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Guaranteeing Stability in Structured Input-Output Models: With Application to System Identification

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***Abstract*— Identifying *structured* discrete-time linear time/parameter-varying (LPV) input-output (IO) models with global stability guarantees is a challenging problem since stability for such models is only implicitly defined through the solution of matrix inequalities (MI) in terms of the model's coefficient functions. In this paper, a structured linear IO model class is developed that results in a quadratically stable model for any choice of coefficient functions, enabling identification using standard optimization routines while guaranteeing stability. This is achieved through transforming the MI-based stability constraints in a necessary and sufficient manner, such that for any choice of transformed coefficient functions the MIs are satisfied. The developed stable LPV-IO model is employed in simulation to estimate the parameter-varying damping of mass-damper-spring system with stability guarantees, while a standard LPV-IO model results in an unstable estimate.**

I. Introduction

Given a stable system, stability of models obtained using system identification [1] is often desirable for their utilization in prediction, simulation, and control. However, even if the underlying data-generating system is stable, the model resulting from the identification process can be unstable due to finite-time effects, modeling errors, or measurement noise [2].

Ensuring stability of identified models has attracted interest from the perspective of different model classes, ranging from linear time-invariant (LTI) [2]–[6] through linear time/parameter-varying (LTV/LPV) [7]–[9] to nonlinear models [10]–[13]. These results have been developed both in continuous time (CT) [13] and in discrete time (DT), and for state-space (SS) [3], [5], [8], [12], [13] and input-output (IO) representations [7], [9], [10].

To ensure stability of an identified model, three different approaches are distinguished in the aforementioned literature. First, certain approaches involve projecting identified parameters back onto the set of stable models post-identification [3], [6]. However, this projection disregards the measured data and may lead to a significant decline in prediction performance [3]. Second, stability can be enforced during optimization by introducing constraints on the model parameters that, when satisfied, imply stability [2], [7], [10], [11]. Nevertheless, these constraints typically take the form of matrix inequalities (MI), which can significantly increase the computational complexity of the optimization process.

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The most recent and third approach, first described by [12], reparameterizes these LMI conditions representing stability in terms of transformed parameters in such a way that the MI conditions are satisfied for any choice of these transformed parameters, i.e., the model is always guaranteed to be stable [5], [8], [9], [12], [13]. This allows for the use of arbitrary functions in the models, such as neural networks or polynomials, while still guaranteeing stability. Additionally, this enables the use of unconstrained optimization methods.

Next to stability, it is often desired to embed structure into the model used in identification. One example is grey-box system identification [14], where models consist of physical equations imposing structural relationships and model parameters representing unknown physical functions. Other examples include embedding prior knowledge, encoding dependence of the model on only a specific subset of delayed inputs and outputs, and in the LPV case, independence of certain coefficient functions from the scheduling signal.

Guaranteeing stability through reparametrization in the structured case is significantly more complex than in the unstructured case [12]. Specifically, the imposed structure results in a stability test in which some parts are fixed. Reparameterization of this stability test then has to adhere to this imposed structure. Additionally, the structure also imposes conditions on possible Lyapunov functions. In contrast, in an unstructured setting all parts of the stability condition can be freely reparametrized. Only in the DT LTI case [5] a scaling argument can be employed to ensure that the eigenvalues of the system are within the unit circle. However, such an approach does not extend to LPV/LTV models, in which it is required to adopt a Lyapunov approach.

The main contribution of this work is a linear IO model class where the model coefficient functions are constrained within a lower dimensional linear subspace and stability of the model is ensured for any choice of coefficient functions. This is achieved through the following subcontributions.

- C1) A criterion based on coupled matrix inequalities to characterize stability of a linear IO model (Section III).
- C2) A reparameterization of the coefficient functions such that the above criterion is satisfied for any choice of the transformed coefficient functions (Section IV).
- C3) A simulation example in which a neural network is employed to learn a position-dependent damping with stability guarantees (Section V).

The stability criterion used in this paper is quadratic stability under a parameter-invariant Lyapunov function (QS). Consequently, only systems that are QS can be represented by the developed stable LPV-IO model. While this could be conservative since not all stable LPV-IO systems are also QS

[15], the developed stable LPV-IO already represents a significant improvement over current methods which consider only deviations from a fixed QS model or incorporate QS as a constraint during optimization [7].

II. Problem Formulation

Consider the discrete-time linear system $G : u \to y$ with input $u_k \in \mathbb{R}$ and output $y_k \in \mathbb{R}$ with y resulting from the parameter-varying input-output difference equation

$$y_k = -\sum_{i=1}^{n_a} a_i(\rho_k) y_{k-i} + \sum_{i=0}^{n_b-1} b_i(\rho_k) u_{k-i}, \quad (1)$$

with coefficients functions $a_i, b_i : \mathbb{P} \to \mathbb{R}$ describing the dependence of the difference equation on a scheduling signal $\rho : \mathbb{Z}_{\geq 0} \to \mathbb{P} \subseteq \mathbb{R}^{n_\rho}$ at time index $k \in \mathbb{Z}_{\geq 0}$. Note that for constant $\rho_k = \rho_c\ \forall k$, (1) is equivalent to the LTI transfer function $G(z) = \beta(z)/\alpha(z)$ with $\beta(z) = \sum_{i=0}^{n_b-1} b_i(\rho_c) z^{-i}$ and $\alpha(z) = 1 + \sum_{i=1}^{n_a} a_i(\rho_c) z^{-i}$. Thus, $a_i(\rho_k)$ can be interpreted as describing the variation in poles of (1), and $b_i(\rho_k)$ as the variation in zeros. For $\rho_k = k\ \forall k$, an LTV-IO model is recovered. Define the collection of a_i, b_i as

$$A(\rho_k) = \begin{bmatrix} a_1(\rho_k) & a_2(\rho_k) & \dots & a_{n_a}(\rho_k) \end{bmatrix} \in \mathbb{R}^{1\times n_a}, \quad (2)$$
$$B(\rho_k) = \begin{bmatrix} b_0(\rho_k) & b_1(\rho_k) & \dots & b_{n_b-1}(\rho_k) \end{bmatrix} \in \mathbb{R}^{1\times n_b}. \quad (3)$$

