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Co-evolution of Dual Opinions under Asynchronous Updating

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Abstract— **Inspired by the dual attitudes theory that implicit opinions are individuals' inner evaluations affected by experience while explicit opinions are external expressions of these evaluations, we propose an asynchronous co-evolution model of dual opinions, where individuals update explicit opinion at each time step but change their implicit opinion based on their own clock. Furthermore, we introduce the after-effect of observed opinion information in this model, which enables individuals to update implicit opinions not only based on the opinion information observed at the current time but also on the information received from the past period of time. We analyze the dynamics of dual opinions in two discussion scenarios: a group of individuals with similar and opposite initial opinions. In the former scenario, rigorous analysis suggests that dual opinions are polarized to extreme opinions, mathematically verifying the empirical finding that group discussion intensifies individuals' preferences, resulting in group polarization. In the latter scenario, our investigation shows that individuals with low bias show acceptance (inward conformity) while those with high bias exhibit compliance (outward conformity). We further analyze the influence of parameters on the co-evolution of dual opinions.**

I. Introduction

The study of opinion dynamics has emerged as a focal subject for further study, providing a framework for better understanding the mechanism behind social opinion evolution and for explaining public perceptions about commonly concerned events, such as the attitudes of the American public towards the Iraq war [1], political elections [2] and opinion polarization [3]. Many remarkable opinion dynamics models have been proposed, such as the F-J model [4], the bounded confidence model [5], and the biased assimilation model [3].

In recent years, the co-evolution of internal and external opinions has received considerable attention. Inspired by Ash's experiment [6] that individuals may not express their true thought under social pressure, [4] propose the Expressed and Private Opinion (EPO) model to investigate the discrepancy between expressed and private opinions in social networks. Building on this work, followed-up studies [5], [7]–[9] combines the bounded confidence model with the EPO model to explore the influence of bounded confidence on the opinion co-evolution. Also, [10] studies the resilience consensus of expressed and private opinions. [11] studies the problem of inferring individuals' inherent opinions from their declared opinions. Additionally, [12] studies the EPO model with asynchronous update, where only one agent is activated to change opinions at each time. Most studies of opinion co-evolution adopt a synchronous updating rule, i.e., individuals update internal and external opinions simultaneously. However, [13] suggests that implicit attitudes are influenced by experience and are hard to change compared with explicit attitudes.

Motivated by the dual attitudes theory [13], this paper combines the biased assimilation model [3] with the EPO model [4] to develop a model of dual opinions under asynchronous updating. In this model, every individual updates explicit opinion at each time step while changes implicit opinions based on the own clock. Furthermore, since implicit opinions are influenced by experience [13], we introduce after-effects to the observed opinion information in the model. That is, when individuals change implicit opinions, they are affected not only by the opinions received at the current time, but also by the opinion information obtained from the past period of time. We theoretically analyze the dual opinion dynamics in the following two typical group discussion scenarios.

Homogeneous Discussion Scenario: the group consists of individuals with similar initial opinions, such as topic communities on Twitter. Our theoretical analysis suggests that the implicit and explicit opinions of individuals in the group converge to extreme opinions in connected communication networks, thereby mathematically verifying the empirical finding that group discussion among like-minded individuals intensifies their initial preferences, resulting in group polarization.

Adversarial Discussion Scenario: the group consists of two types of individuals with opposite initial opinions, such as political debates between Democrats and Republicans in the USA. A two-island network is used to model the communication network in this scenario, where interactions are more within the group. When individuals are with low bias, group discussion promotes the consensus of dual opinions, resulting in a consensus at the neutral opinion. At this point, individuals exhibit an inward conformity called acceptance. However, when individuals are with high bias, group discussion intensifies their implicit opinions, leading to polarization. Despite this, individuals exhibit a certain level of outward conformity that express a more neutral explicit opinion, showing an outward compliance phenomenon. Besides, we discuss the impact of parameters on the dual

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opinion co-evolution, and demonstrate by simulations that the asynchronous updates of implicit opinions and the after-effect of received neighbors' explicit opinions affect the convergence states of dual opinions.

Outline Section II introduces the co-evolution model of dual opinions under asynchronous updating. Section III presents theoretical results and Section IV exhibits numerical simulations. Section V concludes the paper.

II. Preliminaries and Model Description

We will first introduce some notation and graph theory used in this paper, and then describe the co-evolution model of dual opinions under asynchronous updating.

Notation $\mathbb{R}, \mathbb{N}$ and $\mathbb{N}^{+}$ are the sets of real numbers, non-negative integers and positive integers, respectively. $M = [m_{i,j}]_{n\times n} \in \mathbb{R}^{n\times n}$ represents the matrix with elements $m_{i,j} \in \mathbb{R}$. $|S|$ is the cardinality of the set S. $\mathcal{P}_l = (p_{l,1}, \cdots, p_{l,k})$ is an ordered k-tuple with $l \in \mathbb{N}_{+}$. Two k-tuples $\mathcal{P}_{l_1} \geq \mathcal{P}_{l_2}$ if $p_{l_1,i} \geq p_{l_2,i}$ holds for all $i \in \{1, \cdots, k\}$.

Graph Theory Consider a social network represented by a graph $\mathcal{G} = \{V, E, W\}$, where $V = \{1, 2, \cdots, n\}$ is the set of agents (individuals), $E \subset \{e_{i,j} : i, j \in V\}$ is the edge set and $W = [w_{i,j}]_{n\times n} \in \mathbb{R}^{n\times n}$ is the adjacency matrix of the graph $\mathcal{G}$. The edge $e_{i,j} \in E \Leftrightarrow w_{i,j} = 1$ for $i, j \in V$, which indicates that the agent j's opinions will affect the evolution of the agents i's opinions; otherwise, $e_{i,j} \notin E \Leftrightarrow w_{i,j} = 0$. Moreover, assume that the graph $\mathcal{G}$ is undirected, i.e., $w_{i,j} = w_{j,i}$ for every $i, j \in V$. $N_i = \{j \in V, j \neq i : w_{i,j} > 0\}$ represents the neighbors set of agent $i \in V$.

