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QSR-dissipativity-based stabilization of non-passive nonlinear discrete-time systems by linear static output feedback

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Abstract— **In this technical note, we study the relations between the local stabilizability of a class of input-affine discrete-time nonlinear systems and their local *Quadratic-Supply-Rate (QSR)-dissipativity* properties. Focusing on stabilizability by *linear* Static Output Feedback (SOF), we derive several sufficient conditions for Lyapunov stabilizability based on QSR-dissipativity. A closed-form expression for the SOF stabilizing gain is derived from the QSR matrices. Additionally, we prove that necessity also holds in some special cases. The QSR-dissipativity-based conditions provide an alternative to the traditional Passivity-Based Control (PBC) by allowing for a more general input-output behavior, i.e., non-passive dynamics. Numerical examples illustrate their applicability for designing stabilizing controllers for open-loop unstable systems.**

I. Introduction

Nonlinear difference equations are used to model a wide range of dynamical systems, including the logistic map in population dynamics, the Hénon map in chaos theory, and epidemic models, making them of great theoretical importance [1], [2]. Unlike their continuous-time counterparts, the control of *nonlinear* discrete-time systems has not been extensively studied. Developing effective control strategies for discrete-time systems is essential due to the increasing trend toward digital controller implementation. However, the analysis of system properties in the discrete-time setting introduces unique technicalities and intricacies, distinguishing it from the continuous-time setting [3], [4, Chs. 13-14]. Some of the classical techniques for controlling nonlinear discrete-time systems are based on passivity and lossless properties of the open system. However, in general, these techniques cannot be directly applied to non-passive systems. In these cases, when the system is feedback equivalent to a passive one, a first controller is designed to "*passivate*" the system. Seminal works by Byrnes and co-authors [5]–[7] discuss this approach in depth. For a recent work on the Passivity-Based Control (PBC) approach, see [8].

To overcome certain limitations of PBC, researchers have explored the regulation of discrete-time nonlinear systems through broader concepts of dissipativity. This includes the application of general Quadratic Supply Rate (QSR) functions, known as QSR-dissipativity. For instance, [9] demonstrates how to optimally control discrete-time nonlinear systems by leveraging more comprehensive dissipativity concepts. This approach uses nonlinear regulators designed to optimize specific performance criteria, resulting in nonlinear state feedback that depends on the accurate knowledge of the system's state. The case of finite-time stability and stabilization of discrete-time nonlinear systems using optimal feedback has also been recently developed [10], [11]. Another recent development is the concept of Control Dissipation Functions (CDF), presented in [12], which enables the design of stabilizing receding horizon controllers by minimizing CDFs subject to a cyclically negative supply condition.

Recently, the stabilizability of continuous-time nonlinear systems by *linear* Static Output Feedback (SOF) has been studied, resulting in the proof of the equivalence between the existence of a *linear* SOF rendering the system locally exponentially stable and the exponential QSR-dissipativity of the open-loop system [13] under some conditions on the QSR matrices, leading to a stabilizing controller.

In this work, we extend our focus to discrete-time nonlinear systems. To address this challenge, we present a collection of theoretical conditions establishing the links between the *local* stabilizability of nonlinear discrete-time systems by *linear* SOF and their QSR-dissipativity properties. More specifically, we developed new sufficient conditions for the local asymptotic and geometric stabilizability by *linear* SOF based on local QSR-dissipativity properties of the open-loop system. The conditions are based on new inequalities involving the QSR matrices and the storage function. When the dissipativity-based conditions hold, a stabilizing gain derived from the QSR matrices ensures the stabilization of the system in closed loop. Additionally, we characterize a set of conditions where *necessity* also holds. Our work offers, thus, new insights and advancements in developing control strategies for nonlinear discrete-time systems.

Transitioning to the discrete-time setting introduces new significant challenges in comparison with [13]. For the class of continuous-time nonlinear systems considered in [13], necessary and sufficient conditions for dissipativity which are affine in the control input u exist, stemming from the derivative of the storage function, expressed as $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}(f(x) + g(x)u)$. In contrast, in the discrete-time setting considered on this note, the forward-difference relation $V(f(x) + g(x)u) - V(x)$ introduces sufficient conditions for dissipativity that generally involve non-explicit, more complex interactions with the control input u. This difference fundamentally affects the analytical approach, as it requires handling non-linear dependencies not present in continuous-time formulations, thereby complicating both the theoretical

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analysis and practical applications. The uniqueness of the challenges in both discrete and continuous-time also imply slightly different results: In the continuous-time case [13] necessary and sufficient stabilizability conditions are obtained for general $\mathcal{C}^1$ storage functions, while herein necessity holds under a quadratic restriction on the storage only. Additionally, the key stabilizing Δ-*condition* (13) in this paper is different from the one obtained for continuous time [13, eq. (14)]. This further underscores the specific differences in the results and the development of dissipativity-based dissipativity-based stabilization of discrete-time systems.