Now consider that it is desired that both R_1) the coefficient functions $A(\rho), B(\rho)$ are structured, and R_2) the dynamics represented by (1) are stable. With respect to R_1, in this paper the considered structure is of the form

$$A(\rho) = \bar{A}(\rho) H, \qquad B(\rho) = \bar{B}(\rho) H_b, \quad (4)$$

with predefined full rank structure matrices $H \in \mathbb{R}^{n_z \times n_a}$, $n_z < n_a$, $H_b \in \mathbb{R}^{n_l \times n_b}$, $n_l < n_b$ and lower-dimensional coefficient functions $\bar{A} : \mathbb{P} \to \mathbb{R}^{1\times n_z}$, $\bar{B} : \mathbb{P} \to \mathbb{R}^{1\times n_l}$. In other words, $A(\rho)$ and $B(\rho)$ are linearly constrained to some lower-dimensional space of coefficient functions $\bar{A}(\rho)$ and $\bar{B}(\rho)$. Some examples of this structure are as follows.

- Enforcing dependency of (1) on only specific y_{k-i}, u_{k-i}, i.e., sparsity in $A(\rho)$. For example, $y_k = -a_1(\rho_k) y_{k-1} - a_3(\rho_k) y_{k-3}$ can be realized with $A(\rho) = \begin{bmatrix} a_1(\rho) & a_3(\rho) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- Grey-box models in which some physical coefficients are known, and it is only of interest to estimate specific coefficient functions, see Section V.

With respect to R_2, to characterize stability, the standard notion of quadratic Lyapunov stability (QS) is adapted to the IO case using a state vector of delayed outputs.

Definition 1 *Given coefficient functions $A(\rho)$, (1) is said to be quadratically stable if there exists a $P \in \mathbb{S}^{n_a}_{\succ 0}$ such that*

$$x_{k+1}^\top P x_{k+1} < x_k^\top P x_k \quad \forall \rho_k \in \mathbb{P},\ u_k = 0\ \forall k, \quad (5)$$

where $x_k = \begin{bmatrix} y_{k-1} & y_{k-2} & \dots & y_{k-n_a} \end{bmatrix}^\top \in \mathbb{R}^{n_a}$ evolves according to (1) with $u_k = 0\ \forall k$ for any initial $x_0 \in \mathbb{R}^{n_a}$.

Quadratic stability as in Definition 1 implies that y asymptotically approaches zero for all ρ if the input u is uniformly zero after some time $\bar{k}$, i.e., $\lim_{k\to\infty} y_k = 0$ for any ρ, any x_0 and any u with $u_k = 0\ \forall k > \bar{k}$ [16]. Note that QS is both necessary and sufficient for stability in the LTI case, while it is only sufficient for LPV/LTV IO representations.

Given a structure matrix H, the goal of this paper is to parametrize all structured coefficient functions $A(\rho) = \bar{A}(\rho) H$ such that the system (1) is QS as in Definition 1. In other words, the goal is to describe the set of functions

$$\bar{\mathcal{A}} = \{\bar{A}(\rho) \mid A(\rho) = \bar{A}(\rho) H,\ \exists P \in \mathbb{S}^{n_a}_{\succ 0} \text{ s.t. } (5)\}. \quad (6)$$

The approach in this paper is to develop a parametrization of $\bar{A}(\rho)$ such that by construction of $\bar{A}(\rho)$ there always exists a $P \succ 0$ for which (5) is satisfied over the whole domain $\mathbb{P}$, avoiding the need for testing (5) during identification. Specifically, $\bar{A}(\rho)$ and P are jointly constructed from transformed coefficient functions $\xi(\rho) : \mathbb{P} \to \mathbb{R}^{1\times n_a}$ and full rank $X_5 \in \mathbb{R}^{n_a \times n_a}$ that can be chosen freely and an auxilary $\mu \in \mathbb{R}_{(0,1]}$ found through bisection.

This parametrization without constraints on the coefficient functions can subsequently be employed in system identification, in which now any bounded functional parametrization for $\xi(\rho)$ can be chosen, e.g., a linear function or a neural network, while stability is guaranteed by construction.

III. Quadratically Stable Linear IO Systems

To obtain a parameterization of all structured coefficients $\bar{A}(\rho)H$ such that by construction (1) is QS, first further conditions on $\bar{A}(\rho)$ and P are required to determine if (1) is QS. This section provides necessary and sufficient conditions for QS of (1) in terms of matrix inequalities for the Lyapunov certificate P given H, constituting Contribution C1.

Before developing the main equivalence result, first define

$$F = \begin{bmatrix} 0 & 0 \\ I_{n_a} & 0 \end{bmatrix} \in \mathbb{R}^{n_a+1 \times n_a+1}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_a+1}. \quad (7)$$

Additionally, given full row rank $H \in \mathbb{R}^{n_z \times n_a}$, define $V_2 \in \mathbb{R}^{n_a \times n_a - n_z}$ as an orthonormal basis for $\ker H$, i.e., $V_2^\top V_2 = I_{n_a - n_z}$ and $H V_2 = 0$, and define $V_1 \in \mathbb{R}^{n_a \times n_z}$ as an orthonormal basis for Im $H^\top$, i.e., $V_1^\top V_1 = I_{n_z}$ with $H V_1 \in \mathbb{R}^{n_z \times n_z}$ full rank. Matrices V_i can be obtained from, e.g., a singular value decomposition of H. With the provided definitions, the main result of this section can now be stated.