Inspired by the dual attitudes theory, this paper considers the scenario where at each time $t \in \mathbb{N}$ every individual i in group V has dual opinions to an object simultaneously: implicit opinion and explicit opinion, represented by $x_i(t) \in [0, 1]$ and $y_i(t) \in [0, 1]$, respectively. The value of dual opinions represents the level of support that individuals have for the object, with higher opinion values indicating greater levels of support. [13] suggests that implicit attitudes change more slowly than explicit attitudes that are susceptible to the environment. Hence, the update timelines are separated for implicit and explicit opinions, assuming that individuals update, express and exchange their explicit opinions at each time step, but update their implicit opinions based on their own clocks. Let $\mathbb{T}_i = \{t^i_k\}_{k=0}^{\infty}$ represent the time set for individual $i \in V$ to update implicit opinions satisfying

- $t^i_k \in \mathbb{N}$ and $0 = t^i_0 < t^i_k < t^i_{k+1}$ hold for all $k \in \mathbb{N}^{+}$;
- there exists $\tau_{\max} \in \mathbb{N}$ such that $t^i_{k+1} - t^i_k < \tau_{\max}$ holds for all $k \in \mathbb{N}$;
- $\lim_{k\to\infty} t^i_k = \infty$.

The EPO model framework [4] is used to model the co-evolution of dual opinions, and the biased assimilation model [3] is used to update implicit opinions. For agent $i \in V$ and $t^i_k, t^i_{k+1} \in \mathbb{T}_i$, the update rules of dual opinions are as follows:

$$x_i(t^i_{k+1}) = \frac{x_i(t^i_k)^b s_i(t^i_k, t^i_{k+1})}{x_i(t^i_k)^b s_i(t^i_k, t^i_{k+1}) + (1 - x_i(t^i_k))^b (d_i - s_i(t^i_k, t^i_{k+1}))},$$
$$y_i(t) = \phi x_i(t^i_k) + (1 - \phi)\hat{y}_{i,avg}(t - 1), \forall t \in [t^i_k, t^i_{k+1}), t \in \mathbb{N}. \tag{1}$$

In the update rules (1), $b > 0$ is the bias parameter,

$$s_i(t^i_k, t^i_{k+1}) = \sum_{t=t^i_k}^{t^i_{k+1}-1} \alpha_i(t) \sum_{j\in N_i} w_{i,j} y_j(t), \tag{2}$$

is the weighted sum of explicit opinions of neighbors obtained by agent i in time period $[t^i_k, t^i_{k+1})$, the after-effect function $\alpha_i(t) \in [0, 1]$ measures the influence of the opinion information observed by agent i at time t on implicit opinion change and satisfies $\sum_{t=t^i_k}^{t^i_{k+1}-1} \alpha_i(t) = 1$, $d_i = \sum_{j\in N_i} w_{i,j}$ is the total influence of i's neighbors, the resilience parameter $\phi \in (0, 1)$ describes the resilience to resist group pressure of conforming with public opinions, $1 - \phi$ refers to social pressure of the social network, and

$$\hat{y}_{i,avg}(t) = \sum_{j\in N_i} m_{i,j} y_j(t), \forall t \in \mathbb{N},$$

is the public opinion viewed by agent i at time t, where the matrix $M = \{m_{i,j}\}_{n\times n}$ is row stochastic, i.e., $\sum_{j\in N_i} m_{i,j}\bar{1}, \forall i \in V$. We assume that for $j \in V, j \neq i$, $m_{i,j} > 0 \Leftrightarrow w_{i,j} > 0$. Clearly, this model is now well-defined. More detailed descriptions of the EPO model and the biased assimilation model can be found in [4] and [3], respectively.

Remark 1: We use the bias assimilation model to update implicit opinions because it can generate opinion consensus, disagreement and especially polarization. [3] defines opinion polarization as a verb describing the increment of difference of opinions and also shows that opinions evolving according to the weighted average algorithm do not polarize. The polarization in this paper refers to an evolutionary result, where all individuals' opinions evolve to more extreme states compared with initial opinions. The opinions evolve by other linear opinion dynamics models do not reach polarization since the maximum and minimum opinions become neutral in evolution.

Remark 2: As implicit opinions change more slowly than explicit opinions [14], an asynchronous update rule is chosen for dual opinions. Specifically, individuals share their explicit opinions at each time step, while update in their implicit opinions occurs in a pace determined by their own clocks. In [12], an asynchronous linear EPO model is studied under the assumption that only one agent is activated at each time step. In contrast, our dual opinions co-evolution model allows for the update of implicit opinions to occur in each individual's own pace, and the update rule is nonlinear.

Remark 3: In this paper, assuming that individuals possess memories, previously obtained information has an after-effect that influences the updates of implicit opinions. This effect is captured by (2), where during the time period $[t^i_k, t^i_{k+1})$, the observed information at each time impacts the update of implicit opinions through an influence factor $\alpha_i(t)$. To ensure normalization, impose the constraint $\sum_{t=t_k}^{t_{k+1}-1} \alpha_i(t) = 1$.

III. Theoretical Analyses

The two discussion scenarios, namely homogeneous discussion and adversarial discussion mentioned in the Introduction, are common both in real-life and on the Internet.