This paper is organized as follows: Section II introduces the considered class of input-affine nonlinear discrete-time systems and provides theoretical preliminaries on Lyapunov and dissipativity theory for nonlinear discrete-time systems. In Section III, we present the main results of our work, which establish conditions linking the local stabilization of such systems by linear SOF and their QSR-dissipativity properties. In Section IV, we showcase the practical applicability of the developed conditions with numerical examples. Finally, in Section V, we discuss our findings and outline future directions for research in this field.

Notation. $f : \mathcal{X} \to \mathcal{Y}$ is a (vector) function with domain $\mathcal{X}$ and codomain $\mathcal{Y}$. $\mathcal{X} \times \mathcal{Y}$ is the Cartesian product of sets $\mathcal{X}$ and $\mathcal{Y}$. The notation $\overset{o}{\mathcal{X}}$ is used to denote the interior of set a $\mathcal{X}$. $\mathcal{C}^2$ denotes the class of twice-continuously-differentiable functions. $\|\cdot\|$ is the Euclidean norm of a vector. $\mathbb{S}_n^+$ and $\mathbb{S}_n$ stand for the set of symmetric positive definite matrices and symmetric positive semidefinite matrices, respectively. For a matrix M of dimension $n \times n$, $M \succ 0$, $M \prec 0$, $M \preceq 0$ stand respectively for $M \in \mathbb{S}_n^+$, $-M \in \mathbb{S}_n^+$ and $-M \in \mathbb{S}_n$. For a matrix or vector, $(\cdot)^\top$ denotes its transpose. In a block matrix, $\star$ stands for a symmetric block. Finally, $\mathbb{R}^+$, $\mathbb{R}_0^+$, and $\mathbb{N}$ stand for the sets of positive real numbers, nonnegative real numbers, and natural numbers (including 0), respectively.

II. Preliminaries

A. System description

Consider the following nonlinear discrete-time system

$$\begin{cases} x(k+1) = f(x(k)) + g(x(k))u(k), \\ y(k) = h(x(k)), \end{cases} \tag{1}$$

defined for $k \in \mathbb{N}$, where $x(k) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, $\mathcal{X}$ is an open set with $0 \in \mathcal{X}$. The control input $u(k) \in \mathcal{U} \subseteq \mathbb{R}^m$ for all $k \in \mathbb{N}$ is such that $0 \in \mathcal{U}$. Moreover, $f : \mathcal{X} \to \mathbb{R}^n$, $g : \mathcal{X} \to \mathbb{R}^{n\times m}$, are such that $(f, g) \in \mathcal{C}^2$, with $f(0) = 0$. Therefore, the resulting mapping $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$, $F(x(k), u(k)) = f(x(k)) + g(x(k))u(k)$, satisfies $F(0, 0) = 0$, that is, the origin $(x(k), u(k)) \equiv (0, 0)$ is an equilibrium point of (1), where $F(\cdot, \cdot)$ is $\mathcal{C}^2$ in a neighborhood $\mathcal{X} \times \mathcal{U}$ of the origin. Additionally, the output $y(k) \in \mathcal{Y} \subset \mathbb{R}^p$, with $h : \mathcal{X} \to \mathbb{R}^p$, $h(0) = 0$ and $h \in \mathcal{C}^2$, is available for feedback.

In this work, we are interested in investigating the stabilizability of (1) subject to a linear static output feedback (SOF) control law

$$u(x) = \phi(x(k)) = Ky(k) = Kh(x(k)), \tag{2}$$

where $K \in \mathbb{R}^{m\times p}$ is a matrix gain to be designed. The interconnection (1)-(2) generates the closed-loop system

$$x(k+1) = f(x(k)) + g(x(k))Kh(x(k)) = \overline{F}(x(k)), \tag{3}$$

where $\overline{F} : \mathcal{X} \to \mathbb{R}^n$, with $\overline{F}(x) = F(x, Kh(x))$, is such that $\overline{F} \in \mathcal{C}^2$, so that existence and uniqueness properties are straightforward and can be established by interactively constructing a *solution sequence* $x(k) = s(k, x_0)$ to (3), which is uniquely defined for a given initial condition $x(0) = x_0$ (see, for example, [4, p. 764]).

B. Preliminaries on stability/stabilizability notions

In this subsection, we present some notions regarding the stability and stabilizability of the closed-loop system (3), i.e., of the interconnection (1)-(2). For completeness of the presentation, the following definition is recalled.

Definition 1: (Stability notions) [4, Definition 13.1, p. 765]- [2, Definition 4.2, p. 176] The zero solution $x(k) \equiv 0$ to (3) is asymptotically stable if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{k\to\infty} x(k) = 0$. Furthermore, it is geometrically (also known as exponentially) stable if there exist positive constants α, $\beta > 1$, and δ such that if $\|x(0)\| < \delta$, then $\|x(k)\| \leq \alpha \|x(0)\| \beta^{-k}$, $k \in \mathbb{N}$.

We employ the following notion of stabilizability.

Definition 2: (Stabilizability by SOF) The system (1) is SOF asymptotically (respectively, geometrically) stabilizable if there exists gain $K \in \mathbb{R}^{m\times p}$ such that the zero solution $x(k) \equiv 0$ to (3) is rendered asymptotically (respectively, geometrically) stable.