Theorem 2 *Given a structure matrix H, associated V_1, V_2 and structured coefficient functions $A(\rho) = \bar{A}(\rho)H$, (1) is QS as in Definition 1 if and only if there exists a $P \in \mathbb{S}^{n_a}_{\succ 0}$ and $M : \mathbb{P} \to \mathbb{R}^{1\times n_z}$ with $\|M(\rho)\|_2 < 1\ \forall \rho \in \mathbb{P}$ such that*

$$F^\top P F - P - F^\top P G (G^\top P G)^{-1} G^\top P F \prec 0, \quad (8)$$
$$V_2^\top (F^\top P F - P) V_2 \prec 0, \quad (9)$$

and $M(\rho)$ is related to $\bar{A}(\rho)$ as

$$\bar{M}_1^\top(\rho) = X_4^{-1} M^\top(\rho) X_3 + (V_1^\top Q^{-1} V_1)^{-1} V_1^\top Q^{-1} V_2 V_2^\top F^\top P G X_2^{-1}, \quad (10)$$

$$\bar{A}(\rho) = (X_2^{-1} \bar{M}_1(\rho) + (G^\top P G)^{-1} G^\top P F V_1)(H V_1)^{-1}, \quad (11)$$

with Cholesky decompositions $X_1^\top X_1 = Q$, $X_2^\top X_2 = G^\top PG$, $X_3^\top X_3 = \hat{Q}$ *and* $X_4^\top X_4 = V_1^\top Q^{-1} V_1$, *where* (8)-(9) *guarantee that* $Q \succ 0$, $\hat{Q} \succ 0$ *with*

$$Q = -F^\top PF + P + F^\top PG(G^\top PG)^{-1}G^\top PF, \tag{12}$$

$$\hat{Q} = I - X_2^{-\top} G^\top PFV_2(V_2^\top QV_2)^{-1}V_2^\top F^\top PGX_2^{-1}. \tag{13}$$

The proof of Theorem 2 is provided in Section VII and is based on embedding (1) as $x_{k+1} = (F - G\bar{A}(\rho)H)x_k$ with state x_k as in Definition 1 and proving that $V(x) = x^\top Px$ is a Lyapunov function. Theorem 2 states that if (1) is QS, then there exists a P that satisfies (8)-(9), which are necessary conditions for P to be a Lyapunov function. Given such a P, transforming coefficient functions $\bar{A}(\rho)$ according to (10)-(11) results in transformed coefficient functions $M(\rho)$, which have to reside in the unit ball if (1) is QS. The other way around, which is more interesting for system identification, Theorem 2 states that if a P can be found which satisfies (8)-(9), then all coefficient functions $\bar{A}(\rho)$ that satisfy (5) with this P, i.e., all $\bar{A}(\rho)$ for which this P proves QS, can be constructed from functions $M(\rho)$ contained in the unit ball. This set of functions constrained to the unit ball is easy to describe in an unconstrained fashion, which is exploited for system identification purposes in the next section, coupled with a method to construct a P that satisfies (8)-(9).

IV. Translation To A Stable IO Model Class

Given the equivalence result of Theorem 2 characterizing all coefficient functions $\bar{A}(\rho)$ such that (1) is QS, in this section, a model parameterization is developed that by construction satisfies the conditions of Theorem 2, i.e., a model that is guaranteed to be QS without constraints on the coefficient functions, constituting Contribution C2. This model parametrization can then be used in system identification to ensure stability of the identified model.

A. Reparametrizing All QS Linear IO Systems

To satisfy the conditions of Theorem 2 by construction, it is required to parametrize both an $M(\rho)$ contained in the unit ball as well as a P that satisfies (8)-(9).

First, $M(\rho)$ is reparametrized to be inside the unit ball.

Lemma 3 *Given* $M(\rho) : \mathbb{P} \to \mathbb{R}^{1\times n_a}$, $\|M(\rho)\|_2 < 1\ \forall \rho \in \mathbb{P}$ *if and only if there exist bounded matrix functions* $D(\rho) : \mathbb{P} \to \mathbb{R}$, $Z(\rho) : \mathbb{P} \to \mathbb{R}^{n_a-1}$ *such that*

$$N(\rho) = D^\top(\rho)D(\rho) + Z^\top(\rho)Z(\rho) + \epsilon I, \tag{14}$$

$$M^\top(\rho) = \begin{bmatrix} (I - N(\rho))(I + N(\rho))^{-1} \\ -2Z(\rho)(I + N(\rho))^{-1} \end{bmatrix}. \tag{15}$$

with $0 < \epsilon \ll 1$ *a small positive constant.*

For a proof, see [8, Lemma 1]. By Lemma 3, any $M(\rho)$ contained in the unit ball can be represented by unconstrained $D(\rho), Z(\rho)$. Consequently, $D(\rho), Z(\rho)$ can be chosen as any bounded function, e.g., radial basis functions, a neural network or a Fourier expansion, and $M(\rho)$ constructed as (14)-(15) is guaranteed to be in the unit ball.

Second, it is required to parametrize all P that satisfy (8)-(9). Current methods to obtain P solve (8)-(9) or equivalent conditions in an alternating fashion, but provide no convergence guarantees and are computationally complex [17]. Instead, here all P are parameterized by tracing rays and exploiting that at least one P_0 is known that satisfies (8)-(9).