In real life, people tend to communicate with like-minded people [15]. Blogs and online communities also provide a platform to find and communicate with like-minded people. On the other hand, communication between individuals with different opinions are also common, such as political debates between Democratic and Republican parties in the USA.

If not specifically stated, the time step t belongs to $\mathbb{N}$, and $t_k^i \in \mathbb{T}_i$ represents the time for agent i to update its implicit opinion. According to the dual opinions' update rules (1), implicit opinions of agent $i \in V$ remain unchanged for a period of time $t \in [t_k^i, t_{k+1}^i)$, thus denote $x_i(t) = x_i(t_k^i), \forall t \in [t_k^i, t_{k+1}^i)$. Define functions $\mathcal{T}_i : \mathbb{N} \to \mathbb{T}_i$, $\mathcal{T}_i(t) = \max_{t_k \in \mathbb{T}_i, t_k \le t} t_k$, where $i = 1, \cdots, n$, so that $y_i(t) = \phi x_i(\mathcal{T}_i(t)) + (1-\phi)\hat{y}_{i,avg}(t-1), \forall i \in V, t \in \mathbb{N}^+$. Denote moreover $f_i(t_k^i, t_{k+1}^i) = x_i(t_k^i)^b s_i(t_k^i, t_{k+1}^i) + (1 - x_i(t_k^i))^b (d_i - s_i(t_k^i, t_{k+1}^i))$ for all $i \in V$ and $t_k^i, t_{k+1}^i \in \mathbb{T}_i$.

A. Homogeneous Discussion Scenario

Theorem 1: Consider a connected social network $\mathcal{G}$, where the bias parameter $b > 0$ and dual opinions evolve according to (1). If either $x_i(0) > 0.5$, $y_i(0) > 0.5, \forall i \in V$, or $x_i(0) < 0.5, y_i(0) < 0.5, \forall i \in V$ holds, then

$$\lim_{k\to\infty} x_i(t_k^i) = \frac{1 + \operatorname{sgn}(x_i(0) - 0.5)}{2}, \quad \forall i \in V$$
$$\lim_{t\to\infty} y_i(t) = \frac{1 + \operatorname{sgn}(y_i(0) - 0.5)}{2}, \quad \forall i \in V.$$

Proof: Our proof is inspired by [16]. We will prove the case that $x_i(0) > 0.5, y_i(0) > 0.5$ holds for all $i \in V$, and the other case can be proved in a similar way. Define the function $g_{\min}(t) = \min_{i\in V}\{x_i(t), y_i(t)\}, \forall t \in \mathbb{N}$.

First, we will show that $g_{\min}(t)$ is monotonically increasing. If there exists $i \in V$ such that $\mathcal{T}_i(1) = 1$, i.e., $t_1^i = 1$, then $s_i(0,1) = \sum_{l\in V} w_{i,l} y_l(0) \ge d_i g_{\min}(0)$, leading to

$$\begin{aligned} x_i(1) - g_{\min}(0) =& \frac{1}{f_i(0,1)}[x_i(0)^b s_i(0,1)(1 - g_{\min}(0)) \\ & - g_{\min}(0)(1 - x_i(0))^b (d_i - s_i(0,1))] \ge 0, \\ y_i(1) - g_{\min}(0) \ge& (1-\phi)(\hat{y}_{i,avg}(0) - g_{\min}(0)) \ge 0, \end{aligned}$$

which implies that $g_{\min}(1) \ge g_{\min}(0)$. Otherwise, i.e., $\mathcal{T}_i(1) < 1$ for all $i \in V$, we get $x_i(1) = x_i(0) \ge g_{\min}(0)$ and $y_i(1) = \phi x_i(0) + (1-\phi)\hat{y}_{i,avg}(0) \ge g_{\min}(0)$, which implies that $g_{\min}(1) \ge g_{\min}(0)$. Together, we get $g_{\min}(1) \ge g_{\min}(0)$. By iteration, we obtain that $g_{\min}(t+1) \ge g_{\min}(t)$ holds for all $t \in \mathbb{N}$. According to the monotonic bounded convergence theorem, the limit of $g_{\min}(t)$ exists. Let $g^*_{\min} = \lim_{t\to\infty} g_{\min}(t)$.

Second, we will show by contradiction $g^*_{\min} = 1$. Assume that $g^*_{\min} < 1$. Then, choosing $\epsilon = \phi g^*_{\min}(1 - g^*_{\min})(g_{\min}(0)^b - (1 - g_{\min}(0))^b) > 0$, there exists $T_\epsilon > 0$ such that for all $t \ge T_\epsilon$, $|g_{\min}(t) - g^*_{\min}| < \epsilon$. It follows that for $\tau_1 > \tau_2 \ge T_\epsilon$,

$$g_{\min}(\tau_1) - g_{\min}(\tau_2) \le g^*_{\min} - g_{\min}(\tau_2) < \epsilon. \tag{3}$$

Let $t_{k_i}^i = \mathcal{T}_i(T_\epsilon + \tau_{\max})$ for every agent $i \in V$, $\tau_1 = \max_{i\in V} t_{k_i+1}^i$, and $\tau_2 = T_\epsilon$, so that for every $i \in V$, we have $\tau_2 < t_{k_i}^i < t_{k_i+1}^i \le \tau_1 < \tau_2 + 2\tau_{\max}$. For every agent $i \in V$ and $t \in [t_{k_i+1}^i, \tau_1]$,