Concerning the stabilizability notion from Definition 2, we state the following proposition.

Proposition 1: System (1) is asymptotically stabilizable in the sense of Definition 2 if and only if there exist a SOF gain $K \in \mathbb{R}^{m\times p}$, a set $\mathcal{X}_0 \subseteq \mathcal{X}$, with $0 \in \overset{o}{\mathcal{X}}_0$, and a $\mathcal{C}^2$ positive-definite function $V : \mathcal{X}_0 \to \mathbb{R}_0^+$ such that

$$V(x) > 0, \ x \in \mathcal{X}_0, \ x \neq 0, \tag{4a}$$

$$V(\overline{F}(x)) - V(x) < 0, \ x \in \mathcal{X}_0, \ x \neq 0. \tag{4b}$$

Furthermore, it is geometrically stabilizable if and only if there exist a SOF gain K, a set $\mathcal{X}_0 \subseteq \mathcal{X}$, with $0 \in \overset{o}{\mathcal{X}}_0$, a $\mathcal{C}^2$ positive-definite function $V : \mathcal{X}_0 \to \mathbb{R}_0^+$, and scalars c_1, c_2, $c_2 \geq c_1 > 0$, and $\rho > 1$ such that

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2, \ x \in \mathcal{X}_0, \tag{5a}$$

$$V(\overline{F}(x)) - \frac{1}{\rho} V(x) \leq 0, \ x \in \mathcal{X}_0. \tag{5b}$$

Remark 1: The proof of Proposition 1 follows directly from the application of Lyapunov stability arguments. See [4, Theorem 13.2, p. 765] for sufficiency, and the converse Lyapunov results in [4, Theorem 13.6, p. 772] and [4, Theorem 13.7, p. 774-775] for necessity; In [4, Theorem 13.7, p. 774-775], the proof follows by constructing a Lyapunov function that depends on the solution of the system and using the bounds in Definition 1 by assumption. For Proposition 1, the same steps apply, *mutatis-mutandis*, and since the closed-loop function $\overline{F} \in \mathcal{C}^2$, twice-continuously-differentiable Lyapunov functions V are constructed from the system's solution. Furthermore, the quadratic forms bounding the Lyapunov function $V(x)$ can be assumed without loss of generality due to [4, Corollary 13.2, p. 775].

Next, we present preliminaries on dissipativity theory.

C. Dissipativity notions

A discrete-time nonlinear dynamical system (1) with state $x(k) \in \mathcal{X} \subseteq \mathbb{R}^n$ is *locally dissipative with respect to a given supply rate* $r(k) = r(u(k), y(k))$ if there exists a continuous nonnegative storage function $V(x) : \mathcal{X} \to \mathbb{R}_0^+$ such that for all $u(k) \in \mathcal{U}$ and all $k \in \mathbb{N}$ the relation

$$V(F(x(k), u(k))) - V(x(k)) \leq r(u(k), y(k))$$

holds. If $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, and $\mathcal{Y} = \mathbb{R}^p$, the system is said to be simply *dissipative with respect to* r [6].

This work employs the definition of strict QSR-dissipativity below.

Definition 3: System (1) is locally *strictly QSR-dissipative* if there exists a nonnegative storage function V and a positive definite function T, such that the dissipation inequality

$$V(F(x(k), u(k))) - V(x(k)) + T(x(k)) \leq r(u(k), y(k)), \quad (6)$$

with supply rate

$$r(u, y) = y^\top Q y + 2y^\top S u + u^\top R u, \quad (7)$$

holds along all possible trajectories of (1) starting at $x(0)$, for all $k \in \mathbb{N}$, and for all $u \in \mathcal{U}$, where $S \in \mathbb{R}^{p \times m}$, and matrices $Q \in \mathbb{R}^{p \times p}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric.

The following definition applies to geometric dissipativity.

Definition 4: System (1) is locally *geometrically QSR-dissipative* if (6)-(7) holds with $T(x) = \frac{\rho - 1}{\rho} V(x)$, $\rho > 1$.

It is interesting to remark that, when the storage function is positive definite, geometric QSR-dissipativity implies strict QSR-dissipativity, while the converse is not true. A sufficient condition to certify the local strict QSR-dissipativity of system (1) is given in the sequel.

Lemma 1: [4, Theorem 13.21, p. 808] If there exist a set $\mathcal{X}_0 \subseteq \mathcal{X}$, a nonnegative $\mathcal{C}^2$ function $V : \mathcal{X}_0 \to \mathbb{R}_0^+$, a positive-definite function $T : \mathcal{X}_0 \to \mathbb{R}_0^+$, and functions $P_1 : \mathcal{X}_0 \to \mathbb{R}^{1 \times m}$, $P_2 : \mathcal{X}_0 \to \mathbb{S}_m$, such that

$$V\left(f(x) + g(x)u\right) = V(f(x)) + P_1(x)u + u^\top P_2(x) u, \quad (8)$$

and, for all $x \in \mathcal{X}_0$, the following conditions hold

$$\overline{R}(x) \equiv R - P_2(x) \succ 0, \quad (9)$$

$$\begin{aligned} &\left(h^\top(x)S - \frac{1}{2}P_1(x)\right)\overline{R}^{-1}(x)\left(h^\top(x)S - \frac{1}{2}P_1(x)\right)^\top \\ &+ V(f(x)) - V(x) - h^\top(x)Qh(x) + T(x) \leq 0, \quad (10) \end{aligned}$$