Define $\Omega = \Omega_1 \cap \Omega_2$ as the set of P satisfying (8)-(9) with

$$\Omega_1 = \{P \succ 0 \mid F^\top PF - P - F^\top PG(G^\top PG)^{-1}G^\top PF \prec 0\},$$
$$\Omega_2 = \{P \succ 0 \mid V_2^\top (F^\top PF - P)V_2 \prec 0\}, \tag{16}$$
$$\Omega_3 = \{P \succ 0 \mid F^\top PF - P \prec 0\}.$$

Lemma 4 $\Omega_3 \subset \Omega_1$ *and* $\Omega_3 \subset \Omega_2$*, giving that* $\Omega_3 \subset \Omega$. *Moreover, there exists a* $P_0 \in \Omega_3$*, thus* Ω *is not empty.*

Proof. Follows as $-F^\top PG(G^\top PG)^{-1}G^\top PF \preceq 0$ and V_2 is full rank. F is stable ($\lambda_i(F) = 0\ \forall i$), i.e., $\exists P_0 \in \Omega_3$. □

In other words, the non-empty set Ω_3 of Lyapunov functions for F is a subset of all P satisfying (8)-(9). Now, denote by $R^\dagger(Q)$ the solution $P \in \mathbb{S}^{n_a}_{\succ 0}$ to Riccati equation (12). Then the following lemma holds.

Lemma 5 *Given a* $P \in \mathbb{S}^{n_a}_{\succ 0}$, $P \in \Omega_1$ *if and only if there exists a* $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ *such that* $P = R^\dagger(Q_-)$.

Proof. (F, G) is controllable, thus for any $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$, (12) has a unique solution $P \in \mathbb{S}^{n_a}_{\succ 0}$ [18], and $P \in \Omega_1$ as $Q_- \succ 0$. Conversely, if $P \in \Omega_1$, $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ by definition. □

By Lemma 5, any $P \in \Omega_1$ can be implicitly represented by some $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ through solving (12), which allows for eliminating constraint (8). However, not all Q_- ensure that $R^\dagger(Q_-) \in \Omega_2$. Yet, a point Q_0 for which $P_0 = R^\dagger(Q_0) \in \Omega$ is available by Lemma 4, such that any $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ can be projected back into the direction of Q_0 to eventually obtain a Q for which $R^\dagger(Q) \in \Omega_2$, as detailed next.

Lemma 6 *Given a* Q_0 *with* $R^\dagger(Q_0) \in \Omega$*, then for any* $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ *there exists a* $\mu \in \mathbb{R}_{(0,1]}$ *such that* $R^\dagger(Q) \in \Omega$ *with*

$$Q = \mu Q_- + (1 - \mu)Q_0. \tag{17}$$

Proof. If $R^\dagger(Q_-) \in \Omega_2$, simply set $\mu = 1$, $Q = Q_-$. If $R^\dagger(Q_-) \notin \Omega_2$, continuity ensures that there exists a small enough $\mu > 0$ such that $R^\dagger(\mu Q_- + (1 - \mu)Q_0) \in \Omega$. Specifically, by continuity of the solution $R^\dagger(Q)$ [19], continuity of eigenvalues and strictness of $F^\top R^\dagger(Q_0)F - R^\dagger(Q_0) \prec 0$, there exists a ball $\mathbb{B}_\epsilon = \{Q_0 + \Delta \mid \|\Delta\|_2 < \epsilon\}$ such that for any $Q \in \mathbb{B}_\epsilon$, also $R^\dagger(Q) \in \Omega$. Then any $\mu > 0$ such that $\|\mu(Q_- - Q_0)\|_2 < \epsilon$ results in $R^\dagger(\mu Q_- + (1 - \mu)Q_0) \in \Omega$. □

Now, by varying Q_- over $\mathbb{S}^{n_a}_{\succ 0}$, all $Q \in \{Q \succ 0 \mid R^\dagger(Q) \in \Omega_2\} \subset \mathbb{S}^{n_a}_{\succ 0}$ can be covered, as the latter is simply a subset of $\mathbb{S}^{n_a}_{\succ 0}$. Specifically, if already $R^\dagger(Q_-) \in \Omega_2$ simply set $\mu = 1$ to obtain $Q = Q_-$. If $R^\dagger(Q_-) \notin \Omega_2$, Lemma 6 guarantees that a μ can be obtained such that $R^\dagger(\mu Q + (1 - \mu)Q_0) \in \Omega$.

In summary, the above lemmas guarantee that any $P \in \Omega$, i.e., any P satisfying (8)-(9), can be represented through a $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$, Q_0 and $\mu \in \mathbb{R}_{(0,1]}$. Translating this to system identification, these lemmas enable optimization over $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ combined with a simple bisection which is guaranteed to converge, as opposed to solving (8)-(9) in an alternating fashion. Last, $Q_- \in \mathbb{S}^{n_a}_{\succ 0}$ is ensured by parameterizing $Q_- = X_5^\top X_5$ with X_5 full rank.

B. A Stable Linear IO Model Class

Above lemmas enable the use of Theorem 2 in system identification. Specifically, they allow for representing $\bar{A}(\rho)$ in terms of free bounded functions $D(\rho), Z(\rho)$ and full rank X_5 that construct $\bar{A}(\rho), P$ such that (5) is satisfied, i.e., such that (1) with $\bar{A}(\rho)$ is guaranteed to be stable. This construction is summarized in the next algorithm.