- if $\mathcal{T}_i(t) = t$, let $k = \mathcal{T}_i(t-1)$, so that $k, t \in \mathbb{T}_i$ and $\tau_2 < t_{k_i}^i \le k < t \le \tau_1$, and thus

$$\begin{aligned} & x_i(t) - g_{\min}(\tau_2) \\ =& \frac{1}{f_i(k,t)}[x_i(k)^b s_i(k,t)(1 - g_{\min}(\tau_2)) \\ & - (1 - x_i(k))^b (d_i - s_i(k,t)) g_{\min}(\tau_2)] \\ \ge& \frac{d_i g_{\min}(\tau_2)(1 - g_{\min}(\tau_2))}{f_i(k,t)}(x_i(k)^b - (1 - x_i(k))^b) \quad (4) \\ \ge& g^*_{\min}(1 - g^*_{\min})(g_{\min}(0)^b - (1 - g_{\min}(0))^b) > \epsilon, \quad (5) \end{aligned}$$

where (4) and (5) hold since $d_i g_{\min}(\tau_2) \le s_i(k,t) \le d_i$, and $y_i(t) - g_{\min}(\tau_2) \ge \phi(x_i(t) - g_{\min}(\tau_2)) \ge \epsilon$.

- otherwise, i.e., $t > \mathcal{T}_i(t) \ge t_{k_i+1}^i$, we have $x_i(t) - g_{\min}(\tau_2) = x_i(\mathcal{T}_i(t)) - g_{\min}(\tau_2) > \epsilon$ and $y_i(t) - g_{\min}(\tau_2) \ge \phi(x_i(\mathcal{T}_i(t)) - g_{\min}(\tau_2)) \ge \epsilon$.

In conclusion, we get $g_{\min}(\tau_1) - g_{\min}(\tau_2) \ge \epsilon$, which contradicts inequality (3). Thus, the above assumption must be wrong, so we get that $g^*_{\min} = 1$. Consequently, if $x_i(0) > 0.5, y_i(0) > 0.5$ hold for every agent $i \in V$, then $\lim_{k\to\infty} x_i(t_k^i) = \frac{1+sgn(x_i(0)-0.5)}{2} = 1$ and $\lim_{t\to\infty} y_i(t) = \frac{1+sgn(y_i(0)-0.5)}{2} = 1$, which completes the proof. ■

The empirical study [17] groups students with similar opinions and then let them discuss some racial issues and show their opinions before and after the discussion, and [17] indicates that the group discussion intensifies their initial opinions. Theorem 1 vividly reflects this empirical study that the group discussion among individuals with similar opinions will strengthen their initial preference leading to opinion polarization.

B. Adversarial Discussion Scenario

In this part, we investigate the co-evolution of dual opinions in the adversarial discussion scenario, where two types of individuals with opposite initial opinions participate in group discussions. Since more interactions are between like-minded agents [15], we use the two-island network [3] to model the communication network in this discussion scenario.

Definition 1: [3] Given integers $n_1, n_2 \ge 0$ and real numbers $p_s, p_d \in (0,1)$, a (n_1, n_2, p_s, p_d)-two-island network is a weighted undirected graph $G = (V_1, V_2, E, W)$, where

- $|V_1| = n_1$, $|V_2| = n_2$, and $V_1 \cap V_2 = \emptyset$.
- Each node $i \in V_l$ has $n_l p_s \in \mathbb{N}$ neighbors in V_l and $n_l p_d \in \mathbb{N}$ neighbors in V_k, $l, k \in \{1,2\}, l \ne k$.
- $p_s > p_d$, $h_G = p_s/p_d$ is the degree of homophily of G.

Assumption 1: The social network $\mathcal{G}$ is a (m, m, p_s, p_d)-two-island network, where $V = V_1 \cup V_2$, $|V_1| = |V_2| = m$.

- Parameter settings: $\forall i, j \in V$, $m_{i,j} = w_{i,j}/d_i$, $\alpha_i(t) = \alpha(t), \forall t \in \mathbb{N}$.
- Initial states: $\forall i \in V_1$, $x_i(t_0) = y_i(0) = x_0 \in (\frac{1}{2}, 1)$ and $\forall j \in V_2$, $x_j(t_0) = y_j(0) = 1 - x_0$.
- Update time: $\forall i \in V$, $\mathbb{T}_i = \mathbb{T} = \{t_k\}_{k=0}^{\infty}$.

The parameter settings and initial state condition in the above assumption is adopted in [3], where the standard biased assimilation model is established. If Assumption 1 holds, then $\mathbb{T}_i = \mathbb{T}$ for all $i \in V$, which means that all agents update implicit opinions at the same time. Let $z_i(t_k) = \frac{x_i(t_k)}{1-x_i(t_k)}$ for all agents $i \in V$ and $t_k \in \mathbb{T}$.

Lemma 1: Consider a social network $\mathcal{G}$, where dual opinions evolve according to (1). Suppose that Assumption 1 holds. For individuals i, j in the same group, it holds that

$$x_i(t_k) = x_j(t_k) \text{ and } y_i(t) = y_j(t), \forall t \in \mathbb{N}, t_k \in \mathbb{T}. \quad (6)$$

For individuals $i \in V_1$ and $j \in V_2$, it holds that

$$x_i(t_k) + x_j(t_k) = 1, \ 0.5 < x_i(t_k) < 1, \forall t_k \in \mathbb{T},$$
$$y_i(t) + y_j(t) = 1, \ 0.5 < y_i(t) < 1, \forall t \in \mathbb{N}.$$

The proof is simple and so omitted here. Lemma 1 shows that the dual opinions of individuals in different groups are opposite. For brevity, the following Theorems 2-3 will only show the results of individuals in group V_1, and we can prove the results of individuals in group V_2 in a similar way. We divide the individual bias into the following two levels:

$$\textit{Low} : b < \frac{2}{h_G + 1}, \ \textit{High} : b \geq 1.$$

The following analyzes the influence of group discussions on the co-evolution of dual opinions under the above two levels of individual bias b, while the case of $b \in [\frac{2}{h_G+1}, 1)$ will be discussed in Section IV.