then (1) is locally strictly dissipative with respect to the quadratic supply rate (7). Alternatively, thanks to the strict inequality (9), (10) can be equivalently re-written as

$$\begin{bmatrix} V(x) - V(f(x)) + h^\top(x)Qh(x) - T(x) & \star \\ \left(h^\top(x)S - \frac{1}{2}P_1(x)\right)^\top & \overline{R}(x) \end{bmatrix} \succeq 0 \quad (11)$$

The following lemma holds in the case of geometric dissipativity.

Lemma 2: If there exist a set $\mathcal{X}_0 \subseteq \mathcal{X}$, a nonnegative $\mathcal{C}^2$ storage function $V : \mathcal{X}_0 \to \mathbb{R}_0^+$, and functions $P_1 : \mathcal{X}_0 \to \mathbb{R}^{1 \times m}$, $P_2 : \mathcal{X}_0 \to \mathbb{S}_m$, such that conditions (8) and (10) (or, alternatively, (11)) hold with $T(x) = \frac{\rho-1}{\rho}V(x)$ (T then only nonnegative), $\rho > 1$, the system (1) is locally geometrically dissipative with respect to the quadratic supply rate (7).

Remark 2: While necessary and sufficient conditions for the existence of storage functions in continuous-time dissipative nonlinear systems exist[1], only sufficient conditions are generally available for discrete-time nonlinear systems. The conditions in Lemmas 1 and 2 are such examples, specifically sufficient for the dissipativity of system (1). This stems primarily from the restriction (8), which in general does not need to hold for the storage function $V(x)$. On the other hand, necessity holds if system (1) is lossless, i.e., if (10) holds with equality. This discussion first appeared in [6] and was further elaborated in subsequent studies [9, Remark 2.1] and [11, Remark 4.1].

III. Dissipativity-based feedback stabilization

In this section, we provide dissipativity-based conditions for the stabilizability of system (1) by the linear SOF (2).

Proposition 2: Suppose the conditions for local strict QSR-dissipativity in Lemma 1 are feasible, the associated storage function V is positive definite, and, additionally,

$$R \preceq \gamma I + P_2(x), \quad x \in \mathcal{X}_0, \quad (12)$$

$$\Delta \succeq S\gamma^{-1}P_2(x)\gamma^{-1}S^\top, \quad x \in \mathcal{X}_0, \quad (13)$$

hold for some $\gamma \in \mathbb{R}^+$, where

$$\Delta \equiv S\gamma^{-1}S^\top - Q. \quad (14)$$

Then, the system (1) is *locally* asymptotically stabilizable by the *linear* SOF (2). A stabilizing gain K is given by

$$K = -\gamma^{-1}S^\top. \quad (15)$$

Proof: Suppose (10) holds, which guarantees that the system (1) is locally strictly QSR-dissipative. Additionally, suppose (12) also holds[2] for some $\gamma \in \mathbb{R}^+$. Thus

$$\begin{aligned} &V(f(x)) - V(x) - h^\top(x)Qh(x) + T(x) \\ &+\left(h^\top(x)S - \tfrac{1}{2}P_1(x)\right)\gamma^{-1}\left(h^\top(x)S - \tfrac{1}{2}P_1(x)\right)^\top \leq 0, \end{aligned} \quad (16)$$

also holds for $x \in \mathcal{X}_0$. The main idea for the rest of the proof is to show that with the gain (15), the closed-loop given by the interconnection (1)-(2) is stable if the Δ-*condition* (13) holds. Applying the relation (15) in (16), one obtains

$$\begin{aligned} &V(f(x)) - V(x) + T(x) - h^\top(x)Qh(x) \\ &+ h^\top(x)S\gamma^{-1}S^\top h(x) + \frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2} \\ &+ \frac{1}{2}P_1(x)Kh(x) + \frac{1}{2}h^\top(x)K^\top P_1(x)^\top \leq 0, \ x \in \mathcal{X}_0, \end{aligned}$$

[1] See, for example, [4, Theorem 5.6, p. 347, Theorem 5.7, p. 349].
[2] Note that such γ always exist if $\mathcal{X}_0$ is a compact set.

