such that $\upsilon_2\upsilon_3 \leq \upsilon_0$. According to the design procedure presented in §III, we could design a multi-linear state feedback control law such that the norm of the state transition matrix is less than or equal to υ_3 along a base switching path θ_1.

Denote

$$\upsilon_4 = \|\bar{\Phi}(|\theta_1|)\|.$$

Note that υ_1 could be arbitrarily assigned, so we design the observer gain properly such that $\upsilon_1\upsilon_4 \leq \upsilon_0$.

Define a new base switching path $\theta = \theta_1 \sqcup \theta_2$. Accordingly, we could design the observer-driven control law $u(t)$, where state is substituted by the observer. It is clear that

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_t)x(t) - B_{\sigma(t)}F_t e(t), \tag{12}$$

where $\sigma = \mathcal{P}_\theta$, e is the error between the state and the observer, and F_t is the multi-linear feedback gain matrices designed in §III.

From the above analysis, both the error system and the nominal system

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}F_t)x(t)$$

are norm contractive along the switching signal θ. Under the periodic switching signal $\mathcal{P}_\theta$, both systems are exponentially convergent. Due to the boundedness of F_j for any j, system (12) is also exponentially convergent. So we have the following conclusion.

Theorem 2: Under Assumption 2.1, switched system (1-2) is dynamic output feedback stabilizable.

C. Example

As a numerical example, we examine the controllable system (4) with

$$\begin{aligned} &a = 0.5, \\ &b_1 = -1, b_2 = -2, b_3 = 2, b_4 = 2, b_5 = 1, \\ &c_1 = -3, c_2 = 1.5, c_3 = 1, c_4 = -2, c_5 = 2. \end{aligned}$$

Let $h_{1,1} = h_{1,2} = h_{1,3} = h_2 = h_3 = 0.2$ and $\delta = 0.1$. Applying the design procedure presented in Proposition 1, we have

$$\begin{aligned} F_1 &= [-4.1892 \;\; 19.3240 \;\; -12.8772], \\ F_2 &= [-3.4160 \;\; -7.9846 \;\; -8.0486], \\ F_3 &= [-3.8026 \;\; 0 \;\; 0.1181]. \end{aligned}$$

Figure 1 shows the dynamics with initial state $x_0 = [1, -1, -1]^T$. While the state trajectory is with large overshoot, it is clear that the system is exponentially convergent.

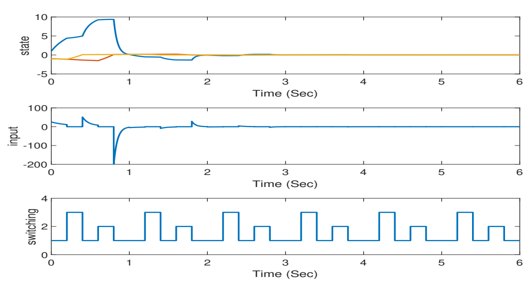

Fig. 1. System dynamics with state feedback

Next, suppose that the whole state information is not available, and the measured output is $y = x_1$. By applying the reduced-order observer design procedure, we have the gain matrix $L_2 = \begin{bmatrix} 93.2484 \\ -71.6242 \end{bmatrix}$ with base switching path

$$\theta(t) = \begin{cases} 2 & \text{if } t \in [0, \bar{h}) \\ 3 & \text{if } t \in [\bar{h}, 2\bar{h}) \\ 2 & \text{if } t \in [2\bar{h}, 3\bar{h}), \end{cases}$$

where $\bar{h} = 0.1$. A trajectory of the observer is depicted in Figure 2 along the error trajectory. It can be seen that the error trajectory is exponentially convergent.

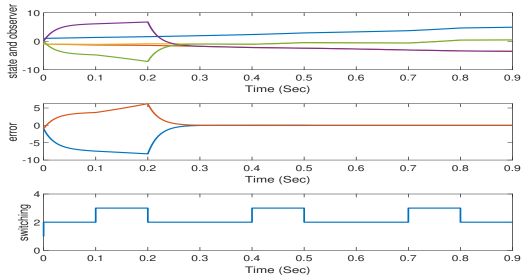

Fig. 2. System dynamics with state observer

Finally, we examine the overall system dynamics with observer-driven feedback control/switching laws. Due to the high-gain nature of the control law, we need to prevent the state transition matrix from quick growth when the state is substituted by the observer. For this, we take smaller $h_{1,1} = h_{1,2} = h_{1,3} = h_2 = h_3 = 0.05$, and keep other parameters unchanged. Figure 3 depicts the state and the observer trajectories, both of which are exponentially convergent. It is clear that, the transient performance is worse than that of the state feedback case, which is caused partly by the error between the observer and the state and partly by increased dimension of the extended system.

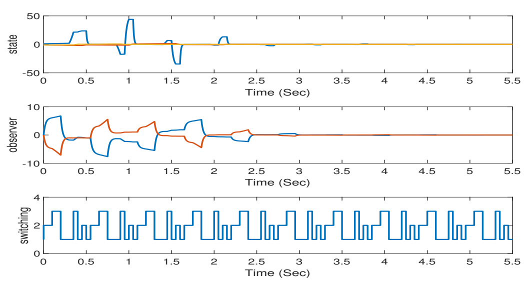

Fig. 3. System dynamics with observer-driven feedback

V. Conclusion

In this work, the problem of stabilization has been addressed for third-order continuous-time switched linear control systems. We proved that, any controllable system is stabilizable by means of (at most) three linear state feedbacks and a periodic switching law, and rate of convergence could be assigned in a constructive manner. When the switched system is observable, we proposed a reduced-order observer, which is also a switched system with two switches and one non-zero observer gain in a base period. To solve the dynamic output feedback stabilization problem, we developed observer-driven control/switching laws that steer the extended system exponentially convergent. A numerical example was presented to show the effectiveness of the proposed design scheme.