Theorem 2: Consider a social network $\mathcal{G}$, where dual opinions evolve according to (1). Suppose that $b < \frac{2}{h_G+1}$ and Assumption 1 holds. For every individual $i \in V_1$, it holds that

$$x_i(t_k) > x_i(t_{k+1}), \forall t_k, t_{k+1} \in \mathbb{T}, \quad (7)$$
$$y_i(t) > y_i(t+1), \forall t \in \mathbb{N}, \quad (8)$$
$$x_i(t_k) > y_i(t), \forall t_k \in \mathbb{T}, t \in [t_k, t_{k+1}), t \in \mathbb{N}^+. \quad (9)$$

Additionally,

$$\lim_{k\to\infty} x_i(t_k) = 0.5 \text{ and } \lim_{t\to\infty} y_i(t) = 0.5, \forall i \in V_1. \quad (10)$$

Proof: First, we prove (7)-(9) by induction. *Base case:* for every individual $i \in V_1$ and $t \in [t_0, t_1 - 1)$, we have

$$\begin{aligned} y_i(t+1) - y_i(t) &= (1-\phi)(\hat{y}_{i,avg}(t) - \hat{y}_{i,avg}(t-1)) \\ &= \frac{(1-\phi)(h_G - 1)}{h_G + 1}(y_i(t) - y_i(t-1)) \\ &= \frac{(1-\phi)^{t+1}(h_G-1)^t}{(h_G+1)^{t+1}}(1 - 2y_i(0)) < 0. \end{aligned} \quad (11)$$

It follows that for individual $i \in V_1$ and $t \in (t_0, t_1)$,

$$\begin{aligned} y_i(t) - x_i(t_0) &= (1-\phi)(\hat{y}_{i,avg}(t-1) - y_i(t_0)) \\ &< (1-\phi)(y_i(t-1) - y_i(t_0)) \leq 0, \end{aligned} \quad (12)$$

which implies that $s_i(t_0, t_1) = \sum_{t=t_0}^{t_1-1} \alpha(t) \sum_{l\in N_i} w_{i,l} y_l(t) \leq \sum_{l\in N_i} w_{i,l} x_l(t_0)$, so that

$$z_i(t_1) \leq z_i(t_0)^b \frac{\sum_{l\in N_i} w_{i,l} x_l(t_0)}{d_i - \sum_{l\in N_i} w_{i,l} x_l(t_0)} < z_i(t_0), \quad (13)$$

where the second inequality holds according to Lemma 3.4 in the Supplemental Information of [3], i.e., $x_i(t_1) < x_i(t_0)$. By this, we have

$$y_i(t_1) - y_i(t_1 - 1) < \phi(x_i(t_1) - x_i(t_0)) < 0, \forall i \in V_1. \quad (14)$$

Inductive assumption: assume that for some time $t_k \in \mathbb{T}$, $x_i(t_k) > x_i(t_{k+1})$, $x_i(t_k) > y_i(t)$ and $y_i(t) > y_i(t+1)$ hold for every $i \in V_1$ and $t \in [t_k, t_{k+1})$. It follows that for every $i \in V_1$, similar to (11)-(12), we have $y_i(t+1) < y_i(t)$ for any $t \in [t_{k+1}, t_{k+2} - 1)$, and consequently $y_i(t) < x_i(t_{k+1})$ and $s_i(t_{k+1}, t_{k+2}) < \sum_{j\in V} w_{i,j} x_i(t_{k+1})$ hold, thus we have $x_i(t_{k+2}) < x_i(t_{k+1})$ by (13). Then, similar to (14), we get that $y_i(t_{k+2}) < y_i(t_{k+2} - 1)$ holds for every individual $i \in V_1$, which completes the inductive proof. Hence, (7)-(9) hold, so that for every $i \in V_1$, $x_i(t_k)$ and $y_i(t)$ are decreasing, thus the limits of $x_i(t_k)$ and $y_i(t)$ exist. Denote $x^* = \lim_{k\to\infty} x_i(t_k)$ and $y^* = \lim_{t\to\infty} y_i(t)$ for $i \in V_1$, so that $\frac{1}{2} \leq y^* \leq x^*$ holds. Taking the limits on both sides of (1), we obtain

$$\begin{aligned} x^* &= \frac{(x^*)^b (h_G y^* + 1 - y^*)}{(x^*)^b (h_G y^* + 1 - y^*) + (1 - x^*)^b (h_G - h_G y^* + y^*)}, \\ y^* &= \phi x^* + \frac{(1-\phi)}{h_G + 1}[(h_G - 1)y^* + 1]. \end{aligned}$$

When $b < \frac{2}{h_G+1}$, Lemma 3 in [18] suggests that the above equations have an unique solution $x^* = y^* = \frac{1}{2}$, which completes the proof. ■

Theorem 2 shows that if individuals have low bias ($b < \frac{2}{h_G+1}$), their implicit and explicit opinions become more neutral as the group discussion proceeds, and finally dual opinions of two types of individuals with opposing initial opinions converge to the neutral opinion 0.5. At this point, individuals exhibit an inward conformity phenomenon [14] that they gradually accept the opinions of other group members and achieve the agreement.