Algorithm 7 Stable structured LPV-IO model.

inputs: Structure matrices H, H_b, full rank matrix $X_5 \in \mathbb{R}^{n_a \times n_a}$, a $Q_0 \in \mathbb{S}^{n_a}_{\succ 0}$ such that $F^\top R^\dagger(Q_0)F - R^\dagger(Q_0) \prec 0$, and functions $D : \mathbb{P} \to \mathbb{R}$, $Z : \mathbb{P} \to \mathbb{R}^{n_a-1}$, $\bar{B}(\rho_k) : \mathbb{P} \to \mathbb{R}^{1\times n_l}$, and data $\{u_k, \rho_k\}_{k=1}^N$.
calculate $Q_- = X_5^\top X_5 \in \mathbb{S}^{n_a}_{\succ 0}$.
if $P_- = R^\dagger(Q_-) \in \mathbb{S}^{n_a}_{\succ 0}$ satisfies (9), set $P \leftarrow P_-$.
else bisect $\mu \in \mathbb{R}_{(0,1]}$ such that $P_- = R^\dagger(\mu Q_- + (1-\mu)Q_0)$ satisfies (9) and set $P \leftarrow P_-$.
calculate X_1, X_2, X_3, X_4 according to Theorem 2.
for $k \in \mathbb{Z}_{[1,N]}$ **do**
  calculate $M(\rho_k)$ according to (14)-(15).
  calculate $\bar{A}(\rho_k)$ according to (10)-(11).
  calculate $A(\rho_k) = \bar{A}(\rho_k)H$, $B(\rho_k) = \bar{B}(\rho_k)H_b$.
  calculate y_k according to (1).
end for

The functional parameterizations for $D, Z, \bar{B}$ in Algorithm 7 are intentionally left free, and can be chosen arbitrarily, e.g., as a neural network, while QS of the model is guaranteed by Theorem 2. Any structured QS IO system can then be represented up to the approximation capabilities of the functional parametrization of $D, Z, \bar{B}$. Naturally, a too limited functional parameterization can introduce structural bias in the representation of a specific system.

Since all these transformations have well-defined gradients, this reparameterization can be used both in curve fitting algorithms using frequency domain data [20] as well as in prediction error methods using time domain data [1].

V. System Identification Example

In this section, the developed stable structured LPV-IO model is used to identify the parameter-varying damping of a mass-damper-spring system with known mass and stiffness, resulting in a structured identification setting. This system is illustrative of, e.g., varying damping due to configuration-dependent contact forces. Whereas the developed model is guaranteed to be stable, a baseline model results in unstable estimates due to the effects of measurement noise[1].

The data-generating system $\mathcal{G}$ is given by (1) with

$$\begin{aligned} A(\rho_k) &= \begin{bmatrix}-2 & 1+km^{-1}T_s^2\end{bmatrix} + d(\rho_k)m^{-1}T_s\begin{bmatrix}1 & -1\end{bmatrix} \\ B(\rho_k) &= \begin{bmatrix}0 & 0 & m^{-1}T_s^2\end{bmatrix} \\ \tilde{y}_k &= y_k + v_k, \end{aligned} \tag{18}$$

with $k = 10^3$, $m = 0.2$, $T_s = 0.05$, $\rho \in \mathbb{R}_{[0,1]} = \mathbb{P}$ and $d(\rho) = 6 + 4\mathrm{e}^{-10\rho^2}$, resulting in frozen LTI dynamics as shown in Fig. 1. These parameters have been chosen such that the system is at the boundary of stability. Furthermore, $\tilde{y}_k \in \mathbb{R}$ is a measurement of the true output $y_k \in \mathbb{R}$ perturbed by zero-mean i.i.d. white noise v_k with $\mathbb{E}(v_k^2) = \sigma_v^2$, resulting in an output-error (OE) identification setup.

[1]The code for this example and other more general LTI/LPV examples are available at `https://gitlab.tue.nl/kon/stable-io-identification/-/tree/main/Structured`

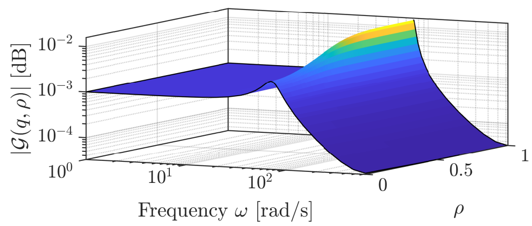

Fig. 1. Bode plot of the frozen LPV dynamics, i.e., the frequency response of the LPV dynamics for constant scheduling ρ.

Note that, instead of a linear constraint as in (4), the true system (18) contains an affine structural constraint $A(\rho) = h + d(\rho)H$ with $h = \begin{bmatrix}-2 & 1+km^{-1}T_s^2\end{bmatrix}$ and $H = m^{-1}T_s\begin{bmatrix}1 & -1\end{bmatrix}$. To convert this constraint to a linear one, F is redefined as $F \leftarrow F + G(h + d_0H)$, such that it again holds that $x_{k+1} = Fx_k + Gd_s(\rho_k)H$ with $d_s(\rho_k) = d(\rho_k) - d_0$. For $d_0 > 5$, F is stable, giving that Lemma 4-6 are applicable.

A dataset $\mathcal{D} = \{u_k, \rho_k, \tilde{y}_k\}_{k=1}^N$ of length $N = 1000$ is generated with $u_k = \sum_{i=\ell}^{10} \sin(2\pi\frac{\ell}{20}t)$ and $\rho_k = 1 - kN^{-1} + w_k$, where w_k is zero-mean white noise with $\mathbb{E}(w_k^2) = 10^{-2}$. The noise variance is set to $\sigma_v^2 = 3.185$, resulting in a signal-to-noise ratio of $10\log_{10}\|y\|_{\ell_2}^2/\|v\|_{\ell_2}^2 = 6$ dB.