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References

[1] D. Antunes and M. Heemels, "Linear quadratic regulation of switched systems using informed policies," *IEEE Trans. Automat. Contr.*, vol. 62, no. 6, pp. 2675-2688, Jun. 2016.
[2] S. Aranovskiy, D. Efimov, D. Sokolov, *et al.*, "Switched observer design for a class of locally unobservable time-varying systems," *Automatica*, vol. 130, Aug. 2021, Art. no. 109715.
[3] P. Bernard and R. G. Sanfelice, "Observers for hybrid dynamical systems with linear maps and known jump times," in Proc. IEEE Conf. Decis. Contr., 2018, pp. 3140-3145.
[4] F. Blanchini and C. Savorgnan, "Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions," *Automatica*, vol. 44, no. 4, pp. 1166-1170, Apr. 2008.
[5] M. Fiacchini, M. Jungers, and A. Girard, "Stabilization and control Lyapunov functions for language constrained discrete-time switched linear systems," *Automatica*, vol. 93, no. 1, pp. 64-74, Jan. 2018.
[6] J. P. Hespanha, D. Liberzon, and A. S. Morse, "Overcoming the limitations of adaptive control by means of logic-based switching," *Syst. Contr. Lett.*, vol. 49, no. 1, pp. 49-65, May 2003.
[7] R. M. Jungers and P. Mason, "On feedback stabilization of linear switched systems via switching signal control," *SIAM J. Contr. Optimiz.*, vol. 55, no. 2, pp. 1179-1198, Feb. 2017.
[8] D. Lee, G. E. Dullerud, and J. Hu, "Graph Lyapunov function for switching stabilization and distributed computation," *Automatica*, vol. 116, Jun. 2020, Art. no. 108923.
[9] D. Liberzon, *Switching in Systems and Control*, Boston: Birkhauser, 2003.
[10] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Automat. Contr.*, vol. 54, no. 2, pp. 308-322, Feb. 2009.
[11] K. R. Santarelli,"Comparison between a switching controller and two LTI controllers for a class of LTI plants," *Int. J. Robust Nonlin. Contr.*, vol. 19, pp. 185-217, 2009.
[12] Y. Sun and L. Wang, "Stabilization of planar discrete-time switched systems: Switched Lyapunov functional approach," *Nonlin. Anal. Hybrid Syst.*, vol. 2, no. 4, pp. 1062-1068, 2008.
[13] Z. Sun, "Feedback stabilization of third-order switched linear control systems," *IEEE Contr. Syst. Lett.*, vol. 4, no. 4, pp. 857-861, Oct. 2020.
[14] Z. Sun and S. S. Ge, *Switched Linear Systems: Control and Design*, London: Springer, 2005.
[15] Z. Sun and S. S. Ge, *Stability Theory of Switched Dynamical Systems*, London: Springer, 2011.
[16] A. Tanwani, S. Hyungbo, and D. Liberzon, "Observability for switched linear systems: Characterization and observer design." *IEEE Trans. Automat. Contr.*, vol. 58, no. 4, pp. 891-904, Apr. 2014.
[17] G. Xie and L. Wang, "Periodic stabilizability of switched linear control systems," *Automatica*, vol. 45, no. 9, pp. 2141-2148, Sep. 2009.
[18] X. Xu and P. J. Antsaklis, "Stabilization of second-order LTI switched systems," *Int. J. Contr.*, vol. 73, no. 14, pp. 1261-1279, 2000.
[19] B. Wang and Q. Zhu, "Stability analysis of discrete time semi-Markov jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 65, no. 12, pp. 5415-5421, Dec. 2020.
[20] B. Wang, Q. Zhu, and S. Li, "Stability analysis of discrete-time semi-Markov jump linear systems with time delay," *IEEE Trans. Automat. Contr.*, DOI: 10.1109/TAC.2023.3240926, 2023.
[21] Y. Yang, C. Xiang, and T. H. Lee, "Necessary and sufficient conditions for regional stabilisability of second-order switched linear systems with a finite number of subsystems," *Automatica*, vol. 50, no. 3, pp. 931-939, Mar. 2014.
[22] W. Zhang, A. Abate, J. H. Hu, *et al.*, "Exponential stabilization of discrete-time switched linear systems," *Automatica*, vol. 45, no. 11, pp. 2526-2536, Nov. 2009.
[23] X. Zhao, H. Liu, J. Zhang, *et al.*, "Multiple-mode observer design for a class of switched linear systems," *IEEE Trans. Autom. Sci. Engin.*, vol. 12, no. 1, pp. 272-280, Jan. 2015.
[24] Y. Zhu and Z. Sun, "Stabilizing design for discrete-time reversible switched linear control systems: A deadbeat control approach," *Automatica*, vol. 129, Jul. 2021, Art. no. 109617.