Theorem 3: Consider a social network $\mathcal{G}$, where individuals update their dual opinions by (1). Suppose that $b \geq 1$ and Assumption 1 holds. For every individual $i \in V_1$, it holds that

$$x_i(t_{k+1}) > x_i(t_k), \forall t_k, t_{k+1} \in \mathbb{T}, \quad (15)$$
$$x_i(t_k) > y_i(t), \forall t \in [t_k, t_{k+1}), t_k, \in \mathbb{T}, t \in \mathbb{N}^+, \quad (16)$$

and there exists $\tau > 0$ such that $y_i(t)$ is monotonic for all $t > \tau, t \in \mathbb{N}$. Additionally,

$$\lim_{k\to\infty} x_i(t_k) = 1 \text{ and } \lim_{t\to\infty} y_i(t) = \frac{\phi h_G + 1}{\phi h_G + 2 - \phi}, \forall i \in V_1.$$

Proof: First, we will show that (15) holds. According to Lemma 1, for $i \in V_1$ and $j \in V_2$, we get that $y_i(t) = 1 - y_j(t) > 0.5$ holds for all $t > 0$, thus $s_i(t_k, t_{k+1}) > \frac{1}{2} d_i$ holds for all $t_k, t_{k+1} \in \mathbb{T}$, which follows that

$$z_i(t_{k+1}) = z_i(t_k)^b \frac{s_i(t_k, t_{k+1})}{d_i - s_i(t_k, t_{k+1})} > z_i(t_k), \quad (17)$$

that is, $x_i(t_{k+1}) > x_i(t_k)$.

Second, we will show by contradiction that there exists a time T_e such that for every $i \in V_1$, $y_i(t)$ is monotonic for all $t > T_e$. Assume that there exists a increasing time sequence $\{\tau_k\}_{k=0}^{\infty}$ fulfilling $\tau_k \in \mathbb{N}$ such that for $i \in V_1$, $y_i(\tau_{2k}) < y_i(\tau_{2k} + 1)$ and $y_i(\tau_{2k+1}) > y_i(\tau_{2k+1} + 1)$ hold for all $k \in \mathbb{N}$. It follows that $y_i(\tau_0) < y_i(\tau_0 + 1)$ holds for every $i \in V_1$. Similar to (11), for $i \in V_1$, $y_i(\tau_0 + 2) - y_i(\tau_0 + 1) \geq \frac{(1-\phi)(h_G-1)}{h_G+1}(y_i(\tau_0 + 1) - y_i(\tau_0)) > 0$. By iteration, we can get that for $i \in V_1$, $y_i(t+1) > y_i(t)$ holds for $t \geq \tau_0$, which means that no such τ_1 exists fulfilling $y_i(\tau_1) > y_i(\tau_1 + 1)$. Hence, the above assumption must be wrong, and thus there exists T_e such that for every $i \in V_1$, $y_i(t)$ is monotonic for all $t > T_e$. By Lemma 1, for all $l \in V$, $x_l(t_k)$ and $y_l(t)$ converge. Denote $x^* = \lim_{k\to\infty} x_i(t_k)$ and $y^* = \lim_{t\to\infty} y_i(t)$ for $i \in V_1$.

Third, we will show by contradiction that $x^* = 1$. Assume $x^* < 1$. For every $i \in V_1$, since $x_i(t_k)$ is increasing when $t_k \in \mathbb{T}$, we have $x_i(t_k) < x^*$. Choosing $\epsilon = x^*(1-x^*)[x_0^{b-1} - (1-x_0)^{b-1}] > 0$, there exists time $k_e \in \mathbb{T}$ such that for every $i \in V_1$, $|x_i(t) - x^*| < \epsilon$ holds for all $t > k_e$. It follows that for $i \in V_1$, $x_i(t_{k+1}) - x_i(t_k) < x^* - x_i(t_k) < \epsilon$ holds for any $t_k > k_e, t_k \in \mathbb{T}$. However, for $i \in V_1$, we have

$$\begin{aligned}
x_i(t_{k+1}) - x_i(t_k) =& \frac{1}{f_i(t_k, t_{k+1})} \Big[x_i(t_k)^b s_i(t_k, t_{k+1})(1 - x_i(t_k)) \\
&- x_i(t_k)(1 - x_i(t_k))^b (d - s_i(t_k, t_{k+1})) \Big] \\
>& \frac{x_i(t_k)(1 - x_i(t_k))}{s_i(t_k, t_{k+1})} \Big[x_i(t_k)^{b-1} s_i(t_k, t_{k+1}) \\
&- (1 - x_i(t_k))^{b-1} (d - s_i(t_k, t_{k+1})) \Big] \\
>& x^*(1 - x^*)[x_i(t_k)^{b-1} - (1 - x_i(t_k))^{b-1}] \\
>& x^*(1 - x^*)[x_0^{b-1} - (1 - x_0)^{b-1}] = \epsilon,
\end{aligned}$$

which leads to a contradiction. Hence, the above assumption must be wrong, and thus we must have $x^* = 1$. Taking the limits on both sides of (1), we get that $y^* = \frac{\phi h_G + 1}{\phi h_G + 2 - \phi}$.

Finally, we will prove by induction that for every $i \in V_1$, (16) holds. *Base case*: $x_i(t_0) = y_i(t_0) = x_0$ for every $i \in V_1$ by Assumption 1, which implies that for any $t \in (t_0, t_1)$,

$$\begin{aligned}
y_i(t) - x_i(t_0) <& \frac{(1 - \phi) h_G}{h_G + 1} (y_i(t - 1) - x_i(t_0)) \\
<& (\frac{(1 - \phi_l) h_G}{h_G + 1})^t (y_i(t_0) - x_i(t_0)) = 0.
\end{aligned}$$

namely, (16) holds when $t \in (t_0, t_1)$. *Inductive assumption*: assume that (16) holds when $t \in [t_k, t_{k+1})$. Based on this, for $i \in V_1$, $\hat{y}_{i,avg}(t_{k+1} - 1) < y_i(t_{k+1} - 1) \leq x_i(t_k)$, so that

$$\begin{aligned}
x_i(t_{k+1}) - y_i(t_{k+1}) =& (1 - \phi)(x_i(t_{k+1}) - \hat{y}_{i,avg}(t_{k+1} - 1)) \\
\geq& (1 - \phi)(x_i(t_{k+1}) - x_i(t_k)) > 0.
\end{aligned}$$