which can be rewritten as

$$V(f(x)) - V(x) + T(x) + h^\top(x)\Delta h(x) + \frac{1}{2}P_1(x)Kh(x) + \frac{1}{2}h^\top(x)K^\top P_1(x)^\top + \frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2} \le 0,\ x \in \mathcal{X}_0,$$

with the matrix Δ defined in (14). Then, if (13) holds, i.e., if $\Delta \succeq S\gamma^{-1}P_2(x)\gamma^{-1}S^\top,\ \forall x \in \mathcal{X}_0$, the last inequality implies, due to the fact that T is a positive-definite function, that

$$\begin{aligned} V(f(x)) - V(x) + h^\top(x)S\gamma^{-1}P_2(x)\gamma^{-1}S^\top h(x) \\ +\frac{1}{2}P_1(x)Kh(x) + \frac{1}{2}h^\top(x)K^\top P_1(x)^\top \\ < -\frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2},\ x \in \mathcal{X}_0,\ x \neq 0, \end{aligned}$$

which (with relation (15)) implies

$$\begin{aligned} V(f(x)) - V(x) + h^\top(x)K^\top P_2(x)Kh(x) + \frac{1}{2}P_1(x)Kh(x) \\ +\frac{1}{2}h^\top(x)K^\top P_1(x)^\top < 0,\ x \in \mathcal{X}_0,\ x \neq 0, \end{aligned}$$

and, due to (8), $V(F(x, Kh(x))) - V(x) < 0,\ x \in \mathcal{X}_0,\ x \neq 0$. The last inequality and the fact that V is positive definite assure the asymptotic stability of the closed-loop system given by the interconnection (1)-(2). ∎

Corollary 1: Suppose that the conditions for *local* geometric dissipativity enunciated in Lemma 2 are feasible with an associated storage function that is positive definite. Furthermore, suppose there exists $\gamma \in \mathbb{R}^+$ such that the inequalities (12)-(13) hold. Then, the closed-loop system is rendered geometrically stable by (2) with the gain (15).

Proof: The proof follows similarly the steps of the proof of Proposition 2, but considering $T(x) = \frac{\rho-1}{\rho}V(x)$, $\rho > 1$, and thus is not repeated here. ∎

Next, we showcase the special case when the conditions of Corollary 1 are also *necessary*.

Proposition 3: Consider (1) with $g(x) = B \in \mathbb{R}^{n\times m}$, that is, the special case of constant $g(x)$. Suppose that the conditions for *local* geometric stabilizability enunciated in Proposition 1 hold in a *compact set*[3] $\mathcal{X}_1 \subseteq \mathcal{X}$, with $0 \in \overset{o}{\mathcal{X}_1}$, for some quadratic Lyapunov function $V(x) = x^\top Px$, $P \in \mathbb{S}_n^+$, and stabilizing gain K. Then, the dissipativity-based stabilization conditions enunciated in Corollary 1, i.e.,

$$(\text{Satisfaction of conditions in Lemma 2} +\ (12) + (13))$$

are both *sufficient and necessary*. A stabilizing gain K can be computed with (15).

Proof: The sufficiency is a direct consequence of Corollary 1. Here, we aim to demonstrate necessity. First, consider condition (5b) is fulfilled for some K, Lyapunov function $V(x) = x^\top Px$, $P \in \mathbb{S}_n^+$, and scalar $\rho > 1$, in a compact set $\mathcal{X}_1$. Condition (5b) can be equivalently rewritten as

$$V(\overline{F}(x)) - \frac{1}{\bar\rho}V(x) \le \frac{1}{\rho}V(x) - \frac{1}{\bar\rho}V(x),\ x \in \mathcal{X}_1, \quad (17)$$

for any $\bar\rho$ such that $1 < \bar\rho < \rho$.

The function f being $\mathcal{C}^2$, with $f(0) = 0$, and $V(x) = x^\top Px$ being quadratic with P positive definite, there exists $\gamma \in \mathbb{R}^+$ large enough such that

$$\begin{aligned} &f^\top(x)PB\gamma^{-1}B^\top Pf(x) \\ &= \frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2} \le \left(\frac{1}{\bar\rho} - \frac{1}{\rho}\right)V(x),\quad x \in \mathcal{X}_1, \end{aligned} \quad (18)$$

where $P_1(x) = 2f^\top(x)PB$. Thus, from (17) and (18)

$$V(\overline{F}(x)) - \frac{1}{\bar\rho}V(x) \le -\frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2},\ x \in \mathcal{X}_1. \quad (19)$$

The Lyapunov function being a quadratic function, we can expand the expression of $V(\overline{F}(x))$ to obtain

$$\begin{aligned} V(\overline{F}(x)) &= V(f(x) + BKh(x)) \\ &= V(f(x)) + P_1(x)Kh(x) + h^\top(x)K^\top P_2(x)Kh(x), \end{aligned}$$

where $P_2(x)$ is, in this case, constant and given by $P_2(x) = B^\top PB$. Therefore, the condition (19) reads

$$\begin{aligned} &V(f(x)) - V(x) + \frac{1}{2}P_1(x)Kh(x) + \frac{1}{2}h^\top(x)K^\top P_1(x)^\top \\ &+ h^\top(x)K^\top P_2(x)Kh(x) + \frac{\bar\rho - 1}{\bar\rho}V(x) \\ &+ \frac{1}{2}P_1(x)\gamma^{-1}P_1(x)^\top\frac{1}{2} \le 0,\quad x \in \mathcal{X}_1. \end{aligned} \quad (20)$$