Given this data, two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are identified. $\mathcal{M}_1$ is a standard structured LPV-IO model that directly parametrizes $\bar{A}(\rho) = \sum_{i=0}^5 \theta_i\rho^i$, i.e., as a 5^{th} order polynomial [21] with parameters $\theta = [\theta_0, \ldots, \theta_5]$. $\mathcal{M}_2$ is the developed stable structured LPV-IO model with $D(\rho) = \sum_{i=0}^5 \psi_i\rho^i$ and $X_5 \in \mathbb{R}^{2\times 2}$ as an upper triangular matrix, resulting in parameters $\psi = [\psi_0, \ldots, \psi_5, \mathrm{vec}(X_5)] \in \mathbb{R}^9$. Both models have full knowledge of k, m, T_s, such that H is fully known. For $\mathcal{M}_2$, $Q_0 \in \mathbb{S}^{n_a}_{\succ 0}$ is initialized by evaluating (12) for $P_0 \in \mathbb{S}^{n_a}_{\succ 0}$ satisfying $F^\top P_0 F - P_0 = -I$. Note that $\mathcal{M}_2$ does not need $Z(\rho)$ as $\bar{A}(\rho_k) \in \mathbb{R}$.

Both models are identified using prediction-error based minimization based on gradient-based optimization [1], which in the OE setting corresponds to minimizing the ℓ_2 loss of the simulation error, i.e. for $\mathcal{M}_2$ according to

$$V_N(\psi) = \frac{1}{N}\sum_{k=1}^N (y_k - \mathcal{M}_2(u_k;\psi))^2, \tag{19}$$

with $\mathcal{M}_2(u_k;\psi)$ the simulated model response of $\mathcal{M}_2$ to u_k, and similarly for $\mathcal{M}_1$. Criterion (19) is minimized in MATLAB using the Levenberg-Marquardt algorithm [22].

Fig. 2 shows the true and estimated parameter-varying damping for both models, together with the boundary for d that results in stable frozen LTI dynamics. Note that this boundary is not at $d = 0$ due to the Euler discretization. It is observed that for $p \in [0, 0.03] \subset \mathbb{P}$, $\mathcal{M}_1$ results in unstable frozen LTI dynamics, even though the true system is quadratically stable. This is a consequence of the measurement noise and finite-data effects. In contrast, $\mathcal{M}_2$ is guaranteed to be quadratically stable, and consequently it results in stable frozen LTI dynamics for any choice of ρ. In terms

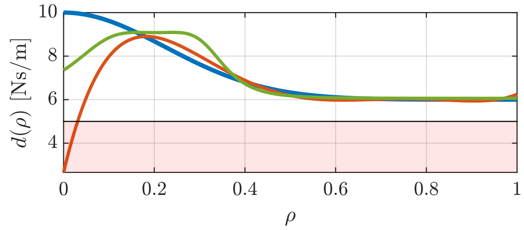

Fig. 2. The true parameter-varying damping $d(\rho)$ (—) is accurately described by the stable LPV-IO model (—) while stability is guaranteed by construction. As a consequence, its coefficient function is guaranteed to result in a stable LTI system for each constant ρ, i.e., it stays outside the region for which $d(\rho)$ results in an unstable LTI model (▭). In contrast, due to the effects of measurement noise, the standard LPV-IO model (—) is unstable: for constant $\rho \in [0, 0.03] \subset \mathbb{P}$ it results in unstable LTI models.

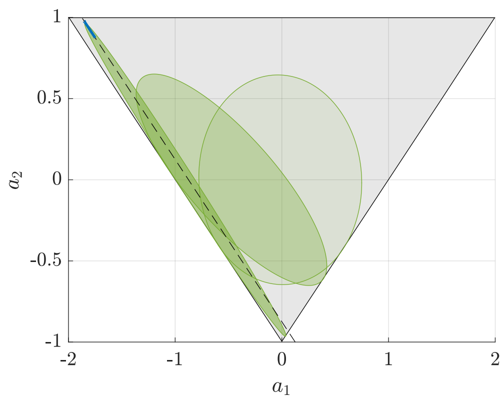

Fig. 3. Coefficient set $\mathcal{A}_P = \{A = [a_1 \quad a_2] \mid (F - GA)^\top P(F - GA) - P \prec 0\}$, i.e., the set of unstructured coefficients for which P proves stability at step 1 (⬭), 10 (⬭) and 23 (⬭) of the optimization. At every step, the intersection of $\mathcal{A}_P$ with the affine structure set (---) is not empty, i.e., the structured coefficient set $\bar{\mathcal{A}}_P = \{\bar{A} = d \mid (F - G(h + \bar{A}H))^\top P(F - G(h + \bar{A}H)) - P \prec 0\}$ is not empty. During identification, P is optimized in such a way that the true system with $A(\rho)$ as in (18) (—) is contained in $\mathcal{A}_P$. Each $\mathcal{A}_P$ is contained within $\mathcal{A}$ (▽), i.e., the set of unstructured coefficients that result in a stable system in the LTI case.

of cost function, the models achieve $V_N(\theta^*) = 3.218$ and $V_N(\psi^*) = 3.199$ on the training dataset, which is very close to the noise variance $\sigma_v^2 = 3.185$. However, on a validation dataset, $\mathcal{M}_1$ only obtains $V_N(\theta^*) = 55.78$ as its predictions diverge for the validation scheduling trajectory due to the instability, while $\mathcal{M}_2$ still achieves $V_N(\psi^*) = 3.220$.

Taking a closer look into the stability set, Fig. 3 shows all unstructured coefficients a_1, a_2 for which P at the current iteration of the optimization of (19) proves stability. By varying P, the complete set of structured coefficients that result in a QS IO model can be represented.

VI. Conclusion

In this paper, a linear IO model class is developed that is both guaranteed to be stable by construction and allows for a user-specified linear structure constraint on the coefficient functions. This stable structured linear IO model class allows for identifying stable models without enforcing LMI conditions during optimization or projecting the identified model onto the set of stable models afterwards. It can be used in any gradient-based system identification procedure since all operations in the model have well defined gradients, e.g., it can be used to fit frequency response measurement data using a curve-fitting algorithm or to fit time domain data using prediction-error methods as is illustrated in the paper.