Thus, for every $i \in V_1$, $x_i(t_{k+1}) > y_i(t_{k+1}) > \hat{y}_{i,avg}(t_{k+1})$ holds, and consequently for any $t \in (t_{k+1}, t_{k+2})$, we have

$$\begin{aligned}
x_i(t_{k+1}) - y_i(t) =& (1 - \phi)(x_i(t_{k+1}) - \hat{y}_{i,avg}(t - 1)) \\
>& (1 - \phi)(x_i(t_{k+1}) - y_i(t - 1)) \\
>& (1 - \phi)^{t - t_{k+1}} (x_i(t_{k+1}) - y_i(t_{k+1})) > 0.
\end{aligned}$$

In conclusion, (16) holds when $t \in [t_{k+1}, t_{k+2})$, which completes the inductive proof. ■

Theorem 3 shows that if individuals have high bias ($b \geq 1$), the group discussion among two types of individuals with opposite initial opinions will worsen the differences, leading to the implicit opinion polarization. Despite this, individuals exhibit more neutral explicit opinions than their implicit opinions in the evolution, showing an outward compliance phenomenon. Besides, explicit opinions may not be monotone at the beginning but eventually converge monotonically.

Remark 4: In contrast to previous analyses on the biased assimilation model [3], [16], [19], which considered that each agent has only one opinion, our study explores the tightly coupled co-evolution of implicit and explicit opinions. Additionally, while previous analyses [4], [9], [18] considered synchronous updates, we adopt the asynchronous update rule, where individuals update implicit opinions in their own pace. Moreover, we introduce the concept of after-effects of observed opinion information, in which individuals are influenced not only by information received at the update time but also by opinion received in the past period of time.

Next, we will analyze the influence of parameters on the co-evolution of dual opinions. Let an ordered tuple $\mathcal{P} = (b, h_G, \phi, \alpha(t))$ represent parameter settings for model (1) under Assumption 1. Denote $x_i^{\mathcal{P}}(t)$, $y_i^{\mathcal{P}}(t)$ as implicit and explicit opinions of agent i at time t in group discussions under parameters settings $\mathcal{P}$.

Proposition 1: Consider two group discussions with parameter setting $\mathcal{P}_1 = (b^1, h_G^1, \phi^1, \alpha^{\mathcal{P}_1}(t))$ and $\mathcal{P}_2 = (b^2, h_G^2, \phi^2, \alpha^{\mathcal{P}_2}(t))$, respectively, where individuals update dual opinions by (1). Suppose that Assumption 1 holds and $b^1, b^2 \in (0, \frac{2}{h_G + 1})$. If $x_l^{\mathcal{P}_1}(0) = x_l^{\mathcal{P}_2}(0)$ holds for all $l \in V$ and $\mathcal{P}_1$,$\mathcal{P}_2$ fulfill one of following conditions

c1. $(b^1, h_G^1, \phi^1) \geq (b^2, h_G^2, \phi^2)$ and $\alpha^{\mathcal{P}_1}(t) = \alpha^{\mathcal{P}_2}(t)$;

c2. $(b^1, h_G^1, \phi^1) \geq (b^2, h_G^2, \phi^2)$ and for all $t_k, t_{k+1} \in \mathbb{T}$,

$$\alpha^{\mathcal{P}_1}(t) = \hat{\alpha}(t) = \begin{cases} 1 & \text{if } t = t_k, \\ 0 & \text{otherwise,} \end{cases}, \ \forall t \in [t_k, t_{k+1});$$

c3. $(b^1, h_G^1, \phi^1) \geq (b^2, h_G^2, \phi^2)$ and for all $t_k, t_{k+1} \in \mathbb{T}$,

$$\alpha^{\mathcal{P}_2}(t) = \check{\alpha}(t) = \begin{cases} 1 & \text{if } t = t_{k+1} - 1, \\ 0 & \text{otherwise,} \end{cases} \forall t \in [t_k, t_{k+1});$$

then for $i \in V_1$, $t_k \in \mathbb{T}$ and $t \in \mathbb{N}$,

$$x_i^{\mathcal{P}_1}(t_k) \geq x_i^{\mathcal{P}_2}(t_k), \text{ and } y_i^{\mathcal{P}_1}(t) \geq y_i^{\mathcal{P}_2}(t). \tag{18}$$

Proof: We will use the superscript $\mathcal{P}_l$ denote variables in group discussion under parameter settings $\mathcal{P}_l$. For $i \in V_1$ and $t \in [0, t_1)$, similar to (11), $y_i^{\mathcal{P}_1}(t) \geq y_i^{\mathcal{P}_2}(t)$ holds since $(h_G^1, \phi^1) \geq (h_G^2, \phi^2)$ and $x_i^{\mathcal{P}_1}(0) = x_i^{\mathcal{P}_2}(0)$, which implies that $s_i^{\mathcal{P}_1}(0, t_1) \geq s_i^{\mathcal{P}_2}(0, t_1)$ and $\hat{y}_{i,avg}^{\mathcal{P}_1}(t) \geq \hat{y}_{i,avg}^{\mathcal{P}_2}(t)$ hold. Based on this, we get that $z_i^{\mathcal{P}_1}(t_1) \geq z_i^{\mathcal{P}_2}(t_1)$ holds for $i \in V_1$ since $b^1 \geq b^2$, i.e., $x_i^{\mathcal{P}_1}(t_1) \geq x_i^{\mathcal{P}_2}(t_1)$. By iteration, we obtain that for $i \in V_1$, (18) holds for any $t_k \in \mathbb{T}$ and $t \in \mathbb{N}$. ■

Proposition 1 builds on Theorem 2 and further demonstrates the influence of parameters on dual opinions coevolutionary process when individuals have low bias. Specifically, if individuals are with lower b, ϕ and h_G, dual opinions are closer to the neutral state 0.5, and for any given $\alpha(t)$, dual opinions are more (less) close to the neutral state 0.5 compared to the case where $\alpha(t)$ is set to $\check{\alpha}(t)$ ($\hat{\alpha}(t)$).