Let us now introduce $R = \gamma I + P_2(x) = \gamma I + B^\top PB \in \mathbb{S}_m^+$, which leads to $\overline{R} = \gamma I$ being clearly invertible, and to satisfaction of (12) by definition. Next, we also select $S = -\gamma K^\top = -K^\top\overline{R} \in \mathbb{R}^{p\times m}$, so that $K = -\overline{R}^{-1}S^\top$. The inequality (20) then rewrites

$$\begin{aligned} &V(f(x)) - \frac{1}{2}P_1(x)\overline{R}^{-1}S^\top h(x) - \frac{1}{2}h^\top(x)S\overline{R}^{-1}P_1(x)^\top \\ &+ h^\top(x)S\overline{R}^{-1}P_2(x)\overline{R}^{-1}S^\top h(x) - V(x) + \frac{\bar\rho - 1}{\bar\rho}V(x) \\ &+ \frac{1}{2}P_1(x)\overline{R}^{-1}P_1(x)^\top\frac{1}{2} \le 0,\quad x \in \mathcal{X}_1. \end{aligned} \quad (21)$$

Selecting $Q = S\overline{R}^{-1}P_2(x)\overline{R}^{-1}S^\top - S\overline{R}^{-1}S^\top$ leads to $Q = S\overline{R}^{-1}B^\top PB\overline{R}^{-1}S^\top - S\overline{R}^{-1}S^\top$ and thus to the satisfaction of $\Delta = S\gamma^{-1}P_2(x)\gamma^{-1}S^\top = S\gamma^{-1}B^\top PB\gamma^{-1}S^\top$, i.e., the fulfillment of condition (13) with equality. To see that condition (10) is also satisfied, note that (21) can be rewritten as (10) by using this particular choice of Q. Therefore, the proof is concluded. ∎

IV. Numerical Examples

A. Example – constant $g(x)$

Consider a system with state $x(k) = \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix}^\top \in \mathbb{R}^2$, given by:

$$\begin{cases} x_1(k+1) = x_1^2(k) + x_2(k), \\ x_2(k+1) = ax_2(k) + u(k), \quad a > 1, \\ y(k) = x_2(k) \end{cases}$$

[3]Compactness is fundamental to the proof of the proposition.

i.e, system (1) with $f(x(k)) = \begin{bmatrix} f_1(x(k)) & f_2(x(k)) \end{bmatrix}^\top = \begin{bmatrix} x_1^2(k) + x_2(k) & ax_2(k) \end{bmatrix}^\top$, $g(x(k)) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, and $h(x(k)) = x_2(k)$, implying that only the second state is available for feedback. The constant $a > 1$ ensures that the open-loop system ($u \equiv 0$) is unstable and non-passive. Here we aim to show that the system can be stabilized using the proposed QSR-dissipativity-based approach for $a < \bar{a}$, $\bar{a} = 2/\sqrt{3}$.

Strict QSR-dissipativity holds for some positive definite T and $R - P_2(x) \succ 0$ if, for example, inequalities

$$2h^\top S = P_1(x) \tag{22}$$

$$V(f(x)) - V(x) - h^\top(x)Qh(x) < 0, \tag{23}$$

are satisfied. From (22) and (8) one gets that, for example, $V(x_1, x_2) = \frac{x_2^2 S}{a} + V_1(x_1)$ with $S > 0$ and $V_1(x_1) = v_1 x_1^2$, for some $v_1 \in \mathbb{R}^+$, is a potential positive-definite storage function. Considering this V, we can rewrite (23) as

$$V(f_1(x), f_2(x)) - V(x_1, x_2) - x_2^2 Q =$$
$$V_1(f_1(x)) - V_1(x_1) + bx_2^2 S - x_2^2 Q < 0,$$

where $b = \frac{a^2-1}{a} > 0$. The previous equation leads to

$$z = v_1\left(x_1^2 + x_2\right)^2 - v_1 x_1^2 + bx_2^2 S - x_2^2 Q < 0. \tag{24}$$

By evaluating the Hessian matrix of z at the origin, one can note that (24) is locally satisfied for some region $\mathcal{X}_0 \subset R^2$ for v_1 and Q such that

$$Q > v_1 + bS > 0. \tag{25}$$

Since $P_2(x) = \frac{S}{a}$, the constraint $\Delta = S\gamma^{-1}P_2(x)\gamma^{-1}S^\top$ can be satisfied for positive numbers $(Q, S, \gamma) > 0$ by choosing any positive tuple $(v_1, \gamma, S) > 0$ that satisfy $v_1 + bS < S^2\gamma^{-1} - S^3\gamma^{-2}$, and then computing Q as $Q = S^2\gamma^{-1} - S^3\gamma^{-2}$. The stabilization procedure then amounts to:

Step 1: For given a, finding the positive values of (γ, S) that satisfy $aS\gamma - S^2 - a^2\gamma^2 + \gamma^2 > 0$. This problem has solutions for all a such that $0 < a < 2/\sqrt{3}$ by choosing any $S > 0$ and then any γ such that

$$\underline{\gamma}(a, S) < \gamma < \bar{\gamma}(a, S), \tag{26a}$$

where

$$\bar{\gamma}(a, S) = \frac{0.5\left(a^2 d(a,s) - d(a,s) + aS\right)}{a^2 - 1}, \tag{26b}$$

$$\underline{\gamma}(a, S) = \frac{0.5\left(-a^2 d(a,s) + d(a,s) + aS\right)}{a^2 - 1}, \tag{26c}$$

with $d(a,s) = \sqrt{\frac{-(3a^2-4)S^2}{(a^2-1)^2}}$.