The results derived in this paper straightforwardly generalize towards the multi-input-multi-output and state-space case as long as the structure specification remains unchanged. An important future direction is to extend the results towards less conservative stability criteria [15].

VII. Proof of Theorem 2

Before considering the proof, first note that the evolution of (1) for $u_k = 0 \; \forall k \in \mathbb{Z}_{\geq 0}$ can equivalently be expressed as

$$x_{k+1} = \begin{bmatrix} -a_1(\rho_k) & \dots & -a_{n_a-1}(\rho_k) & -a_{n_a}(\rho_k) \\ 1 & & & 0 \\ & \ddots & & 0 \\ & & 1 & 0 \end{bmatrix} x_k$$
$$= (F - GA(\rho_k))x_k = (F - G\bar{A}(\rho_k)H)x_k, \tag{20}$$

see also [9], with $x_k = \begin{bmatrix} y_{k-1} & y_{k-2} & \dots & y_{k-n_a} \end{bmatrix}^\top \in \mathbb{R}^{n_a}$. Consequently, (1) is QS, see Definition 1, if and only if

$$(F - G\bar{A}(\rho_k)H)^\top P(F - G\bar{A}(\rho_k)H) - P \prec 0 \;\; \forall \rho_k \in \mathbb{P}. \tag{21}$$

Necessity: Given an $\bar{A}(\rho)H = A(\rho)$ such that (1) is QS, i.e., (21) holds, it is shown that 1) P satisfies (8)-(9), 2) that (8)-(9) imply that also Q, $\hat{Q}$ in (12)-(13) are positive definite, 3) that there exists an $\bar{M}_1(\rho)$ related to $\bar{A}(\rho)$ as in (11), and 4) that there exists an $M(\rho)$ related to $\bar{M}_1(\rho)$ as in (10).

1) Necessary conditions on P: First, since H has a kernel, simply projecting (21) onto the basis for this kernel V_2 gives (9). Second, note that G is full rank and thus $G^\top PG$ is invertible. Then, completing the squares in (21) gives

$$(F - G\bar{A}(\rho_k)H)^\top P(F - G\bar{A}(\rho_k)H) - P = F^\top PF - P$$
$$- F^\top PG(G^\top PG)^{-1}G^\top PF + R^\top(\rho_k)G^\top PGR(\rho_k), \tag{22}$$

with $R(\rho_k) = \bar{A}(\rho_k)H - (G^\top PG)^{-1}G^\top PF$. Now note that $R(\rho_k)^\top G^\top PGR(\rho_k) \succeq 0$, thus (8) must hold if (21) holds.

2) Positive definiteness of Q,$\hat{Q}$ in (12)-(13): Given a P that satisfies (8)-(9), Q as defined in (12) is trivially positive definite since P satisfies (8). To prove that $\hat{Q} \succ 0$, note that $V_2^\top(F^\top PF - P)V_2 \prec 0$ and (8) imply that

$$V_2^\top QV_2 - V_2^\top F^\top PG(G^\top PG)^{-1}G^\top PFV_2 \succ 0. \tag{23}$$

Now, it holds that $G^\top PG \succ 0$ and $V_2^\top QV_2 \succ 0$ such that taking the Schur complement of (23) implies

$$G^\top PG - G^\top PFV_2(V_2^\top QV_2)^{-1}V_2^\top F^\top PG \succ 0. \tag{24}$$

Then a congruence with X_2^{-1} gives $\hat{Q} \succ 0$.

3) Relation between $\bar{A}(\rho)$ and $\bar{M}_1(\rho)$: Since $Q \succ 0$, it has a factorization $Q = X_1^\top X_1$. Similarly $G^\top PG = X_2^\top X_2$. Then a congruence of (22) with X_1^{-1} gives that

$$I - X_1^{-\top}\left(\bar{A}(\rho_k)H - (G^\top PG)^{-1}G^\top PF\right)^\top X_2^\top X_2 \tag{25}$$
$$\left(\bar{A}(\rho_k)H - (G^\top PG)^{-1}G^\top PF\right)X_1^{-1} \succ 0 \quad \forall \rho_k \in \mathbb{P},$$

or equivalently $I - \hat{M}^\top(\rho_k)\hat{M}(\rho_k) \succ 0 \; \forall \rho_k \in \mathbb{P}$ by defining

$$\hat{M}(\rho) = X_2\left(\bar{A}(\rho)H - (G^\top PG)^{-1}G^\top PF\right)X_1^{-1}. \tag{26}$$

In other words, if (1) with $A(\rho) = \bar{A}(\rho)H$ is QS, there exists an $\hat{M}(\rho)$ that is contained in the unit ball. Additionally, $\hat{M}(\rho)$ is structured. Specifically, a multiplication of (26) with $X_1 \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ reveals

$$\begin{aligned}\hat{M}(\rho)X_1 \begin{bmatrix} V_1 & V_2 \end{bmatrix} =& X_2 \begin{bmatrix} \bar{A}(\rho)HV_1 & 0 \end{bmatrix} \\ &- X_2^{-\top} G^\top PF \begin{bmatrix} V_1 & V_2 \end{bmatrix}.\end{aligned} \tag{27}$$

where it has been used that $(G^\top PG)^{-1} = X_2^{-1}X_2^{-\top}$. Now express $\hat{M}(\rho)$ in new coordinates $\bar{M}(\rho)$ as

$$\hat{M}(\rho) = \begin{bmatrix} \bar{M}_1(\rho) & \bar{M}_2(\rho) \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^\top X_1^{-1}, \tag{28}$$

such that (27) results in

$$\bar{M}_2(\rho) = \bar{M}_2 = -X_2^{-\top} G^\top PFV_2, \tag{29}$$

$$\bar{M}_1(\rho) = X_2\bar{A}(\rho)HV_1 - X_2^{-\top}G^\top PFV_1. \tag{30}$$

Now, since HV_1 is invertible, (30) uniquely determines the relationship between $\bar{M}_1(\rho)$ and $\bar{A}(\rho)$, which is equivalent to (11). Additionally, (29) states that $\bar{M}_2(\rho)$ is ρ-independent and only determined by P to adhere to the structure specification H, and is thus denoted explicitly as $\bar{M}_2$.