IV. Discussion

The influence of the after-effect function and asynchronous update on the evolution is intricately linked to the other parameters involved. To exhibit the impact of these two features on convergence results, we conduct the following simulations.

Consider a group discussion, denoted as GD1, with parameters set in Theorem 2, except for the agents in different groups having distinct implicit opinion update times. In Fig. 1(a), we depict the trajectories of dual opinions during the evolution, where $b = 0.5$ and the update time of implicit opinions is defined as follows: $t_k^i = 2k$ for $i \in V_1$, $t_k^j = 4k$ for $j \in V_2$, and $\alpha_l(t) = 1/(t_{k+1}^l - t_k^l)$ for all $l \in V$ and $t \in [t_k^l, t_{k+1}^l)$. Consequently, agents in V_2 exhibit greater resistance to changes in implicit opinions compared to agents in V_1. As shown in Fig. 1(a), influenced by agents in group V_2, the dual opinions of agents in group V_1 are guided from support to disapproval. Eventually, consensus is reached among all agents' dual opinions at the value 0, which differs from the convergence state observed in Theorem 2. This exhibits the impact of implicit opinion update times on convergence states of dual opinions.

Moreover, the after-effect function $\alpha(t)$ can also affect the co-evolution results. Fig 1(b) and (c) show opinion evolution in two group discussions, called GD2 and GD3, under the same parameter setting except for the after-effect function, where individuals have the same $b = 0.805$, $\phi = 0.5$ and $h_G = 2$, and implicit opinion updating time is the same as GD1. The after-effect function in GD2 is the same as it in GD1, while in GD3, the after-effect function $\alpha_i(t) = \hat{\alpha}(t)$ for $i \in V_1$ and $\alpha_j(t) = \check{\alpha}(t)$ for $j \in V_2$. We observe the opposite convergence states in Fig. 1(b) and (c), which implies the impact of the function $\alpha(t)$ on the evolution of dual opinions.

Besides, we have theoretically analyzed the dual opinion co-evolution under the adversarial discussion scenario when agents have low and high bias in Theorems 2 and 3, respectively. For $b \in [\frac{2}{h_G+1}, 1)$, we conduct numerical simulations of the evolution of dual opinions under Assumption 1, where $h_G = 2, \mathbb{T} = \{2k\}_{k=0}^{\infty}, \alpha(t) = \frac{1}{2}$ and parameters $b \in [0.5, 1]$, $\phi \in (0, 1)$ are taken with step size 0.01. Fig. 1(d) shows an example for $b = 0.805$, $\phi = 0.5$. Simulation results show that the trajectories of the evolutionary process vary under different parameter settings but dual opinions all converge eventually. Fig. 1(e) and (f) show the convergence states of implicit and explicit opinions of agents in group V_1 under different bias parameters b and resilience ϕ, and critical value b^* exists that changes dual opinions from consensus to disagreement.

V. Conclusion

In this paper, we investigated the dynamics of coupled implicit and explicit opinions under asynchronous updating. Slowing down the update rate of implicit opinions accurately reflects the difficulty of changing implicit opinions compared to explicit opinions as demonstrated in the dual attitude theory. Additionally, we considered that the neighbors' explicit opinions can have after-effects on the update of implicit opinions. We theoretically analyzed the co-evolution of dual opinions in both homogeneous and adversarial discussion scenarios, which are common in reality and on the Internet. We recognized that more research is needed to explore the co-evolution of dual opinions in a more general setting and influence of implicit updating time and the after-effect on the evolution.

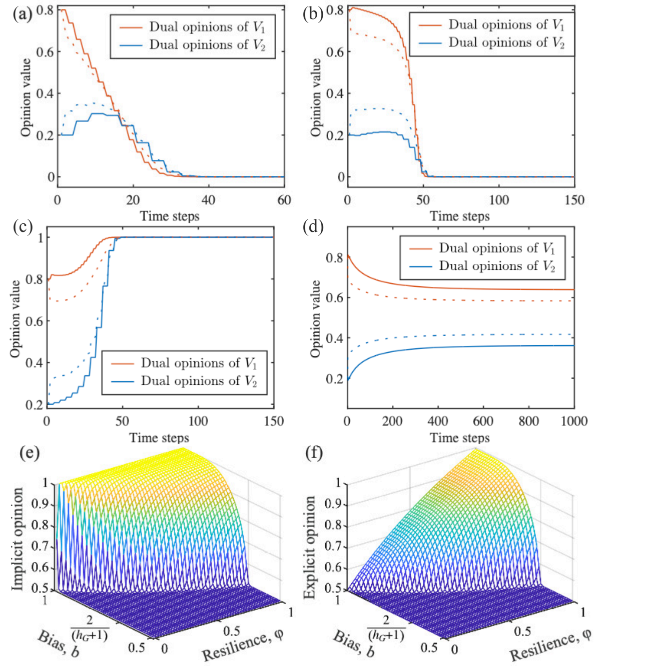

Fig. 1. The trajectories of dual opinions in evolution, $\phi = 0.5, h_G = 2$. Solid and dotted lines are implicit and explicit opinions, respectively.

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