Step 2: Choosing any v_1 such that $0 < v_1 < S^2\gamma^{-1} - a^{-1}S^3\gamma^{-2} - bS$ and then $Q = S^2\gamma^{-1} - S^3\gamma^{-2}$. One can check that v_1 small enough always exists satisfying that inequality since the procedure in Step 1 guarantees that $aS^2\gamma^{-1} - S^3\gamma^{-2} - abS$ is positive. Furthermore, the procedure in Step 1 also guarantees that Q is such that (25) holds.

Step 3: Finally, it is straightforward to check that there exists R small enough such $R \leq \gamma + P_2(x) = \gamma + S/a$ and the system is locally strictly QSR-dissipative in $\mathcal{X}_0 \times \mathbb{R}$, where $\mathcal{X}_0 \subset \mathbb{R}^2$ is the set where $z < 0$ is attained. Then, the stabilizing gain can be computed by $K = -\gamma^{-1}S^\top$.

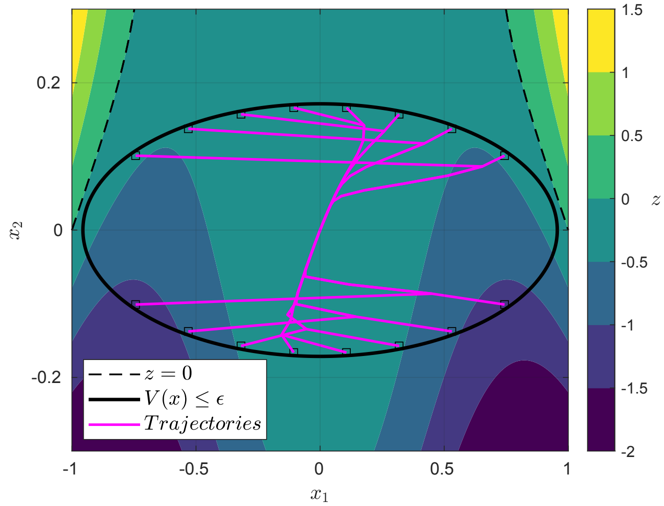

Fig. 1: Contour plot of $z = v_1\left(x_1^2 + x_2\right)^2 - v_1 x_1^2 - x_2^2 Q$ illustrating region where $z < 0$, a sublevel set of the Lyapunov function $V(x) \leq \epsilon$ included in the region where $z < 0$, and closed-loop trajectories.

To illustrate this procedure, consider $a = 1.05$ and choose $S = 100$. Use (26a) to choose $\gamma = 0.5(\underline{\gamma}(a,S) + \bar{\gamma}(a,S)) = 512.1951$ and then take $v_1 = 0.5a^{-1}(aS^2\gamma^{-1} - S^3\gamma^{-2} - abS) = 3.0658$. Compute $Q = S^2\gamma^{-1} - S^3\gamma^{-2} = 15.8935 > v + bS$. Then local assymptotic stability holds with the control $K = -0.1952$. As $V(x)$ (and $V(\overline{F}(x)) - V(x)$) are positive (respectively, negative) -definite continuously-differentiable polynomials, there also exist α, β, and ρ such that (5a)-(5b) hold and geometric stability of the origin can also be concluded.

The region of the state space $\mathcal{X}_0 \subset \mathbb{R}^2$ where (24) holds with the computed Q and v_1 is illustrated in Fig. 1 by the region where $z < 0$. To improve visualisation, the line contours where $z = 0$ are also plotted in the figure. Furthermore, we show the sublevel set $\varepsilon = \{x \in \mathbb{R}^2 : V(x) \leq \epsilon\}$, $\epsilon = 2.8$, included in the region where $z < 0$, and which is a region of attraction of the closed-loop system with the computed control. Furthermore, we illustrate several trajectories of the closed-loop system. The associated inputs (not plotted), are upper bounded by $\|u(k)\| \leq |K| \|h(x)\|$. Since the state is in $V(x) \leq \epsilon$, which is a compact set, and $h(x) = x_2$, we have $|h(x)| \leq 0.2$, leading to $|u(k)| \leq 0.039$.

B. Example – nonlinear $g(x)$

Now suppose we replace $g(x) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ with the nonlinear matrix $g(x) = \begin{bmatrix} 0 & (1 + x_1^2) \end{bmatrix}^\top$ in the example. Relation (22) leads to V being of the form $V(x) = \frac{Sx_2^2}{a(1+x_1^2)} + L(x_1)$ for some function $L(x_1)$. Let us take $L(x_1) = v_1 x_1^2$, $v \in \mathbb{R}^+$, to ensure the storage function is positive definite.