4) Relation between $\bar{M}_1(\rho)$ and $M(\rho)$: Now $\hat{M}(\rho) = (\bar{M}_1(\rho)V_1 + \bar{M}_2V_2)X_1^{-1}$ is structured with $\bar{M}_2$ fixed as in (29), and $\hat{M}(\rho)$ satisfies $I - \hat{M}^\top(\rho_k)\hat{M}(\rho_k) \succ 0 \; \forall \rho_k \in \mathbb{P}$, or $I - \hat{M}(\rho_k)\hat{M}^\top(\rho_k) \succ 0 \; \forall \rho_k \in \mathbb{P}$. Combining the two gives

$$\begin{aligned} I - (\bar{M}_1(\rho_k)V_1^\top + \bar{M}_2V_2^\top)X_1^{-1} \\ X_1^{-\top}(V_1\bar{M}_1^\top(\rho_k) + V_2\bar{M}_2^\top) \succ 0 \quad \forall \rho_k \in \mathbb{P}. \end{aligned} \tag{31}$$

Expanding this expression and completing the squares gives

$$\begin{aligned} I - \bar{M}_2V_2^\top(Q^{-1} - Q^{-1}V_1(V_1^\top Q^{-1}V_1)^{-1}V_1^\top Q^{-1})V_2\bar{M}_2^\top \\ - T^\top(\rho_k)V_1^\top Q^{-1}V_1T(\rho) \succ 0 \quad \forall \rho_k \in \mathbb{P}, \end{aligned} \tag{32}$$

$$T(\rho) = \bar{M}_1^\top(\rho) + (V_1^\top Q^{-1}V_1)^{-1}V_1^\top Q^{-1}V_2\bar{M}_2^\top. \tag{33}$$

Now first note that with $Q = X_1^\top X_1$, it holds that

$$\begin{aligned} &Q^{-1} - Q^{-1}V_1(V_1^\top Q^{-1}V_1)^{-1}V_1^\top Q^{-1} \\ &= X_1^{-1}(I - X_1^{-\top}V_1(V_1^\top X_1^{-1}X_1^{-\top}V_1)^{-1}V_1^\top X_1^{-1})X_1^{-\top} \\ &= X_1^{-1}(X_1V_2(V_2^\top X_1^\top X_1V_2)^{-1}V_2^\top X_1^\top)X_1^{-\top} \\ &= V_2(V_2^\top QV_2)^{-1}V_2, \end{aligned} \tag{34}$$

where the second identity follows by recognizing that $I - X_1^{-\top}V_1(V_1^\top X_1^{-1}X_1^{-\top}V_1)^{-1}V_1^\top X_1^{-1}$ is a projection operator onto the space complementary to Im $X_1^{-\top}V_1$, i.e., onto Im X_1V_2 since $(X_1V_2)^\top X_1^{-\top}V_1 = 0$. Now substitute (34) and (29) into (32) and note that $V_2^\top V_2 = I$ to obtain

$$\begin{aligned} I - X_2^{-\top}G^\top PAV_2(V_2^\top QV_2)^{-1}V_2^\top A^\top PGV_2X_2^{-1} \\ - T^\top(\rho_k)V_1^\top Q^{-1}V_1T(\rho_k) \succ 0 \quad \forall \rho_k \in \mathbb{P}, \end{aligned} \tag{35}$$

where the first line can be recognized as $\hat{Q}$. Now $T^\top(\rho_k)V_1^\top Q^{-1}V_1T(\rho_k) \succeq 0 \; \forall \rho_k \in \mathbb{P}$, such that $\hat{Q}$ is positive definite, which was already shown to be equivalent to (9) under (8). Now decompose $\hat{Q} = X_3^\top X_3$ and $V_1^\top Q^{-1}V_1 = X_4^\top X_4$. A congruence of (35) with X_3^{-1} results in

$$I - M(\rho_k)M^\top(\rho_k) \succ 0 \quad \forall \rho_k \in \mathbb{P}, \tag{36}$$

or $\|M(\rho_k)\|_2 < 1 \; \forall \rho_k \in \mathbb{P}$ where $M(\rho)$ is given by

$$M^\top(\rho) = X_4T(\rho)X_3^{-1}, \tag{37}$$

which, with $T(\rho)$ in (33) and $\bar{M}_2$ in (29), equals (10) .

Sufficiency: follows by following the same arguments in reverse order. Specifically, if P satisfies (8)-(9), then by step 2) $\hat{Q} \succ 0$. If $M(\rho)$ is such that $\|M(\rho_k)\|_2 < 1 \; \forall \rho_k \in \mathbb{P}$, then by constructing $\bar{M}_1(\rho)$ as in (10), or equivalently as in (37), gives that (35) and thus (32) and (31) are satisfied. Now construct $\bar{A}(\rho)$ according to (11), or equivalently according to (30). Then, by construction of $\bar{A}(\rho)$ and since (31) is satisfied, (25) is also satisfied, which implies (21). In other words, P proves QS for the constructed $\bar{A}(\rho)$.

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