By developing (23) with this V, one gets to the conclusion that (23) holds if

$$\begin{aligned} z =& S(ax_2)^2(1+x_1^2) + av_1(x_1^2+x_2)^2(1+x_1^2)l \\ &- Sx_2^2 l - av_1 x_1^2(1+x_1^2)l - aQx_2^2(1+x_1^2)l < 0 \end{aligned} \quad (27)$$

where $l = 1 + (x_1^2 + x_2)^2$. By evaluating the Hessian matrix of z at the origin, one can note that (27) is locally satisfied for some region $\mathcal{X}_0 \subset R^2$ for v_1 and Q such that

$$Q > \frac{Sa^2 + v_1 a - S}{a}. \quad (28)$$

In this case, $P_2(x)$ is not a constant and is given by $P_2(x) = \frac{(1+x_1^2)^2}{l}\frac{S}{a}$. By imposing the bound $w = \frac{(1+x_1^2)^2}{l} < M$ for some positive M, one gets $P_2(x) \leq \frac{MS}{a}$. Using a similar line of though as in the previous case, we can use the following procedure to solve the sufficient condition $\Delta \succeq S\gamma^{-1}P_2(x)\gamma^{-1}S^\top$.

Step 1: For given $a > 1$, compute some M such that $0 < M < \frac{a^2}{4(a^2-1)}$ and choose some $S > 0$. Then compute any γ such that $\underline{\gamma}(a,S) < \gamma < \bar{\gamma}(a,S)$, where $\bar{\gamma}(a,S)$ and $\underline{\gamma}(a,S)$ have a similar form to (26b) but with $d(a,s) = \sqrt{\frac{-S^2(4a^2M-a^2-4M)}{(a^2-1)^2}}$.

Step 2: Compute some v_1 such that $0 < v_1 a < aS^2\gamma^{-1} - S^3\gamma^{-2}M + S(1-a^2)$. This is guaranteed to yield a positive v_1 since the previous step ensures $aS^2\gamma^{-1} - S^3\gamma^{-2}M + S(1-a^2) > 0$.

Step 3: Compute Q using $Q = S^2\gamma^{-1} - \frac{MS^3\gamma^{-2}}{a}$. A positive Q satisfying (28) and the sufficient condition $\Delta \succeq S\gamma^{-1}P_2(x)\gamma^{-1}S^\top$ is ensured for all $x \in \mathcal{W} = \{x : w = \frac{(1+x1^2)^2}{l} < M\}$. Some small enough R satisfying $R \preceq \gamma I + P_2(x)$ for $x \in \mathcal{W}$ can always be computed. Then, the system is locally strictly QSR-dissipative in $\mathcal{X}_0 \times \mathbb{R}$, where $\mathcal{X}_0 \subset \mathbb{R}^2$ is the set where $z < 0 \wedge w < M$ is attained. The stabilizing gain can be computed by $K = -\gamma^{-1}S^\top$.

Fig. 2 illustrates a sublevel set of $V(x)$ included in the set where $z < 0 \wedge w < M$, and closed-loop trajectories for $a = 1.05$, $M = 1.8823$, $S = 100$, $\gamma = 512.1951$, $v_1 = 1.4643$, $Q = 12.6905$, and $\epsilon = 0.8$. The control gain is given again by $K = -0.1952$.

V. Conclusion

In conclusion, this research extends the results obtained for dissipativity-based stabilization of continuous-time nonlinear systems in [13] to the discrete-time setting. The new conditions demonstrate the relations between the asymptotic and geometric SOF stabilizability properties of nonlinear input-affine discrete-time systems and the strict/geometric dissipativity properties of the open system. The conditions allow stabilizing nonlinear discrete-time systems which are open-loop unstable and that cannot be directly stabilized by techniques based on passive and/or lossless systems unless, possibly, a "*passivation*" step can be applied. The next steps in this line of research include addressing the stabilization of some sub-classes of hybrid and switched systems.

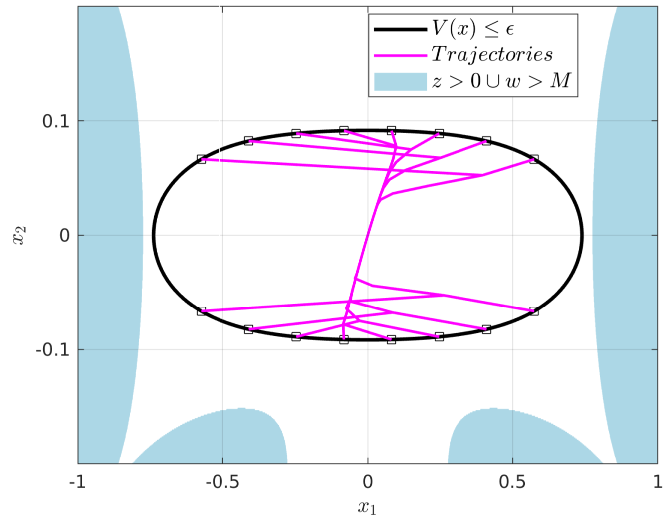

Fig. 2: A sublevel set of the Lyapunov function $V(x) \leq \epsilon$, for the case of non-constant $g(x)$, included in the region where $z < 0 \wedge w < M$, and closed-loop trajectories.

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